N. J., 1963).

[32]The definition of untransformed pair function in Eqs. (2.10) and (2.11) differs from that used in earlier work (see Refs. 1 and 3) through the inclusion of the factor $\exp\{-t_0[\omega(k_1)+\omega(k_2)-\omega(k_3)-\omega(k_4)]\}$. We make this change so that in the following papers of this series the temperature dependence of the pair function will be more apparent [see Appendix B of F. Mohling, I. RamaRao, and D. W. J. Shea, following paper, Phys. Rev. A1, 192 (1970)].

[33]The temperature integrals of the $\mathcal{G}_{\mu,\nu}(t_2, t_1, k)$ are essentially equal to the reduced density matrices. Since $\langle n(p)\rangle = \langle p|\rho_1|p\rangle$, this equality is obvious from Eq. (4.1) for the $(\mu,\nu)=(1,1)$ case. See also Ref. 3.

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Microscopic Theory of Quantum Fluids. II. Λ Transformation

F. Mohling
Department of Physics and Astrophysics, University of Colorado, Boulder, Colorado 80302

and

I. RamaRao *
Department of Physics, Stanford University, Stanford, California 94305

and

Dion W. J. Shea †
Aeronomy Laboratory, Environmental Science Services Administration, Boulder, Colorado 80302

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In a previous paper, the master-graph formulation of the quantum-statistical theory of quantum fluids was developed. If this formulation is used to calculate the equilibrium properties of quantum fluids, apparent divergences are encountered in the low-temperature limit. In the present paper, we transform this theory by means of a Λ transformation to overcome these apparent low-temperature divergences. In this transformation, the terms in the theory which gave rise to the apparent low-temperature divergences and which represent the dominant low-temperature contributions are summed explicitly to obtain well-behaved expressions. In addition, a consistent method is developed to obtain the corrections to the dominant low-temperature contributions. Explicit expressions for the Λ-transformed theory are given for the cases of a Bose fluid above the Bose-Einstein condensation temperature and for a Fermi fluid. Finally, the physical implications of the Λ transformation are discussed.

1. INTRODUCTION

In the preceding paper,[1] we developed a quantum-statistical theory of quantum fluids and obtained the master-graph formulation of the theory through a very careful analysis of the self-energy problem. In particular, we expressed the grand potential and the momentum distribution [Eqs. (I. 5.12) and (I. 4.1)] in terms of master graphs. These quantities were functionals of five different types of line factors which arose from the self-energy analysis: the solid-line factors $\mathcal{G}_{\mu,\nu}(t_2, t_1, k)$ with $(\mu, \nu) = (1,1)$, (0, 2), and (2, 0); and the outgoing (incoming) dotted zero-momentum line factors $G_{\text{out}}^{(0)}(t)\,[G_{\text{in}}^{(0)}(t)]$. These line factors were expressed in terms of a set of integral equations (I. 4.4), (I. 4.10)–(I. 4.15), (I. 4.17), (I. 4.18), (I. 3.1), (I. 5.14), and (I.5.15).

The reader might expect that a simple iteration of the integral equations could be used to calculate these line factors, which iteration could, in turn, be applied to obtain meaningful expressions for the grand potential and the momentum distribution. Unfortunately, this iterative procedure cannot be used since, in the low-temperature limit, it gives apparently divergent contributions (ADC) to the line factors, and hence to the grand potential and the momentum distribution. To overcome this problem we first identify the dominant parts, which lead to ADC, of the kernels of the integral equations for the line factors and then solve the resultant approximate integral equations exactly. These solutions, referred to as *characteristic functions*, are well-behaved functions which would otherwise appear in the master-graph formulation as series expansions. The (line-factor) corrections to the characteristic functions can then be obtained by iteration. The purpose of this paper is to show how this procedure can be achieved in a direct manner by applying the Λ transformation to the master-graph formulation.

What we call the Λ transformation actually consists of two parts: (i) the identification of those terms (the characteristic functions) in each master-graph line factor that comprise the ADC for that line factor; and (ii) the systematic summation everywhere (by an integral transformation) of the ADC to obtain a well-behaved formulation of the theory in the low-temperature limit. The integral transformation, or Λ *transformation*, results in a consistent rearrangement of the theory which permits the identification of the well-behaved corrections to the ADC.

In Sec. 2, we discuss the problem of identifying the characteristic functions which represent the ADC to the master-graph line factors. In Sec. 3, we define the basic integral transformations of the line factors and pair functions, which then leads us to define transformed master graphs. In Sec. 4, we apply the general form of the Λ transformation to obtain expressions for the grand potential and the momentum distribution in terms of transformed master graphs. In Sec. 5, we show the explicit form of the Λ transformation for a "normal" fluid, i.e., for a Bose fluid above the Bose-Einstein condensation temperature or for a Fermi fluid. In Sec. 6, we discuss the physical implications of the Λ transformation.

The complete development of the Λ transformation for a "degenerate" fluid, i.e., for a Bose fluid below the Bose-Einstein condensation temperature, will be presented in a subsequent paper.

2. CHARACTERISTIC FUNCTIONS

There are five different types of internal lines that occur in master graphs (see Sec. 4 of I). These are the three different types of solid internal lines with associated line factors $\mathcal{G}_{\mu,\nu}(t_2, t_1, k)$, $(\mu, \nu) = (1, 1)$, $(0, 2)$, or $(2, 0)$, the outgoing dotted (zero-momentum) lines with associated line factor $G_{\text{out}}^{(0)}(t)$, and the incoming dotted (zero-momentum) lines with associated line factor $G_{\text{in}}^{(0)}(t)$. Each of these line factors makes ADC to the grand potential and the momentum distribution. In this section, we study the identification of the characteristic functions that represent the ADC to the line factors. Three such characteristic functions are associated with the line factors: one connected with the $+k$ lines, one with the $-k$ lines, and one with the zero-momentum lines.

Let us examine Eqs. (I. 4.4) and (I. 4.13)–(I. 4.15), which give rise to a set of integral equations for the master-graph solid-internal line factors $\mathcal{G}_{\mu,\nu}(t_2, t_1, k)$. To obtain a consistent treatment of the $+k$ and $-k$ lines, which undergo different Λ transformations, we will consider only the first lines of Eqs. (I. 4.13)–(I. 4.15). In so doing, we associate $\mathcal{G}_{1,1}(t_2,t_1,k)$ only with the $+k$ lines and $\mathcal{G}(t_2,t_1,k)$ only with the $-k$ lines.

Consider the kernel $Q_{1,1}(t_2,t_1,k)$ of the integral equation for $\mathcal{G}_{1,1}(t_2,t_1,k)$ [see Eq. (I. 4.12)]. This kernel contains certain types of terms, which we represent by $Q_0'(t_2,t_1,k)$, that generate all of the ADC in $\mathcal{G}_{1,1}(t_2,t_1,k)$. If we solve exactly the integral equation

$$\mathcal{G}_0'(t_2, t_1, k) = \delta(t_2^{(-)} - t_1) + \epsilon\, \mathcal{L}_0'(t_2, t_1, k), \tag{2.1}$$

$$\mathcal{L}_0'(t_2, t_1, k) = \int_0^\beta ds\; \mathcal{G}_0'(t_2, s, k) \times Q_0'(s, t_1, k), \tag{2.2}$$

which is analogous to Eqs. (I. 4.4) and (I. 4.13), then $\mathcal{G}_0'(t_2,t_1,k)$ gives essentially the series sum of the ADC to $\mathcal{G}_{1,1}(t_2,t_1,k)$. We refer to $\mathcal{G}_0'(t_2, t_1, k)$ as the characteristic function for the $+k$ lines.

Similarly, consider the kernel $M_{1,1}(t_2, t_1, -k)$ of the integral equation for $\mathcal{G}(t_2, t_1, -k)$, Eqs. (I. 4.10) and (I. 4.11). Analogous to $Q_0'(t_2, t_1, k)$, we denote the part of $M_{1,1}(t_2, t_1, -k)$ that generates all of the ADC in $\mathcal{G}(t_2, t_1, -k)$ by $M_0^{(1)}(t_2, t_1, -k)$. If we solve exactly the integral equation

$$\mathcal{G}_0^{(1)}(t_2, t_1, -k) = \delta(t_2^{(-)} - t_1) + \epsilon\mathcal{L}_0^{(1)}(t_2, t_1, -k)\ , \tag{2.3}$$

$$\mathcal{L}_0^{(1)}(t_2, t_1, -k) = \int_0^\beta ds\, \mathcal{G}_0^{(1)}(t_2, s, -k) \times M_0^{(1)}(s, t_1, -k)\ , \tag{2.4}$$

which is analogous to Eqs. (I. 4.10) and (I. 4.11), then $\mathcal{G}_0^{(1)}(t_2, t_1, -k)$, the characteristic function for the $-k$ lines, gives essentially the series sum of the ADC in $\mathcal{G}(t_2, t_1, -k)$. With the identification of the characteristic functions $\mathcal{G}_0'(t_2, t_1, k)$ and $\mathcal{G}_0^{(1)}(t_2, t_1, -k)$, we have, according to Eqs. (I. 4.14) and (I. 4.15), also identified all of the ADC in $\mathcal{G}_{0,2}(t_2, t_1, k)$ and $\mathcal{G}_{2,0}(t_2, t_1, k)$.

We next introduce the characteristic function $\mathcal{G}_0^{(0)}(t_2, t_1)$ for the zero-momentum lines which gives essentially the series sum of the ADC in $G_{\text{out}}^{(0)}(t)$ and $G_{\text{in}}^{(0)}(t)$. Unfortunately, there is no simple integral equation for $G_{\text{out}}^{(0)}(t)$ or $G_{\text{in}}^{(0)}(t)$ which would enable us to derive an expression for

$\mathcal{G}_0^{(0)}(t_2, t_1)$. However, we can deduce the proper form for this characteristic function by considering the $k \to 0$ limit of $\mathcal{G}_0'(t_2, t_1, k)$.

In addition to the above three characteristic functions which are associated with the master-graph line factors, we also have a fourth one denoted by $\mathcal{G}_0^{(T)}(t_2, t_1, k)$. This function is required to transform the $\mathcal{L}_{1,1}^{(t)}(t, t, p)$ term which occurs in Eq. (I. 5. 12) for the grand potential. In analogy with the determination of $\mathcal{G}_0'(t_2, t_1, k)$, we denote by $Q_0^{(T)}{}'(t_2, t_1, k)$ a special class of terms in the kernel $Q_{1,1}^{(T)}(t_2, t_1, k)$ of the integral equation for $\mathcal{G}_{1,1}^{(T)}(t_2, t_1, k)$ [see Eqs. (I. 5. 2) and (I. 5. 7)]. The quantity $Q_0^{(T)}{}'(t_2, t_1, k)$ generates, essentially,[2] all of the ADC in $\mathcal{G}_{1,1}^{(T)}(t_2, t_1, k)$. If we solve exactly the integral equation

$$\mathcal{G}_0^{(T)}{}'(t_2, t_1, k) = \delta(t_2^{(-)} - t_1) + \epsilon \mathcal{L}_0^{(T)}{}'(t_2, t_1, k) \; , \qquad (2.5)$$

$$\mathcal{L}_0^{(T)}{}'(t_2, t_1, k) = \int_0^T ds\, \mathcal{G}_0^{(T)}{}'(t_2, s, k) \times Q_0^{(T)}{}'(s, t_1, k) \; , \qquad (2.6)$$

which is analogous to Eqs. (I. 5. 2) and (I. 5. 7), then $\mathcal{G}_0^{(\tau)}{}'(t_2, t_1, k)$ can be used to arrive at the series sum of the ADC in $\mathcal{G}_{1,1}^{(\tau)}(t_2, t_1, k)$.

We shall not give further consideration in this section to the difficult problem of properly determining the characteristic functions. However, in Sec. 5, we shall illustrate their determination for the case of a "normal" fluid ($\langle x \rangle = 0$).[3] In this connection, we observe that for a normal fluid, only $\mathcal{G}_0'(t_2, t_1, k)$ and $\mathcal{G}_0^{(\tau)}{}'(t_2, t_1, k)$ must be considered; $\mathcal{G}_0^{(0)}(t_2, t_1)$ does not occur and $\mathcal{G}_0^{(1)}(t_2, t_1, -k)$ is identical to $\mathcal{G}_0'(t_2, t_1, -k)$. We shall consider the determination of the characteristic functions for the case of a "degenerate" fluid ($\langle x \rangle \neq 0$) in a subsequent paper.

Assuming that the characteristic functions, introduced in this section, are known in principle, we will proceed with the formulation of the Λ transformation in Secs. 3 and 4.

3. BASIC TRANSFORMATION DEFINITIONS

In I, the quantum-statistical theory was expressed in terms of master graphs (untransformed) which were functionals of the pair functions and five different types of line factors. As discussed in Secs. 1 and 2, each of these line factors contains characteristic functions which represent sums over ADC. In essence, the Λ transformation is a self-consistent method for transferring the well-behaved characteristic functions from the line factors to the vertex, or pair, functions. This transfer results in the replacement of free single-particle energies by renormalized single-particle energies, i.e., by quasiparticle energies. This property of the Λ transformation will be demonstrated for a normal quantum fluid in Sec. 5 and for the degenerate Bose fluid in a subsequent paper.

We first define the transformed line factors, identified with a prime superscript, by

$$\mathcal{G}_{1,1}(t_2, t_1, k) = H'(t_2, k) \int_0^\beta ds\, \mathcal{G}'_{1,1}(t_2, s, k)[(H'(s,k)]^{-1}\mathcal{G}_0'(s, t_1, k), \qquad (3.1)$$

$$\mathcal{G}_{0,2}(t_2, t_1, k) = \int_0^\beta ds_2 ds_1 \mathcal{G}'_{0,2}(s_2, s_1, k)[H'(s_2, k)]^{-1} \times \mathcal{G}'_0(s_2, t_2, k)[H^{(1)}(s_1, -k)]^{-1}\mathcal{G}_0^{(1)}(s_1, t_1, -k), \qquad (3.2)$$

$$\mathcal{G}_{2,0}(t_2, t_1, k) = H'(t_2, k) H^{(1)}(t_1, -k)\, \mathcal{G}'_{2,0}(t_2, t_1, k), \qquad (3.3)$$

$$G_{\text{out}}^{(0)}(t) = h_a^{(0)} \int_0^\beta ds\, G_{\text{out}}^{(0)}{}'(s)[H^{(0)}(s)]^{-1}\mathcal{G}_0^{(0)}(s, t), \qquad (3.4)$$

$$G_{\text{in}}^{(0)}(t) = h_b^{(0)} H^{(0)}(t)\, G_{\text{in}}^{(0)}{}'(t)\, , \qquad (3.5)$$

where the quantities $H'(t,k)$, $H^{(1)}(t,k)$, $H^{(0)}(t)$, $h_a^{(0)}$, and $h_b^{(0)}$ are arbitrary functions introduced for later convenience. We observe that the transformation function associated with a given line depends upon whether the line is an incoming or outgoing line and upon whether the momentum label of the line is $+k$, $-k$, or $k \equiv 0$. Equations (3.1) – (3.5) are represented diagrammatically in Fig. 1.

In addition to the transformed line factors, we also need to define transformed pair functions. These combine most of the quantities that are removed from the line factors (except for the functions $h_a^{(0)}$ and

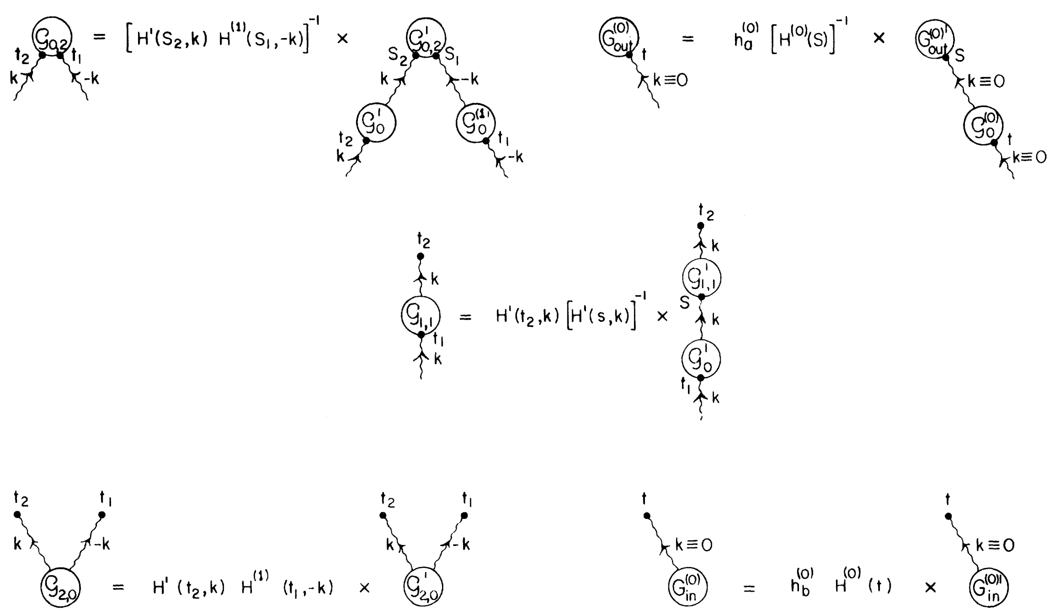

FIG. 1. Diagrammatic representation of the Λ transformation of the master-graph line factors as defined by Eqs. (3.1)–(3.5).

$h_b{}^{(0)}$ associated with the dotted lines) with the untransformed pair functions, Eqs. (I. 2.10) and (I.2.11). Thus, we next define the transformed pair function by

$$
{}^{t_1 t_2}\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}''_{t_0} \equiv [H'(t_1,k_1)H'(t_2,k_2)]^{-1} H'(t_0,k_3)\,H'(t_0,k_4) \times \int_0^{\beta} ds_1 ds_2\, \mathcal{G}'_0(t_1,s_1,k_1)\,\mathcal{G}'_0(t_2,s_2,k_2)\quad {}^{s_1 s_2}\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}_{t_0}, \tag{3.6}
$$

where we used a double-prime superscript to indicate the transformed pair function in Eq. (3.6). This equation has been written for the case in which all of the lines attaching to the pair function are $+k$ lines. If one (or more) of the lines is a $-k$ or $k\equiv 0$ line, then the $+k$ transformation function associated with that line must be replaced with the $-k$ or $k\equiv 0$ transformation function, respectively. Equation (3.6) is represented diagrammatically in Fig. 2.

Transformed Master (μ,α) Graphs

We can now define transformed master (μ, ν) graphs and transformed master (μ, ν) L graphs by making the following changes in the rules for untransformed master (μ, ν) graphs that were given in Sec. 4 of I:

(a) Replace the untransformed solid line factors $\mathcal{G}_{\mu,\nu}(t_2, t_1, k)$ with the transformed solid-line factors $\mathcal{G}'_{\mu,\nu}(t_2, t_1, k)$ defined in Eqs. (3.1)–(3.3).

(b) Replace the untransformed pair functions with the transformed pair functions defined in Eq. (3.6) except in the wiggly-line double-bond correction term of rule (g) which remains essentially untransformed as

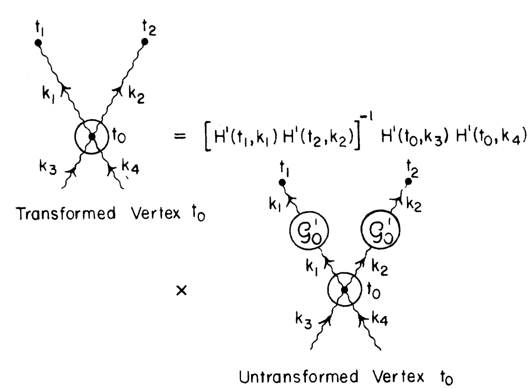

FIG. 2. Diagrammatic representation of the transformed vertex, or pair, function as defined by Eq. (3.6).

$$[H'(t_3,k_1)H'(t_3,k_2)]^{-1}\delta(t_3-t_1)\delta(t_3-t_2)^{t_3 t_3}$$

$$\times\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}_{t_4} H'(t_4,k_3)H'(t_4,k_4)\ .$$

Here we use the *untransformed* pair function.

(c) Replace the untransformed outgoing dotted (zero-momentum) line factors $G_{\text{out}}^{(0)}(t)$ by $h_a^{(0)}G_{\text{out}}^{(0)\prime}(t)$, as defined in Eq. (3.4), and the untransformed incoming dotted (zero-momentum) line factors $G_{\text{in}}^{(0)}(t)$ by $h_b^{(0)}G_{\text{in}}^{(0)\prime}(t)$, as defined in Eq. (3.5).

From Eqs. (3.1)–(3.6) and the rules for transformed master (μ,ν) graphs, it is clear that a given transformed master (μ,ν) graph is equal to the corresponding untransformed master (μ,ν) graph except for the transformation associated with the external lines [see Eq. (3.6)]. Thus, it is simple to obtain the transformation of all the functions in I that were defined in terms of master (μ,ν) graphs. We define

$$\mathcal{K}'_{\mu,\nu}(t_2,t_1,k) \equiv \sum\begin{pmatrix}\text{all different transformed}\\ \text{master }(\mu,\nu)\ L\text{ graphs}\end{pmatrix}_k\ , \qquad (3.7)$$

where these $\mathcal{K}'_{\mu,\nu}$ differ from the corresponding untransformed $K_{\mu,\nu}$ of Eq. (I.4.17) through the transformation of the vertices [see Eq. (3.6)] to which the external lines attach. Similarly, we define

$$\mathcal{K}_{1,1}^{(1)\prime}(t_2,t_1,-k) \equiv \sum\{\text{all different transformed master }(\mu,\nu)\ L\text{ graphs with the outgoing external line transformed by }[H^{(1)}(t_2,-k)]^{-1}\ {}_0^{(1)}(t_2,s_2,-k)\text{ and the incoming external line transformed by }H^{(1)}(t_1,-k)\}_{-k}$$

$$=[H^{(1)}(t_2,-k)]^{-1}\int_0^\beta ds\,\mathcal{G}_0^{(1)}(t_2,s,-k)K_{1,1}(s,t_1,-k)H^{(1)}(t_1,-k)\ . \qquad (3.8)$$

Using Eqs. (I.5.14), (I.5.15), (3.4), and (3.5), we find the transformations of $K_{\text{out}}^{(0)}(t)$ and $K_{\text{in}}^{(0)}(t)$ to be

$$h_a^{(0)}\mathcal{K}_{\text{out}}^{(0)\prime}(t) \equiv [(x\Omega)^{1/2}e^{\beta g}]^{-1}\sum(\text{all different transformed master }(0,1)\ L\text{ graphs})_{k=0}$$

$$=H^{(0)}(t)K_{\text{out}}^{(0)}(t) \qquad (3.9)$$

and $h_b^{(0)}\mathcal{K}_{\text{in}}^{(0)\prime}(t) \equiv (x\Omega)^{-1/2}\sum$ [all different transformed master $(1,0)$ L graphs$]_{k=0}$

$$=[H^{(0)}(t)]^{-1}\int_0^\beta ds\,\mathcal{G}_0^{(0)}(t,s)K_{\text{in}}^{(0)}(s)\ . \qquad (3.10)$$

Finally, we define

$$\Omega F' \equiv \sum\ [\text{all different transformed master }(0,0)\text{ graphs}] = \Omega F\ , \qquad (3.11)$$

where we have used Eq. (I.5.1).

Equations (3.7)–(3.11) complete our basic definitions for the Λ transformation. A detailed outline of the transformation procedure will be presented in Sec. 4, and critical discussion of this approach will be given in Sec. 6.

4. Λ TRANSFORMATION OF THEORY

In Sec. 3, we outlined for the Λ transformation the basic definitions which are needed to treat the ADC properly and to obtain corrections to the ADC in a self-consistent manner. The purpose of this section is to show how the theory presented in I can be expressed entirely in terms of the transformed quantities, defined by Eqs. (3.1)–(3.11). To achieve this goal, we first obtain a set of coupled integral equations for the transformed master-graph line factors directly in terms of transformed quantities. This set is not simply the direct transformation of the set of coupled integral equations developed in Secs. 4 and 5 of I. Rather, a certain part, which is closely related to the characteristic functions for the various types of lines, is removed from the kernels of the integral equations for the transformed line factors [see Eqs. (4.7), (4.14), (4.18), and (4.19), and see also the discussion at the beginning of Sec. 3]. Thus, the transformed line factors give the corrections to the ADC of the untransformed line factors. (The ADC in the untransformed line factors represent the dominant low-temperature contributions to these line factors.) We will complete this section by expressing the momentum distribution and the grand potential in terms of transformed quantities.

We begin by transforming the kernel $Q_{1,1}(t_2, t_1, k)$ [see Eq. (I.4.12)] of the integral equation for $\mathcal{G}_{1,1}(t_2, t_1, k)$. We define

$$\mathfrak{Q}'_{1,1}(t_2, t_1, k) \equiv [H'(t_2, k)]^{-1} \int_0^\beta ds \mathcal{G}'_0(t_2, s, k) \times Q_{1,1}(s, t_1, k) H'(t_1, k)$$

$$= \mathfrak{M}'_{1,1}(t_2, t_1, k) + \int_0^\beta ds_2 ds_1 \mathfrak{M}'_{2,0}(t_2, s_1, k) \times \mathcal{G}^{(1)\prime}(s_2, s_1, -k) \mathfrak{M}'_{0,2}(t_1, s_2, k) , \qquad (4.1)$$

where we have used Eq. (I.4.12), as well as Eqs. (4.2) and (4.3) below, in order to obtain the second line of Eq. (4.1). Using Eqs. (I.4.18) and (3.7), the transformed functions $\mathfrak{M}'_{\mu,\nu}(t_2, t_1, k)$ can be written[4]

$$\mathfrak{M}'_{\mu,\nu}(t_2, t_1, k) \equiv \mathcal{K}'_{\mu,\nu}(t_2, t_1, k) + \delta_{\mu,\nu} \times \delta(\beta - t_1) \exp\{\beta[g - \omega(p)]\} H'(\beta, p) \times [H'(t_2, p)]^{-1} \int_0^\beta ds \mathcal{G}'_0(t_2, s, p). \qquad (4.2)$$

The transformed quantity $\mathcal{G}^{(1)\prime}(t_2, t_1, -k)$ for $-k$ lines in Eq. (4.1) is defined in analogy with Eq. (3.1) by

$$\mathcal{G}(t_2, t_1, -k) \equiv H^{(1)}(t_2, -k) \int_0^\beta ds \;{}^{(1)\prime}(t_2, s, -k) \times [H^{(1)}(s, -k)]^{-1} \mathcal{G}_0^{(1)}(s, t_1, -k) .$$

We next transform the kernel $M_{1,1}(t_2, t_1, -k)$ of the integral equation for $\mathcal{G}(t_2, t_1, -k)$ [see Eq. (I.4.11)]:

$$\mathfrak{M}_{1,1}^{(1)\prime}(t_2, t_1, -k) = [H^{(1)}(t_2, -k)]^{-1} \times \int_0^\beta ds \mathcal{G}_0^{(1)}(t_2, s, -k) \times M_{1,1}(s, t_1, -k) H^{(1)}(t_1, -k)$$

$$= \mathcal{K}_{1,1}^{(1)\prime}(t_2, t_1, -k) + \delta(\beta - t_1) \times \exp\{\beta[g - \omega(-p)]\} \times H^1(\beta, -p)[H^{(1)}(t_2, -p)]^{-1} \times \int_0^\beta ds \mathcal{G}_0^{(1)}(t_2, s, -p) , \qquad (4.4)$$

where we have used Eqs. (I.4.18) and (3.8) to obtain the second line. It is important to observe that $Q_{1,1}(t_2, t_1, k)$ and $M_{1,1}(t_2, t_1, -k)$ are transformed by different characteristic functions; the reasons for this will be indicated in Sec. 6. We also note that functions such as $Q_{1,1}$ and $M_{\mu,\nu}$ transform in the same way as the pair function [see Eq. (3.6), where the characteristic function is associated with the outgoing lines], in contrast with the way in which the line factors transform. In this connection, it is worth remarking that there is no arbitrariness involved in the transformation of $Q_{1,1}$ and $M_{\mu,\nu}$, once the choice of the line-factor transformation equations (3.1)–(3.5) has been made.

We now proceed to derive integral equations for the transformed line factors $\mathcal{G}'_{\mu,\nu}(t_2, t_1, k)$. In analogy with Eq. (I.4.4) we define the functions $\mathcal{L}'_{\mu,\nu}(t_2, t_1, k)$ by

$$\mathcal{G}'_{\mu,\nu}(t_2, t_1, k) \equiv \delta(t_2^{(-)} - t_1)\delta_{\mu,\nu} + \epsilon \mathcal{L}'_{\mu,\nu}(t_2, t_1, k) , \qquad (4.5)$$

for $(\mu, \nu) = (1,1)$, $(0,2)$, or $(2,0)$. From Eqs. (3.1), (4.1), (4.5), (I.4.4), and (I.4.13), we find

$$\mathcal{L}'_{1,1}(t_2, t_1, k) = \int_0^\beta ds \mathcal{G}'_{1,1}(t_2, s, k) \times Q'_{1,1}(s, t_1, k), \qquad (4.6)$$

where

$$Q'_{1,1}(t_2, t_1, k) \equiv \mathfrak{Q}'_{1,1}(t_2, t_1, k) - \Lambda'(t_2, t_1, k), \qquad (4.7)$$

$$\Lambda'(t_2, t_1, k) \equiv \epsilon [H'(t_2, k)]^{-1} [\mathcal{G}'_0(t_2, t_1, k) - \delta(t_2^{(-)} - t_1)] H'(t_1, k) . \qquad (4.8)$$

Also from Eqs. (3.1) – (3.3), (4.2), (4.5), (I.4.4), (I.4.14), and (I.4.15), we find

$$\mathcal{L}'_{0,2}(t_2, t_1, k) = \int_0^\beta ds_2 ds_1 \mathfrak{M}'_{0,2}(s_2, s_1, k)\, \mathcal{G}'_{1,1}(s_2, t_2, k)\, \mathcal{G}^{(1)\prime}(s_1, t_1, -k) - M_{0,2}{}^{(1)\prime}(t_2, t_1, k), \qquad (4.9)$$

and $\mathcal{L}'_{2,0}(t_2, t_1, k) = \int_0^\beta ds_2 ds_1 \mathcal{G}'_{1,1}(t_2, s_2, k)\, \mathcal{G}^{(1)\prime}(t_1, s_1, -k)$

$$\times \mathfrak{M}'_{2,0}(s_2, s_1, k) - \delta(t_2, t_1)\, M_{2,0}{}^{(1)\prime}(t_2, t_1, k), \tag{4.10}$$

where, for $(\mu, \nu) = (0, 2)$ or $(2, 0)$ only,

$M_{\mu,\nu}{}^{(1)\prime}(t_2, t_1, k) \equiv$ [that part of $\mathfrak{M}'_{\mu,\nu}(t_2, t_1, k)$ in which both external lines attach to the same vertex, except that the external lines are untransformed at *both* ends]

$$= M_{\mu,\nu}{}^{(1)}(t_2, t_1, k). \tag{4.11}$$

Equations (4.5)–(4.11) provide a complete set of integral equations for the transformed line factors in terms of transformed quantities. We note that although the role of $\mathfrak{L}'_{1,1}(t_2, t_1, k)$ in the transformed theory is the same as the role of $\mathfrak{L}_{1,1}(t_2, t_1, k)$ in the untransformed theory, $\mathfrak{L}'_{1,1}(t_2, t_1, k)$ is not equal to a simple transformation of $\mathfrak{L}_{1,1}(t_2, t_1, k)$. This fact follows from the definition of the kernel of $\mathfrak{L}'_{1,1}(t_2, t_1, k)$ in Eq. (4.7).

We next repeat the above procedure for $\mathcal{G}^{(1)\prime}(t_2, t_1, -k)$. In analogy with Eq. (I. 4.10), we define the function $\mathfrak{L}^{(1)\prime}(t_2, t_1, -k)$ by

$$\mathcal{G}^{(1)\prime}(t_2, t_1, -k) \equiv \delta(t_2{}^{(-)} - t_1) + \epsilon \mathfrak{L}^{(1)\prime}(t_2, t_1, -k) \quad . \tag{4.12}$$

From Eqs. (4.3), (4.4), (4.12), (I. 4.10), and (I. 4.11), we find

$$\mathfrak{L}^{(1)\prime}(t_2, t_1, -k) = \int_0^\beta ds\; \mathcal{G}^{(1)\prime}(t_2, s, -k)\, M_{1,1}{}^{(1)\prime}(s, t_1, -k), \tag{4.13}$$

$$\text{where} \quad M_{1,1}{}^{(1)\prime}(t_2, t_1, -k) \equiv \mathfrak{M}_{1,1}{}^{(1)\prime}(t_2, t_1, -k) - \Lambda^{(1)}(t_2, t_1, -k)\ , \tag{4.14}$$

$$\Lambda^{(1)}(t_2, t_1, -k) \equiv \epsilon\, [H^{(1)}(t_2, -k)]^{-1} [\, \mathcal{G}_0{}^{(1)}(t_2, t_1, -k) - \delta(t_2{}^{(-)} - t_1)]\, H^{(1)}(t_1, -k). \tag{4.15}$$

The discussion of $\mathfrak{L}'_{1,1}(t_2, t_1, k)$ at the end of the previous paragraph also applies to $\mathfrak{L}^{(1)\prime}(t_2, t_1, -k)$.

Having expressed the transformed solid-line factors in terms of the transformed quantities, we must now do the same for the transformed dotted (zero-momentum) line factors $G_{\text{out}}{}^{(0)\prime}(t)$ and $G_{\text{in}}{}^{(0)\prime}(t)$ which were defined in Eqs. (3.4) and (3.5). In analogy with Eqs. (I. 3.1), we therefore define the functions $K_{\text{out}}{}^{(0)}(t)$ and $K_{\text{in}}{}^{(0)\prime}(t)$ by

$$G_{\text{out}}{}^{(0)\prime}(t) \equiv \delta(\beta - t) + K_{\text{out}}{}^{(0)\prime}(t)\ , \tag{4.16}$$

$$G_{\text{in}}{}^{(0)\prime}(t) \equiv 1 + K_{\text{in}}{}^{(0)\prime}(t)\ . \tag{4.17}$$

Then, from Eqs. (3.4), (3.5), (3.9), (3.10), (4.16), (4.17), and (I. 3.1), we obtain

$$K_{\text{out}}{}^{(0)\prime}(t) = \mathcal{K}_{\text{out}}{}^{(0)\prime}(t) - \int_0^\beta ds\, G_{\text{out}}{}^{(0)\prime}(s)\Lambda^{(0)}(s, t) - \delta(\beta - t)[1 - H^{(0)}(\beta)/h_a{}^{(0)}]\ , \tag{4.18}$$

$$K_{\text{in}}{}^{(0)\prime}(t) = \mathcal{K}_{\text{in}}{}^{(0)\prime}(t) - \int_0^\beta ds\, \Lambda^{(0)}(t, s) G_{\text{in}}{}^{(0)\prime}(s) - \{1 - [h_b{}^{(0)} H^{(0)}(t)]^{-1} \int_0^\beta ds\, \mathcal{G}_0{}^{(0)}(t, s)\}\ , \tag{4.19}$$

$$\text{where} \quad \Lambda^{(0)}(t_2, t_1) \equiv [H^{(0)}(t_2)]^{-1} [\, \mathcal{G}_0{}^{(0)}(t_2, t_1) - \delta(t_2{}^{(-)} - t_1)]\, H^{(0)}(t_1). \tag{4.20}$$

Equations (3.9), (3.10), and (4.16)–(4.20) express $G_{\text{out}}{}^{(0)\prime}(t)$ and $G_{\text{in}}{}^{(0)\prime}(t)$ entirely in terms of transformed quantities. We note that $K_{\text{out}}{}^{(0)\prime}(t)$ plays the same role in the transformed theory as $K_{\text{out}}{}^{(0)}(t)$ does in untransformed theory, even though it is $\mathcal{K}_{\text{out}}{}^{(0)\prime}(t)$ which is the direct transformation of $K_{\text{out}}{}^{(0)}(t)$. A similar statement holds for the corresponding "in" quantities.

Before proceeding further, it is instructive to stress the underlying pattern in the treatment of $+k$, $-k$, and $k \equiv 0$ lines and the conceptual similarity between the three sets of equations for the transformed quantities given in the preceding paragraphs. In each case, the effect of the Λ transformation is to remove the

ADC (which give the dominant low-temperature contributions) from the transformed line factors. (See also discussion at the beginning of Sec. 3.)

Λ Transformation of $\langle n(p)\rangle$ and Ωf

Having obtained the set of coupled integral equations for the transformed line factors, we must now express the momentum distribution and the grand potential in terms of transformed quantities. From Eqs. (I.4.1) and (3.1), we see immediately that the momentum distribution is given by[4]

$$\langle n(p)\rangle = H'(\beta,p)\exp\{\beta[g-\omega(p)]\}\int_0^\beta ds_1\, \mathcal{G}'_{1,1}(\beta,s_1,p)\,[H'(s_1,p)]^{-1}\int_0^\beta ds_2\, \mathcal{G}'_0(s_1,s_2,p)\,. \tag{4.21}$$

Since the case of the grand potential is more involved, we shall first indicate the transformation of some of the individual terms in Eq. (I.5.12) for Ωf. From Eqs. (I.3.1), (3.4), (3.5), (3.10), and (4.17), we find

$$\begin{aligned}(x\Omega)e^{\beta g}[1-\int_0^\beta dt\, K_{\text{out}}^{(0)}(t)k_{\text{in}}^{(0)}(t)] &= (x\Omega)e^{\beta g}[G_{\text{in}}(\beta)-\int_0^\beta dt\, G_{\text{out}}^{(0)}(t)K_{\text{in}}^{(0)}(t)]\\ &= (x\Omega)e^{\beta g}[h_b^{(0)}H^{(0)}(\beta)+h_b^{(0)}(\beta)K_{\text{in}}^{(0)\prime}(\beta)\\ &\quad - h_a^{(0)}h_b^{(0)}\int_0^\beta dt\, G_{\text{out}}^{(0)\prime}(t)\mathcal{K}_{\text{in}}^{(0)\prime}(t)]\,.\end{aligned} \tag{4.22}$$

From Eqs. (I.4.2), (3.1), (4.5), and (4.8), we find

$$\int_0^\beta dt\, \mathcal{L}_{1,1}(t,t,p) = \int_0^\beta dt\, \mathcal{L}'_{1,1}(t,t,p) + \int_0^\beta dt\, ds\, \mathcal{G}_{1,1}(t,s,p)\Lambda'(s,t,p)\,. \tag{4.23}$$

Similarly, we obtain (see beginning of Appendix A)

$$\int_0^\beta dt\, \mathcal{L}_{1,1}^{(t)}(t,t,p) = \int_0^\beta dt\, \tilde{\mathcal{L}}_{1,1}^{(t)\prime}(t,t,p) + \int_0^\beta dt\int_0^t ds\,\tilde{\mathcal{G}}_{1,1}^{(t)\prime}(t,s,p)\,\Lambda^{(t)\prime}(s,t,p). \tag{4.24}$$

The derivation of Eq. (4.24) is identical to the derivation of Eq. (4.23) as can be seen by comparing Eqs. (I.5.2) and (I.5.7)–(I.5.10) with Eqs. (I.4.4) and (I.4.10)–(I.4.13). Using Eqs. (4.22)–(4.24), we obtain the following expression for the grand potential:

$$\begin{aligned}\Omega f(x,\beta,g,\Omega) &= (x\Omega)[h_b^{(0)}H^{(0)}(\beta)e^{\beta g}-1]+\Omega F^{*\prime}(x,\beta,g,\Omega)\\ &\quad +\tfrac{1}{2}\sum_k\int_0^\beta dt_1\,dt_2 M_{2,0}^{(1)\prime}(t_2,t_1,k)M_{0,2}^{(1)\prime}(t_2,t_1,k)\\ &\quad +\sum_p\int_0^\beta dt\,\tilde{\mathcal{L}}_{1,1}^{(t)\prime}(t,t,p)+\sum_p\int_0^\beta dt\int_0^t ds\tilde{G}_{1,1}^{(t)\prime}(t,s,p)\Lambda^{(t)\prime}(s,t,p)\,,\end{aligned} \tag{4.25}$$

where

$$\begin{aligned}\Omega F^{*\prime}(x,\beta,g,\Omega) &\equiv \Omega F'(x,\beta,g,\Omega)+(x\Omega)e^{\beta g}h_b^{(0)}H^{(0)}(\beta)K_{\text{in}}^{(0)\prime}(\beta)\\ &\quad -(x\Omega)e^{\beta g}h_a^{(0)}h_b^{(0)}\int_0^\beta dt\, G_{\text{out}}^{(0)\prime}(t)\,\mathcal{K}_{\text{in}}^{(0)\prime}(t)+\sum_p\langle n(p)\rangle\\ &\quad -\sum_p\int_0^\beta dt\,\mathcal{L}'_{1,1}(t,t,p)-\sum_p\int_0^\beta dt\,ds\,\mathcal{G}'_{1,1}(t,s,p)\,\Lambda'(s,t,p),\end{aligned} \tag{4.26}$$

and we have also used Eqs. (I.5.12), (I.5.13), (3.11), and (4.11). We shall find in one explicit application of the Λ transformation that in Eqs. (4.24) and (4.25) the last term vanishes [see Eq. (A3)].

5. Λ – TRANSFORMED THEORY OF A NORMAL FLUID

In Secs. 2–4, we have presented a somewhat abstract account of the general structure of the Λ transformation applied to the quantum-statistical theory developed in I. Special attention has been given to the proper treatment of the low-temperature divergences. In this section, we make this formalism more transparent by considering the relatively simple case of a normal quantum fluid. For the case of a normal fluid, the density of zero-momentum particles "x," as well as all (0, 2), (2, 0), and zero-momentum quantities, is set equal to zero. In addition, the special treatment given to $-k$ lines [see Eq. (3.8) and the discussion below Eq. (3.6)] is no longer necessary.

Characteristic Function

As indicated by the formulation of Secs. 2–4, the characteristic function $\mathcal{G}_0'(t_2, t_1, k)$ plays a central role in the application of the Λ transformation. We shall now obtain an expression for $\mathcal{G}_0'(t_2, t_1, k)$ for a normal fluid. We first choose the following form for the kernel $Q_0'(t_2, t_1, k)$ in Eq. (2.2):

$$Q_0'(t_2, t_1, k) = [1 - B(k)]^{-1} \times \begin{pmatrix} -\epsilon B(k)\delta(t_2^{(-)} - t_1) - \epsilon\, \Delta(k)\theta(t_2 - t_1) \\ +\ \delta(\beta - t_1)[1 - B(k)] \exp\{\beta[g - \omega(k)]\} \end{pmatrix}, \quad (5.1)$$

where the quantities[5] $B(k)$ and $\Delta(k)$ may have a dependence on β, although it has not been explicitly indicated. The term containing $\delta(\beta - t_1)$ is the first term of Eq. (I. 4.18). The existence of the rest of the terms in Eq. (5.1) can be demonstrated by a simple lowest-order calculation of the function $K_{1,1}(t_2, t_1, k)$ defined by Eq. (I. 4.17). Thus, one has only to used the 1-vertex master (μ, ν) L graph of Fig. 3 to obtain the expression

$$K_{1,1}(t_2, t_1, k_1) \cong \sum_{k_2} \int_0^\beta ds\, \mathcal{G}_{1,1}(t_1^{(-)}, s, k_2)^{t_2 s} \times \begin{pmatrix} k_1 & k_2 \\ k_1 & k_2 \end{pmatrix}_{t_1} . \quad (5.2)$$

Using the explicit form (B1) and (B2) for the pair function and iterating the line factor, one can demonstrate the form of Eq. (5.1). The function $B(k)$ is nonzero only if the two-body interaction

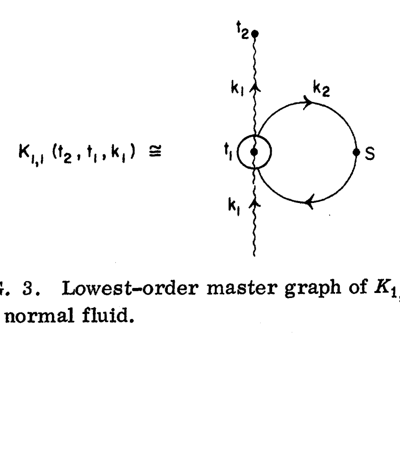

FIG. 3. Lowest-order master graph of $K_{1,1}(t_2, t_1, k)$ for a normal fluid.

includes an infinite hard core.[6]

Returning to the integral equations (2.1) and (2.2) for $\mathcal{G}_0'(t_2, t_1, k)$, it is now easy to see that apparent low-temperature divergences occur if we iterate the right-hand side of Eq. (2.2) using the kernel (5.1). We therefore solve Eqs. (2.1) and (2.2) exactly to get the characteristic function

$$\mathcal{G}_0'(t_2, t_1, k) = [1 - B(k)]\{\delta(t_2 - t_1) - [\theta(t_2 - t_1) + \epsilon n'(k)]\Delta(k) + \delta(\beta - t_1) \times \epsilon n'(k)\} \exp[-(t_2 - t_1)\Delta(k)] , \quad (5.3)$$

which is the series sum of the ADC to $\mathcal{G}'_{1,1}(t_2, t_1, k)$. The quantity $n'(k)$ in Eq. (5.3) is defined to be

$$n'(k) \equiv ([1 - B(k)]^{-1} \exp\{\beta[\omega'(k) - g]\} - \epsilon)^{-1}, \quad (5.4)$$

where $\quad \omega'(k) = \omega(k) + \Delta(k)$. (5.5)

Upon comparing Eq. (5.4) with the free-particle momentum distribution [Eq. (I. 1.1)] we note that, except for the $[1 - B(k)]^{-1}$ factor, $n'(k)$ involves the replacement of the free-particle energy by the quasiparticle energy $\omega'(k)$. Thus, Eq. (5.4) for $n'(k)$ can be referred to as the quasiparticle distribution function. The quantity $\Delta(k)$ in Eq. (5.5) can be identified as the quasiparticle self-energy.

The solution of Eq. (5.3) for $\mathcal{G}_0'(t_2, t_1, k)$ can be checked by substituting the right-hand side of Eq. (5.3) into the right-hand side of Eq. (2.2) and verifying that the correct expression for $\mathcal{G}_0'(t_2, t_1, k)$ is reproduced. Equation (5.3) for $\mathcal{G}_0'(t_2, t_1, k)$ satisfies the useful integral property

$$\int_0^\beta dt_1 \mathcal{G}_0'(t_2, t_1, k) = n'(k) \exp\{\beta[\omega(k) - g]\} \exp[(\beta - t_2)\Delta(k)]. \quad (5.6)$$

The question whether or not the particular choice (5.1) of the kernel $Q_0'(t_2, t_1, k)$ removes ADC to the line factor completely will be discussed below Eq. (5.12).

Application of Λ Transformation

Using the characteristic function (5.3), we next outline the results of the Λ transformation. We also obtain an elegant prescription for the calculation of the quasiparticle self-energy $\Delta(k)$, so far left undetermined by the theory.

It is convenient to choose the arbitrary function $H'(t, k)$ introduced in Sec. 3 to be

$$H'(t, k) = \exp[-t\ \Delta(k)] . \quad (5.7)$$

The transformed pair function is then obtained by substituting Eq. (5.7) into Eq. (3.6)

$$t_1 t_2 \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}_{t_0}'' = \exp[t_1\Delta(k_1)]\exp[t_2\Delta(k_2)]\exp\{-t_0[\Delta(k_3) + \Delta(k_4)]\}\int_0^\beta ds_1 ds_2\ \mathcal{G}_0'(t_1,s_1,k_1) \times \mathcal{G}_0'(t_2,s_2,k_2)\, s_1 s_2 \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}_{t_0} . \tag{5.8}$$

An explicit expression for this transformed pair function is given in Appendix B. The basic Λ-transformation function, i.e., $\Lambda'(t_2, t_1, k)$, obtained by substituting Eqs. (5.3) and (5.7) into Eq. (4.8) is

$$\Lambda'(t_2,t_1,k) = [1-B(k)]\{-[\theta(t_2-t_1)+\epsilon n'(k)] \times \epsilon\,\Delta(k) + n'(k)\delta(\beta-t_1)\} - \epsilon B(k)\delta(t_2-t_1) . \tag{5.9}$$

From Eqs. (4.1), (4.2), (4.5)–(4.7), (5.6), and (5.9), we obtain the following simple integral equation for the transformed master-graph line factor:

$$\mathcal{G}_{1,1}'(t_2,t_1,k) = \delta(t_2^{(-)}-t_1) + \epsilon\,\mathcal{L}_{1,1}'(t_2,t_1,k), \tag{5.10}$$

$$\mathcal{L}_{1,1}'(t_2,t_1,k) = \int_0^\beta ds\,\mathcal{G}_{1,1}'(t_2,s,k)Q_{1,1}'(s,t_1,k) , \tag{5.11}$$

where $$Q_{1,1}'(t_2,t_1,k) = \mathcal{K}_{1,1}'(t_2,t_1,k) + \epsilon\, B(k)[\delta(t_2^{(-)}-t_1)+\epsilon\delta(\beta-t_1)n'(k)] + \epsilon[1-B(k)][\theta(t_2-t_1)+\epsilon n'(k)]\,\Delta(k) . \tag{5.12}$$

The quantity $\mathcal{K}_{1,1}'(t_2,t_1,k)$ is defined in Eq. (3.7). We now observe that Eq. (5.12) allows us to subtract from $\mathcal{K}_{1,1}'(t_2,t_1,k)$ all those terms which, when iterated, would give rise in $\mathcal{G}_{1,1}'(t_2,t_1,k)$ to ADC at low temperatures. With such a choice the integral equation (5.11) can be solved by iteration in actual application to a Bose liquid above the Bose-Einstein condensation temperature or to a Fermi liquid, i.e., the function $Q_{1,1}'(t_2,t_1,k)$ then consists only of small terms. The question as to whether or not the subtracted terms in Eq. (5.12) eliminate all the ADC to $\mathcal{G}_{1,1}'(t_2,t_1,k)$ can be checked in any explicit calculation.

In addition to making iterations possible at low temperature, Eq. (5.12) also gives us a prescription for calculating $B(k)$ and the quasiparticle self-energy $\Delta(k)$. The procedure for determining $B(k)$ and $\Delta(k)$ can be described in two steps: (i) First calculate $\mathcal{K}_{1,1}'(t_2,t_1,k)$, defined in Eq. (3.7), to any desired order (see end of Appendix B for definition of order); (ii) identify in the calculated expression for $\mathcal{K}_{1,1}'(t_2,t_1,k)$ those terms with the same form as the last terms in Eq. (5.12). Then we have

$$\Delta(k) \equiv \{t\text{-independent coefficient of } -\epsilon[1-B(k)][\theta(t_2-t_1)+\epsilon n'(k)] \text{ in } \mathcal{K}_{1,1}'(t_2,t_1,k)\} , \tag{5.13}$$

$$B(k) \equiv \{t\text{-independent coefficient of } -\epsilon[\delta(t_2^{(-)}-t_1)+\epsilon\delta(\beta-t_1)n'(k)] \text{ in } \mathcal{K}_{1,1}'(t_2,t_1,k)\} . \tag{5.14}$$

It is important to observe that Eqs. (5.13) and (5.14) actually give integral equations for $\Delta(k)$ and $B(k)$, since the right-hand sides of these equations are functionals of Δ and B. [To see this, observe that $\mathcal{K}_{1,1}'(t_2,t_1,k)$ is expressed in terms of the transformed pair function (B3) which is a functional of $\omega'(k)$ and $B(k)$.] These integral equations (5.13) and (5.14) can be solved to any desired order in actual calculations. We will not present any explicit results for $\Delta(k)$ and $B(k)$, obtained by such a procedure for the case of a normal Bose or Fermi liquid, as they are identical to results obtained before.[7,8] The prescriptions (5.13) and (5.14) will be discussed further in Sec. 6.

The momentum distribution $\langle n(k)\rangle$ can be calculated from the following relation, obtained by substituting Eqs. (5.6) and (5.7) into Eq. (4.21):

$$\langle n(k)\rangle = n'(k)\int_0^\beta dt\,\mathcal{G}_{1,1}'(\beta,t,k) = n'(k) + \epsilon\, n'(k)\int_0^\beta dt\,\mathcal{L}_{1,1}'(\beta,t,k). \tag{5.15}$$

The second term in the second line of this equation is small in the sense described below Eq. (5.12). Thus, the lowest-order approximation for the momentum distribution gives $\langle n(k)\rangle \cong n'(k)$, the quasiparticle distribution function.

The transformed expression for the grand potential of a normal fluid can be obtained from Eq. (4.25) by simply setting x, as well as the (0, 2), (2, 0), and zero-momentum quantities, equal to zero. The final result, using Eqs. (4.26), (5.9), (5.15), (A3), and (A7), is

$$\Omega f(\beta,g,\Omega) = \Omega F'(\beta,g,\Omega) + \sum_k \int_0^\beta dt\, [\mathcal{L}_{1,1}^{(t)'}(t,t,k) - \{1-B(k)\}\mathcal{L}'_{1,1}(t,t,k)]$$
$$-\epsilon \sum_k \ln\{1-\epsilon[1-B(k)]\exp[\beta(g-\omega'(k))]\} + \epsilon\sum_k [1-B(k)]\Delta(k)$$
$$\times \int_0^\beta dt\, ds\, \mathcal{G}'_{1,1}(t,s,k)[\theta(s-t)+\epsilon n'(k)] + \sum_k B(k)\langle n(k)\rangle \quad . \tag{5.16}$$

When we do not consider the idealized case of an interaction which includes an infinite repulsive core, then $B(k)\equiv 0$, and Eq. (5.16) for Ωf becomes

$$\Omega f(\beta,g,\Omega) = \Omega F'(\beta,g,\Omega) + \sum_k \int_0^\beta dt\, [\mathcal{L}_{1,1}^{(t)'}(t,t,k) - \mathcal{L}'_{1,1}(t,t,k)]$$
$$-\epsilon \sum_k \ln\{1-\epsilon\exp[\beta(g-\omega'(k))]\}$$
$$+\epsilon\sum_k \Delta(k)\int_0^\beta dt\, ds\, \mathcal{G}'_{1,1}(t,s,k)[\theta(s-t)+\epsilon n'(k)] \,. \tag{5.17}$$

Note that the argument of the logarithm appearing in Eqs. (5.16) and (5.17) is never negative for a normal fluid. In fact, assuming that the minimum value of $\omega'(k)$ occurs at $k=0$, for $\epsilon=+1$ (Bose fluid) the occurrence of $g=\omega'(0)$ [when $B(0)=0$] is precisely the condition for the onset of Bose-Einstein condensation.

The theory is now in a form suitable for explicit calculations to obtain well-behaved iterative expressions for the various thermodynamic properties of the system. We shall not give further details here since they lead to results identical with those obtained earlier by Tuttle and Mohling[7] and Sikora.[8]

6. DISCUSSION

In this paper, we have focused our attention on two aspects of the master-graph formulation of quantum-statistical mechanics presented in I: (a) the low-temperature, apparently divergent contributions (ADC) encountered when an iterative procedure is used to calculate thermodynamic properties; (b) the application of the Λ transformation to sum these apparent divergences so that well-behaved expressions for thermodynamic properties can be readily calculated. In general terms, the role of the Λ transformation performed on the quantum-statistical development in I has been twofold: It shifts the ADC from the integral equations for the master-graph line factors to the vertex, or pair, functions giving well-behaved final expressions. It also results in a self-consistent iterative method for calculating the thermodynamic quantities and the self-energies of the quasiparticles. The Λ transformation transforms the original theory of I to a form which suggests a microscopic quasiparticle model of the fluid.

We now make a few remarks on the Λ transformation scheme presented in Secs. 3–5:

(i) The transformation procedure outlined in Sec. 3 associates the characteristic functions introduced in Sec. 2 only with the outgoing line at each vertex in the master-graph expressions for any quantity (see Secs. 3 and 4). This procedure is not unique. One could equally well have associated characteristic functions with the incoming line or with both the incoming and outgoing lines at each vertex. Of course, these other possibilities would give the same final results.

(ii) The transformation is in no sense a unitary transformation of the basic operators in the theory, such as the Bogoliubov transformation for a Bose fluid.[9] It is a temperature-dependent integral transformation on the basic line factors and vertex functions of the theory.

(iii) As can be seen in Eq. (4.7), the role of the function $\Lambda'(t_2,t_1,k)$ after the Λ transformation is essentially the same as that of $Q'_0(t_2,t_1,k)$ before the Λ transformation [see Eq. (2.2) and the preceding discussion]. The similarity between these two functions can be observed for the normal fluid case by comparing Eqs. (5.1) and (5.9). The reason $n'(k)$ appears in $\Lambda'(t_2,t_1,k)$, Eq. (5.9), is that the Λ transformation has in this case summed in all different ways the statistical factor $\exp\{\beta[g-\omega(k)]\}$ which appears in $Q_0'(t_2, t_1, k)$, Eq. (5.1), in addition to replacing the free-particle energy $\omega(k)$ by the quasiparticle energy $\omega'(k)$.

(iv) A general interesting feature of this transformation is that the transformation of any temperature-dependent quantity, e.g., $K_{\mu\nu}(t_2, t_1, k)$, depends on the nature of its external lines (see Sec. 3). Thus, the transformation function for each external line depends on whether the momentum label associated with that external line is $+k$, $-k$, or $k\equiv 0$. In a sense, the Λ transformation acts as a probe for the self-energy structures in the theory. In Sec. 5, we showed for a normal fluid, which has only one Λ function [see Eq. (5.9)], that a single self-energy $\Delta(k)$ results from the Λ transformation. In a subsequent paper, we will show for a degenerate Bose fluid, which has three different Λ functions [see Eqs. (4.8), (4.15), and (4.20)], that a different self-energy is associated with each of these Λ functions. The interpretation of these different self-energies will be given there.

(v) The Λ transformation gives a prescription for calculating the self-energies [see, for example, Eq. (5.13)]. It has been suggested[10] that this prescription is quite similar to the corresponding prescriptions for calculating the self-energies given by Balian, Bloch, and de Dominicis,[11] who use a generalized Hartree-Fock approach, and by Hugenholtz and Pines.[12] A detailed proof of such an equivalence would give further insight into the Λ transformation.

The ultimate goal in developing a microscopic theory of a many-body system at finite temperatures is threefold: (i) to check the phenomenological theories developed for the particular problem of interest, (ii) to obtain explicit analytic expressions for the various thermodynamic quantities of interest, and (iii) most important of all, to provide numerical comparison of the theoretical results with experimental observations. In I and in the present paper, we have developed a general formalism. The first two aspects have been studied earlier in detail for a Fermi liquid by Tuttle and Mohling[7]; the first two aspects of this program for a degenerate Bose system will be completed in the subsequent papers of this series.

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APPENDIX A

In this Appendix, we outline the $\Lambda^{(\tau)}$-transformation procedure for the quantities defined in Eqs. (I.5.2)–(I.5.10) which are identified with a superscript τ. This is necessary in order to properly Λ transform the quantity $\mathcal{L}_{1,1}{}^{(\tau)}(t_2, t_1, k)$ in Eq. (I.5.12) for the grand potential.

The $\Lambda^{(\tau)}$ transformation can be carried out formally in direct analogy with the analysis presented in Secs. 3 and 4. Five steps in this development should be emphasized:

(i) Define the characteristic functions $\mathcal{G}_0{}^{(\tau)\prime}(t_2, t_1, k)$ [see Eq. (2.5)] and $\mathcal{G}_0{}^{(\tau)(1)}(t_2, t_1, k)$ in analogy with $\mathcal{G}_0'(t_2, t_1, k)$ and $\mathcal{G}_0{}^{(1)}(t_2, t_1, k)$.

(ii) Define all the transformed (1, 1), (0, 2) and (2, 0) quantities identified with a superscript τ, and replace β by τ in the integration limits of all $\Lambda^{(\tau)}$-transformation equations.

(iii) The arbitrary functions $H'(t, k)$ and $H^{(1)}(t, k)$, which appear in the Λ-transformation equations of Sec. 3, are also used with the same forms for the τ case.

(iv) Define the τ-transformed pair-function, identified with a superscript τ, using $\mathcal{G}_0{}^{(\tau)\prime}(t_2, t_1, k)$ instead of $\mathcal{G}_0'(t_2, t_1, k)$ in Eq. (3.6).

(v) Include a tilde over each of the $\Lambda^{(\tau)}$-transformed quantities in the transformed versions of Eqs. (I.5.2)–(I.5.10). The need for this notation is clarified at the end of this Appendix for the normal fluid case.

With the above points in mind, we can write down all the results for $\Lambda^{(\tau)}$-transformed quantities. Except for introducing a superscript τ and a tilde, there is no formal difference in the equations for transformed quantities. To avoid duplication we shall not write these results.

Normal Fluid

To clarify the difference between the $\Lambda^{(\tau)}$-transformation and the ordinary Λ transformation, we follow the analysis in the first part of Sec. 5 for the normal fluid case. Thus, we *choose* the kernel $Q_0{}^{(\tau)}(t_2, t_1, k)$ of Eq. (2.6) to be

$$Q_0{}^{(\tau)}(t_2, t_1, k) = [1 - B(k)]^{-1}[-\epsilon B(k) \times \delta(t_2{}^{(-)} - t_1) - \epsilon\,\Delta(k)\theta(t_2 - t_1)] \quad . \quad \text{(A1)}$$

This τ-independent kernel is obtained by omitting the $\delta(\beta - t_1)$ term in Eq. (5.1). [Note that for the case of a normal fluid we set all the (0, 2) and (2, 0) quantities equal to zero, so that $Q_{1,1}{}^{(\tau)}(t_2, t_1, k) = M_{1,1}(t_2, t_1, k)$, which is τ-independent as can be seen from Eq. (I.5.8).] The solution for the characteristic function $\mathcal{G}_0{}^{(\tau)\prime}(t_2, t_1, k)$ can be obtained easily from Eq. (5.3) to be

$$\tilde{\mathcal{G}}_0^{(\tau)\prime}(t_2,t_1,k) = [1-B(k)]\,[\delta(t_2^{(-)}-t_1) - \Delta(k)\theta(t_2-t_1)]\exp[-(t_2-t_1)\Delta k] \quad . \tag{A2}$$

By analogy with the derivation of Eq. (5.9), we find

$$\Lambda^{(\tau)\prime}(t_2,t_1,k) = -[1-B(k)]\,\epsilon\Delta(k)\theta(t_2,-t_1) - \epsilon B(k)\delta(t_2^{(-)}-t_1) \quad . \tag{A3}$$

We next write the equations which are analogous for the τ case to Eqs. (5.10)–(5.12):

$$\tilde{\mathcal{G}}_{1,1}^{(\tau)\prime}(t_2,t_1,k) = \delta(t_2^{(-)}-t_1) + \epsilon\tilde{\mathfrak{L}}_{1,1}^{(\tau)\prime}(t_2,t_1,k)$$

$$\tilde{\mathfrak{L}}_{1,1}^{(\tau)\prime}(t_2,t_1,k) = \int_0^\tau ds\,\tilde{\mathcal{G}}_{1,1}^{(\tau)\prime}(t_2,s,k)\{\tilde{K}_{1,1}^{(\tau)\prime}(s,t_1,k) + [1-B(k)]\exp[\beta(g-\omega'(k))]\,\delta(\beta-t_1)\} \quad , \tag{A5}$$

where for the τ case, the quantity $Q'_{1,1}(t_2,t_1,k)$ of Eq. (4.7) becomes, using Eqs. (4.1), (4.2), (5.7), and (A2),

$$\begin{aligned}\tilde{\mathfrak{M}}_{1,1}^{(\tau)\prime}(t_2,t_1,k) - \Lambda^{(\tau)\prime}(t_2,t_1,k) &= \tilde{\mathcal{K}}_{1,1}^{(\tau)\prime}(t_2,t_1,k) - \Lambda^{(\tau)\prime}(t_2,t_1,k)\\ &\quad + [1-B(k)]\exp[\beta(g-\omega'(k))]\delta(\beta-t_1)\\ &\equiv K_{1,1}^{(\tau)\prime}(t_2,t_1,k) + [1-B(k)]\exp[\beta(g-\omega'(k))]\delta(\beta-t_1)\;.\end{aligned} \tag{A6}$$

The second line of Eq. (A6) defines the quantity $\tilde{K}_{1,1}^{(\tau)\prime}(t_2,t_1,k)$.

The quantity of interest in the calculation of the grand potential is the function $\tilde{\mathfrak{L}}_{1,1}^{(t)\prime}(t,t,k)$, obtained by setting $\tau=t_2=t_1=t$ in Eq. (A5). When setting $\tau=t<\beta$, we must be careful not to lose the statistical factor, which is the second term inside the brackets on the right-hand side of Eq. (A5). The simplest way to deal with the statistical factor is to remember that the $\delta(\beta-t)$ function was introduced in the definition of primary linked-pair (μ,ν) graphs in Sec. 2 of I with the intention that its temperature integral always be unity. We may next use Eqs. (A4), (A5), and the integral identity (I.A4) to express the iterated form of $\int_0^\beta dt\,\tilde{\mathfrak{L}}_{1,1}^{(t)\prime}(t,t,k)$ entirely as a sum over integrated products of $\tilde{K}_{1,1}^{(\tau)\prime}$ and the statistical factor. If we then sum over all ways of including statistical factors between products of $\tilde{K}_{1,1}^{(\tau)\prime}$ and also sum up the series which includes only statistical factors, then upon using the integral identity (I.A4) again, we obtain the following result:

$$\int_0^\beta dt\,\tilde{\mathfrak{L}}_{1,1}^{(t)\prime}(t,t,k) = \int_0^\beta dt\,\mathfrak{L}_{1,1}^{(t)\prime}(t,t,k) - \epsilon\ln\{1-\epsilon[1-B(k)]\exp[\beta(g-\omega'(k))]\} \quad , \tag{A7}$$

where, with $\tau \geq (t_2, t_1,)$,

$$\mathfrak{L}_{1,1}^{(\tau)\prime}(t_2,t_1,k) = \int_0^\tau ds\,\mathcal{G}_{1,1}^{(\tau)\prime}(t_2,s,k)K_{1,1}^{(\tau)\prime}(s,t_1,k), \tag{A8}$$

$$\mathcal{G}_{1,1}^{(\tau)\prime}(t_2,t_1,k) = \delta(t_2^{(-)}-t_1) + \epsilon\mathfrak{L}_{1,1}^{(\tau)\prime}(t_2,t_1,k) \quad , \tag{A9}$$

$$K_{1,1}^{(\tau)\prime}(t_2,t_1,k) \equiv \tilde{K}_{1,1}^{(\tau)\prime}(t_2,t_1,k) + \epsilon n'(k)\tilde{K}_{1,1}^{(\tau)\prime}(\beta,t_1,k) = Q'_{1,1}(t_2,t_1,k) \quad , \tag{A10}$$

where the τ-independent quantity $Q'_{1,1}(t_2,t_1,k)$ is given by Eq. (5.12), and $n'(k)$ is given by Eq. (5.4).

In order to prove the second line of Eq. (A10), we first note that

$$\tilde{\mathcal{K}}_{1,1}^{(\tau)\prime}(t_2,t_1,k) + \epsilon n'(k)\tilde{\mathcal{K}}_{1,1}^{(\tau)\prime}(\beta,t_1,k) = \tilde{\mathcal{K}}'_{1,1}(t_2,t_1,k) \quad , \tag{A11}$$

where $\tilde{\mathcal{K}}'_{1,1}(t_2,t_1,k)$ is defined by Eq. (3.7). Identity (A11) is true, because the sum on the left-hand side is equivalent to replacing the τ-transformed pair functions in $\tilde{\mathcal{K}}_{1,1}^{(\tau)\prime}$ by the ordinary pair functions of Eq. (B3). Finally we note, after referring to the definition of $\tilde{\mathcal{K}}_{1,1}^{(\tau)\prime}$ in Eq. (A6), that the sum

$$-[\Lambda^{(\tau)\prime}(t_2,t_1,k) + \epsilon n'(k)\Lambda^{(\tau)\prime}(\beta,t_1,k)]$$

is precisely equal to the terms which are added to $\mathcal{K}'_{1,1}$ in Eq. (5.12).

We conclude this appendix by noting that Eqs. (A7)–(A10) summarize the distinction which we have made for the τ case between quantities with and without a tilde.

APPENDIX B

In this Appendix, we list the explicit expressions for the untransformed and transformed pair functions. The untransformed pair function is given by

$$\overset{t_1t_2}{\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}}_{t_0} \exp\{t_0[\omega(k_1)+\omega(k_2)-\omega(k_3)-\omega(k_4)]\}$$

$$\equiv \theta(t_1-t_2)\overset{t_2}{\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}}_{t_0}\theta(t_2-t_0)+\theta(t_2-t_1)\overset{t_1}{\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}}_{t_0}\theta(t_1-t_0)\,, \qquad \text{for } t_1\neq t_2$$

$$= \overset{t_1}{\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}}_{t_0}\theta(t_1-t_0), \quad \text{for } t_1=t_2, \tag{B1}$$

where $\theta(y)=0$ for $y\leqq 0$ and $\theta(y)=1$ for $y>0$. Also,

$$\overset{t_2}{\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}}_{t_0} = f_1(k_1k_2|k_3k_4)+\delta(t_2-t_0)f_3(k_1k_2|k_3k_4)$$

$$+P\sum_{k_5,k_6} f_2(k_1k_2|k_5k_6|k_3k_4)[\omega(k_1)+\omega(k_2)-\omega(k_5)-\omega(k_6)]^{-1}$$

$$\times\exp\{(t_2-t_0)[\omega(k_1)+\omega(k_2)-\omega(k_5)-\omega(k_6)]\}, \tag{B2}$$

where $\omega(k)=\hbar^2k^2/2m$ and the functions f_1, f_2, and f_3 can be expressed entirely in terms of two-particle reaction matrices, which are well defined even for an infinite repulsive-core interaction.[6,13]

The transformed pair function for a *normal fluid* is obtained by substituting Eqs. (B1), (B2), and (5.3) into Eq. (5.8), giving

$$\overset{t_1t_2}{\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}}''_{t_0} = \exp\{t_0[\omega'(k_1)+\omega'(k_2)-\omega'(k_3)-\omega'(k_4)]\}\Big\{[\theta(t_2-t_1)+\epsilon n'(p_2)]$$

$$\times\theta(t_1-t_0)\overset{t_1}{\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}}''_{t_0}+[\theta(t_1-t_2)+\epsilon n'(p_1)]\,\theta(t_2-t_0)\overset{t_2}{\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}}''_{t_0}$$

$$+n'(p_1)n'(p_2)\overset{\beta}{\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}}''_{t_0}\Big\}, \tag{B3}$$

where $$\overset{t}{\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}}''_{t_0} \equiv [1-B(k_1)][1-B(k_2)][g''(k_1k_2|k_3k_4)+f''(k_1k_2|k_3k_4;\,tt_0)]\,, \tag{B4}$$

with $g''(k_1k_2|k_3k_4)=f_1(k_1k_2|k_3k_4)-[\Delta(k_1)+\Delta(k_2)]\,f_3(k_1k_2|k_3k_4)$

$$+\sum_{k_5,k_6} f_2(k_1k_2|k_5k_6|k_3k_4)\{P[\omega(k_1)+\omega(k_2)-\omega(k_5)-\omega(k_6)]^{-1}$$

$$-P[\omega'(k_1)+\omega'(k_2)-\omega(k_5)-\omega(k_6)]^{-1}\} \quad , \tag{B5}$$

$$f''(k_1k_2|k_3k_4;tt_0)=P\sum_{k_5k_6} f_2(k_1k_2|k_5k_6|k_3k_4)[\omega'(k_1)+\omega'(k_2)-\omega(k_5)-\omega(k_6)]^{-1}$$

$$\times\exp\{(t-t_0)[\omega'(k_1)+\omega'(k_2)-\omega(k_5)-\omega(k_6)]\}+\delta(t-t_0)f_3(k_1k_2|k_3k_4) \quad . \tag{B6}$$

In Eqs. (B3)–(B6), $\omega'(k_1)$ is as given by Eq. (5.5). The quantity $g''(k_1k_2|k_3k_4)$ of Eq. (B5), which can be expressed[6] as the matrix element of a two-body reaction matrix, is an important expansion function[7] in the Λ-transformed quantum-statistical theory. Moreover, the functions f'' and g'' are of the same order.[14] Thus, an nth-order calculation requires the inclusion of terms proportional to the nth power in products of these functions.

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[*] On leave of absence from Tata Institute of Fundamental Research, Colaba, Bombay 5, India.

[†] NRC/ESSA Postdoctoral Resident Research Associate. On leave of absence from Department of Physics, Creighton University, Omaha, Neb.

[1] F. Mohling, I. RamaRao, and D. W. J. Shea, preceding paper, Phys. Rev. A1, 177 (1970); hereafter referred to as I. Equation (n) of this paper will be referred to as (I.n).

[2] Certain difficulties which arise in this case because $\tau \neq \beta$ are dealt with in Appendix A.

[3] Density of particles in the zero-momentum state is given by $\langle x \rangle$.

[4] We use the convention introduced in I (for $\langle x \rangle \neq 0$), that whenever a momentum k does not take the value zero it is represented by p.

[5] Definitions of $B(k)$ and $\Delta(k)$ used in this paper [see Eqs. (5.13) and (5.14)] differ from those of Ref. 13 by factors of $-\epsilon$.

[6] This fact has been demonstrated in the Appendix of F. Mohling, Phys. Rev. 124, 583 (1961).

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[14] See Eq. (47) in Ref. 7.