Superluminous Waves in Streaming Relativistic Plasmas*

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Superluminous waves propagating transverse to the direction of relative streaming (relativistic or nonrelativistic) of a plasma are investigated in the absence of any external field. It is shown that these waves do not excite any instability, that they are essentially stable.

I. INTRODUCTION

Plasmas which are otherwise stable exhibit the well-known two-stream (TS) instability when a relative streaming velocity is imparted to them. Bünemann[1] showed that besides this TS instability there exists a transverse instability in plasmas with wave propagation transverse to the direction of relative streaming. This instability is important for wave numbers k satisfying the condition $ck/\omega_p \lesssim 1$, where ω_p is the plasma frequency. For other wave numbers, the growth rate of this instability is smaller by a factor of v_e/c (v_e being the thermal speed of the electrons) than the one for the longitudinal TS instability. Momota[2] presented the criterion for the existence of these transverse instabilities in cold as well as warm plasmas and showed that the thermal effects were stabilizing. Lee[3] extended Momota's work to include a uniform external magnetic field along the direction of streaming, but he confined himself to counter-streaming plasmas. He concluded that these waves can be excited only if the streaming velocity is greater than the thermal velocity and that the magnetic field acts as a stabilizer towards this instability.

The superluminous waves (waves with phase velocities exceeding the velocity of light), which are excited in a plasma because of thermal fluctuations, do not exhibit any resonance effects; i.e., there is no Landau damping or growth[4,5] associated with them. However, if there are two relativistic plasmas streaming with relativistic velocities, it would be interesting to examine the stability of these waves. As the streaming and thermal velocities are large, in accordance with the conclusions drawn by Bünemann[1] and Lee,[3] we should expect strong transverse instabilities in such plasmas. In this paper we show that the superluminous waves propagating transverse to the direction of relative streaming are essentially stable, no matter how large the thermal and the streaming velocities are. Such plasmas are encountered in nature, but there are magnetic fields associated with them which we have not taken into account in this study. The external magnetic field could be incorporated into the theory without much difficulty, but then an analytical solution is not possible. (The effect of the field will be reported in a forthcoming paper.) Section II presents the analysis of the dispersion relation for transverse superluminous waves in extreme relativistic plasmas as well as in nonrelativistic plasmas with relativistic streaming.

II. DISPERSION RELATION

We consider two hot homogeneous collisionless plasmas in which the electrons are streaming but the ions are immobile and provide only the neutralizing background. Using the linearized relativistic Vlasov equation for the electrons, namely,

$$\frac{\partial f_1}{\partial t} + \frac{\vec{p}}{m\gamma}\cdot\frac{\partial f_1}{\partial \vec{x}} - e\left(\vec{E}_1 + \frac{\vec{p}\times\vec{B}_1}{mc\gamma}\right)\cdot\frac{\partial f_0}{\partial \vec{p}} = 0\,, \qquad (1)$$

we get the dispersion relation,[5] $|\mathbf{R}| = 0$, where

$$\mathbf{R} = (c^2k^2-\omega^2)\mathbf{I} - c^2\vec{k}\vec{k} + \sum_{\alpha=1}^{2}\frac{m\omega\omega_{p\alpha}^2}{N_\alpha}\times\int\frac{d^3p\,\vec{p}}{(\vec{k}\cdot\vec{p}-m\omega\gamma)}\left\{\frac{\partial f_{0\alpha}}{\partial\vec{p}} + \frac{1}{m\omega\gamma}\left[\vec{p}\left(\vec{k}\cdot\frac{\partial f_{0\alpha}}{\partial\vec{p}}\right) - (\vec{k}\cdot\vec{p})\frac{\partial f_{0\alpha}}{\partial\vec{p}}\right]\right\}\,. \qquad (2)$$

In Eq. (1), the subscripts 0 and 1 represent the unperturbed and the perturbed quantities, respectively, m is the rest mass of the electron, and $\gamma = (1+p^2/m^2c^2)^{1/2}$. The subscript α in Eq. (2) labels the two streams and f_0 is the equilibrium distribution function which we take as drifted Maxwellian,[5] namely,

$$f_{0\alpha}(\vec{p}) = N_\alpha\beta_\alpha\exp[-a_\alpha(\gamma-\vec{U}_\alpha\cdot\vec{p}/mc^2)]\,, \qquad (3)$$

with $a_\alpha = mc^2/(kT_\alpha)$

and $$\beta_\alpha = a_\alpha[4\pi m^3c^3\gamma_\alpha^2K_2(a_\alpha/\gamma_\alpha)]^{-1}\,. \qquad (4)$$

In Eq. (4), K_2 is the Bessel function of second kind with imaginary argument and $\gamma_\alpha = (1-U_\alpha^2/c^2)^{-1/2}$.

If we further take the streaming along the z axis and the wave propagation along the x axis, then the elements $R_{xy}, R_{yx}, R_{yz}, R_{zy}$ vanish and the dispersion relation gives the two modes

$$R_{yy}=0 \quad \text{and} \quad R_{xx}R_{zz}-R_{xz}R_{zx}=0 \quad . \tag{5}$$

The former is purely transverse and the latter is a coupled transverse longitudinal mode. From Eqs. (2) and (3), we can show that the elements R_{xz} and R_{zx}, which are responsible for the coupling, are proportional to χ, where

$$\chi=\sum_{\alpha=1}^{2}\int dp\int_0^{2\pi}d\phi\int_0^1 dy\,F_\alpha(p,y,\phi)\times\sinh(pU_\alpha y/KT_\alpha) \quad , \tag{6}$$

which vanishes for both stationary and identical counterstreaming plasmas. Now for superluminous waves, retaining terms of the order of (c^2k^2/ω^2), we can simplify Eq. (2) to give

$$R_{zz}=c^2k^2-\omega^2-\sum_\alpha 4\pi\beta_\alpha\omega_{p\alpha}^2\int_0^\infty\frac{dp\,p^2}{\gamma}\,e^{-a_\alpha\gamma}\times\left\{\left(1+\frac{2p}{m\gamma U_\alpha\eta}\cosh\eta-\left[\frac{1}{\eta}+\frac{p}{m\gamma U_\alpha}\left(1+\frac{2}{\eta^2}\right)\right]\times\sinh\eta-\frac{k^2p^3}{m^3\omega^2\gamma^3U_\alpha\eta}\times\left[\left(1+\frac{12}{\eta^2}\right)\cosh\eta-\frac{1}{\eta}\left(5+\frac{12}{\eta^2}\right)\sinh\eta\right]\right)\right\} \quad , \tag{7}$$

$$R_{xx}=-\omega^2+\sum_\alpha\frac{4\pi c^2\beta_\alpha\omega_{p\alpha}^2}{a_\alpha^2U_\alpha^2}\int_0^\infty\frac{\alpha p\,p^2}{\gamma}\,e^{-a_\alpha\gamma}\left\{\cosh\eta-\frac{1}{\eta}\sinh\eta+\frac{3k^2p^2}{m^2\omega^2\gamma^2\eta}\times\left[\left(1+\frac{3}{\eta^2}\right)\sinh\eta-\frac{3}{\eta}\cosh\eta\right]\right\} \quad , \tag{8}$$

and

$$R_{yy}=c^2k^2-\omega^2+\sum_\alpha\frac{4\pi c^2\beta_\alpha\omega_{p\alpha}^2}{a_\alpha^2U_\alpha^2}\int_0^\infty\frac{dp\,p^2}{\gamma}\,e^{-a_\alpha\gamma}\times\left\{\cosh\eta-\frac{1}{\eta}\sinh\eta+\frac{k^2p^2}{m^2\omega^2\gamma^2\eta}\times\left[\left(1+\frac{3}{\eta^2}\right)\sinh\eta-\frac{3}{\eta}\cosh\eta\right]\right\} \quad , \tag{9}$$

with $\eta=a_\alpha U_\alpha p/(mc^2)$.

We shall evaluate these elements for arbitrary streaming velocities in the two limiting cases, i.e., the extreme relativistic case ($a\ll 1$) and the nonrelativistic case ($a\gg 1$).

A. $a \ll 1$

Equations (7)–(9), in this case, can be integrated to give

$$R_{zz}=c^2k^2-\omega^2-\sum_\alpha\frac{2\pi m^3c^4}{a_\alpha U_\alpha}\,\beta_\alpha\omega_{p\alpha}^2\left[\left(1+\frac{2c}{a_\alpha U_\alpha}\right)e^{-a_{\alpha+}}+\left(\frac{2c}{a_\alpha U_\alpha}-1\right)e^{-a_{\alpha-}}-\frac{2c^2}{a_\alpha U_\alpha^2}\,[E_1(a_{\alpha-})-E_1(a_{\alpha+})]-\frac{c^3k^2}{\omega^2U_\alpha}\left(\frac{e^{-a_{\alpha-}}}{a_{\alpha-}}-\frac{5c}{a_\alpha U_\alpha}\,[E_1(a_{\alpha-})-E_1(a_{\alpha+})]+\frac{e^{-a_{\alpha+}}}{a_{\alpha+}}+\frac{12c^2}{a_\alpha^2U_\alpha^2}\,[E_2(a_{\alpha-})+E_2(a_{\alpha+})]-\frac{12c^3}{a_\alpha^3U_\alpha^3}\,[E_3(a_{\alpha-})-E_3(a_{\alpha+})]\right)\right] , \tag{10}$$

$$R_{xx}=-\omega^2+\sum_\alpha\frac{2\pi m^3c^5\beta_\alpha\omega_{p\alpha}^2}{a_\alpha^2U_\alpha^2}\,(\Psi_1+3\Psi_2) \quad , \tag{11}$$

and

$$R_{yy}=c^2k^2-\omega^2+\sum_\alpha\frac{2\pi m^3c^5\beta_\alpha\omega_{p\alpha}^2}{a_\alpha^2U_\alpha^2}\,(\Psi_1+\Psi_2) \; ; \tag{12}$$

where

$$\Psi_1=\frac{e^{-a_{\alpha+}}}{a_{\alpha+}}+\frac{e^{-a_{\alpha-}}}{a_{\alpha-}}-\frac{c}{a_\alpha U_\alpha}\,[E_1(a_{\alpha-})-E_1(a_{\alpha+})] \quad , \tag{13}$$

$$\Psi_2=\frac{c^3k^2}{\omega^2a_\alpha U_\alpha}\left(E_1(a_{\alpha-})-E_1(a_{\alpha+})-\frac{3c}{a_\alpha U_\alpha}\,[E_2(a_{\alpha-})+E_2(a_{\alpha+})]+\frac{3c^2}{a_\alpha^2U_\alpha^2}\,[E_3(a_{\alpha-})-E_3(a_{\alpha+})]\right) \quad , \tag{14}$$

with $a_{\alpha\mp}=a_\alpha(1\mp U_\alpha/c)$ and E_n as the exponential integral[6] defined by

$$E_n(z)=\int_1^\infty\frac{dt\,e^{-zt}}{t^n} \quad ; \quad \mathrm{Re}\,z>0 \quad . \tag{15}$$

Taking $U_1=-U_2=U_0$ and using the relation

$$E_{n+1}(z)=(1/n)[e^{-z}-zE_n(z)] \; ; \quad n\geq 1 \quad , \tag{16}$$

Eq. (10) for identical plasmas with density $(\frac{1}{2}N)$ finally reduces to

$$\omega^2=c^2k^2+\chi_1+\chi_2/\omega^2 \quad , \tag{17}$$

where

$$\chi_1=\frac{\omega_p^2c^3}{aU_0^3\gamma_0^2K_2(a/\gamma_0)}\times\left[E_1(a_-)-E_1(a_+)-\frac{U_0}{c}\left(1-\frac{aU_0}{2c}\right)e^{-a_-}-\frac{U_0}{c}\left(1+\frac{aU_0}{2c}\right)e^{-a_+}\right] \tag{18}$$

and

$$\chi_2=\frac{\omega_p^2k^2c^7}{a^2U_0^4\gamma_0^2K_2(a/\gamma_0)}\left[6\left(1-a+\frac{aU_0^2}{c^2}\right)\times\left(\frac{e^{-a_-}}{a_-}-\frac{e^{-a_+}}{a_+}\right)-\frac{6U_0}{c}\left(\frac{e^{-a_-}}{a_-}+\frac{e^{-a_+}}{a_+}\right)-a\left(6-\frac{U_0^2}{c^2}\right)\left(E_1(a_+)-E_1(a_-)\right)\right] \quad . \tag{19}$$

In writing Eq. (17) we have made use of the fact that the coupling is zero for conterstreams, and that Eqs. (10)–(12) represent three independent modes. The roots of Eq. (17) are real unless $\chi_1<0$, because the negative root with χ_1 and χ_2 positive is not a valid root for superluminous waves. It is, however, interesting to note that for $a<1$, χ_1 is always a positive quantity because the term within the square brackets of Eq. (18) can be written as

$$\ln\left(\frac{1+U_0/c}{1-U_0/c}\right)-\frac{2U_0}{c}+O(a^2)\quad,\tag{20}$$

and this is always positive. Similarly, we can show that $\Psi_i>0$ for all values of U_0 and consequently Eqs. (11) and (12) can have only real valid roots. Hence, superluminous waves in relativistic counterstreaming plasmas are always stable. We point out here that for the validity of the expansion used in Eq. (7), we must satisfy the condition $\chi_2\ll\chi_1^2$, which in turn demands that $a\leqslant U_0^4/C^4$.

B. $a\gg 1$

In this case, $\gamma\approx(1+p^2/2m^2c^2)$; so Eq. (7) can be written as

$$R_{zz}=c^2k^2-\omega^2-\sum_\alpha 4\pi\beta_\alpha\omega_{p\alpha}^2e^{-a_\alpha}\int_0^\infty dp\,p^2\times\exp\left(-\frac{a_\alpha p^2}{2m^2c^2}\right)\left\{\left(1+\frac{2p}{m\lambda U_\alpha}\right)\cosh\lambda-\frac{1}{\lambda}\times\left(1+\frac{p\lambda}{mU_\alpha}+\frac{2p}{m\lambda U_\alpha}\right)\sinh\lambda-\frac{k^2p^3}{m^3\omega^2\lambda U_\alpha}\times\left[\left(1+\frac{12}{\lambda^2}\right)\cosh\lambda-\frac{1}{\lambda}\left(5+\frac{12}{\lambda^2}\right)\sinh\lambda\right]\right\},\tag{21}$$

where $\lambda=a_\alpha U_\alpha p/(mc^2)$. Equation (21) can be integrated immediately to yield

$$\omega^2=c^2k^2+\omega_p^2\sigma\left(1+\frac{k^2v_e^2}{\omega^2}+\frac{k^2U_0^2}{\omega^2}\right),\tag{22}$$

where
$$\sigma=\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{KT}{mc^2}\right)^{1/2}\left(\gamma_0^2K_2\,\frac{a}{\gamma_0}\right)^{-1}\times\exp\left(-a+\frac{U_0^2}{2v_e^2}\right).\tag{23}$$

Similarly, the other two modes given by Eqs. (8) and (9), in this limit, simplify to give

$$\omega^2=\sigma\omega_p^2(1+3k^2v_e^2/\omega^2)\ ,\tag{24}$$

and
$$\omega^2=c^2k^2+\sigma\omega_p^2(1+k^2v_e^2/\omega^2)\quad.\tag{25}$$

Now σ being positive for all values of U_0, all these three modes are absolutely stable for superluminous waves. However, it is interesting to note that for slow waves, i.e., $\omega<ck$, one of the modes given by Eq. (22) for a cold plasma is always unstable. This is in conformity with the conclusions of Momota[2] and Lee.[3] For $U_0\ll c$, $\sigma\approx 1$; so two of the modes become independent of U_0, whereas the linearly polarized transverse mode does depend on the streaming velocity.

III. CONCLUSIONS

The superluminous waves propagating transverse to the direction of streaming in counterstreaming plasmas are always stable, no matter what the temperature and the streaming velocities are; they may be nonrelativistic or relativistic. In the nonrelativistic limit for nonrelativistic streaming velocities, only one of the transverse modes, which is linearly polarized, is affected by the streamings.

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[1] D. Bünemann, Ann. Phys. 25, 340 (1963).
[2] H. Momota, Progr. Theoret. Phys. 35, 380 (1966).
[3] Kai Fong Lee, Phys. Rev. 181, 447 (1969).
[4] B. Buti, Phys. Fluids 5, 1 (1962).
[5] B. Buti, Phys. Fluids 6, 89 (1963).
[6] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Natl. Bur. Stds., Washington, D.C., 1965).