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Stimulated Modulational Instabilities of Plasma Waves

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Nonlinear coupling of a modulated wave and a low-frequency mode is shown to produce a resonant interaction and instability when the Cherenkov condition $v_p = v_g \cos\theta$ is satisfied for the phase velocity v_p of the low-frequency mode, the group velocity v_g of the modulated wave, and the angle θ between the two wave vectors. This effect stimulates the self-trapping (modulational instability) or self-focusing of the modulated wave. Examples are shown for the cases of couplings between plasma cyclotron waves and magnetohydrodynamic (MHD) modes.

I. INTRODUCTION

Recently, the propagation of modulated waves in a nonlinear dispersive medium has aroused considerable interest in the self-focusing[1] or self-trapping[2] of laser beams and in the modulational instability of a nonlinear plasma wave.[3,4] Such an effect has been represented[5] by a Schrödinger equation for the amplitude of the modulated wave φ with a nonlinear potential term that is proportional to $|\varphi|^2$, i.e.,

$$i\frac{\partial\varphi}{\partial t} - \frac{\partial\omega}{\partial|\varphi|^2}(|\varphi|^2 - |\varphi_0|^2)\varphi + \frac{1}{2}\frac{\partial v_g}{\partial k}\frac{\partial^2\varphi}{\partial x^2} = 0. \quad (1)$$

The modulation has been shown to become unstable when the potential is attractive[1]; i.e., when

$$\frac{\partial\omega}{\partial|\varphi|^2}\frac{\partial v_g}{\partial k} < 0.$$

In the present paper, we present a new process that leads to a similar instability. In this case, the instability is due to a coupling of the modulated wave with a low-frequency nondispersive mode that may coexist in the same medium.

In this Introduction, we describe the general idea of the process. Consider a wave with slowly varying amplitude $\epsilon\varphi(x,t)\, e^{i(kx-\omega t)}$, where ϵ is a small parameter. A second-order nonlinearity will generate a perturbation of the form $\epsilon^2|\varphi(x,t)|^2$ and $\epsilon^2\varphi^2(x,t)\, e^{2i(kx-\omega t)}$. If the medium can propagate a low-frequency and long-wavelength mode, the slow perturbation, $\epsilon^2|\varphi(x,t)|^2$, will then excite this mode. We represent this mode by $\epsilon^2 V(x,t)$. If the medium is nondispersive at low frequencies, the equation describing V to lowest order may be written as an inhomogeneous linear equation with a source term proportional to $|\varphi|^2$, i.e.,

$$DV(x,t) = \alpha|\varphi(x,t)|^2. \quad (2)$$

In Eq. (2), α is the coupling coefficient, and D is a linear differential operator involving $\partial/\partial t$ and $\partial/\partial x$, having the form

$$D = \prod_{j=1}^{n} d_j\left(\frac{\partial^2}{\partial t^2} - v_{pj}^2\frac{\partial^2}{\partial x^2}\right) \quad , \quad (3)$$

where d_j is a constant, and the v_{pj} are the characteristic phase velocities ($j = 1, 2\cdots n$) of the low-frequency modes. Because a linear wave packet propagates at the group velocity v_g, the space and time dependency of φ in the right-hand side of Eq. (2), should be of the form $(x - v_g t) \equiv \xi$ in the lowest order. Equation (1) then assumes the form

$$\prod_{j=1}^{n} d_j\left(v_g^2 - v_{pj}^2\right)\frac{\partial^{2n}V}{\partial\xi^{2n}} = \alpha|\varphi|^2 \ . \quad (4)$$

On the other hand, if the medium is linear but dispersive for the modulated high-frequency mode the amplitude function $\varphi(x,t)$ will obey a linear Schrödinger equation of the form[5]

$$i\frac{\partial\varphi}{\partial t}+\frac{1}{2}\frac{dv_g}{dk}\frac{\partial^2\varphi}{\partial x^2}=0. \tag{5a}$$

If the medium is nonlinear, then the nonlinear interaction between φ and V (the third-order nonlinearity) will produce a coupling term. Equation (5a) can then be modified to

$$i\frac{\partial\varphi}{\partial t}+\frac{1}{2}\frac{dv_g}{dk}\frac{\partial^2\varphi}{\partial x^2}=\beta V\varphi, \tag{5b}$$

where β is the coupling coefficient. Elimination of V from Eqs. (4) and (5b) immediately results in a nonlinear Schrödinger equation with a nonlinear potential proportional to $|\varphi|^2$ and having a coefficient proportional to $(v_g^2-v_{pj}^2)^{-1}$. In other words, the nonlinear frequency change $\partial\omega/\partial|\varphi|^2$ now becomes proportional to $(v_g-v_{pj})^{-1}$. This indicates that a suitable choice of v_{pj} will change the sign of the coefficient in such a way that the potential becomes attractive. Hence, the instability can be produced by the coupling. Furthermore, if v_g approaches v_{pj} from the attractive potential side, the instability will be enhanced because of the resonant denominator.

If one assumes a two-dimensional wave packet $\varphi(x,y,t)\,e^{i(kx-\omega t)}$, it can be shown in the same way that the resonant condition becomes $v_g\cos\theta=v_{pj}$, where θ is the angle between x and the direction of propagation of the low-frequency mode.

Equation (5b) itself is an ordinary linear Schrödinger equation if V is regarded as the potential. Furthermore, Eq. (4) shows that V is created by the particle density $|\varphi|^2$, represented by the Schrödinger equation. Therefore the entire process can be understood as an interaction between quantum particles (quasiparticles representing the modulated wave) and the self-generated wave (low-frequency mode).

Because the Cherenkov condition $v_g\cos\theta=v_p$ satisfies energy and momentum conservation among the modes, $(\omega,\vec{k})$, $(\Delta\omega,\Delta\vec{k})$ and $(\omega\pm\Delta\omega, \vec{k}\pm\Delta\vec{k})$, if $v_g\cos\theta-v_p<\epsilon$ the process may be interpreted as a decay instability,[6] (the decay instability of a modulated wave). A separate treatment is necessary for this case because the present ordering breaks down.

II. THEORY FOR CYCLOTRON WAVES

We now give examples of the above process employing modulated electron and ion cyclotron waves. We assume a plasma with warm electrons and cold ions. We assume also that the thermal effect on the cyclotron waves is negligible. This is valid so long as we limit the frequency of interest to be well away from the cyclotron frequency.

We first introduce the scale transform in terms of the slowness parameter ϵ through the equations

$$\xi=\epsilon(x-v_g t),\quad \eta=\epsilon y,\quad \tau=\epsilon^2 t, \tag{6}$$

and then consider a modulated wave of the form $\epsilon\varphi(\xi,\eta,\tau)\,e^{i(kx-\omega t)}$, where x is the direction of the dc magnetic field. Because the cyclotron waves are transverse waves in the linear regime, the longitudinal variables, such as the density n or the x components of the field variables, appear only from the second- and higher-order nonlinearity. Thus, we assume the following expansions in ϵ for the transverse variables:

$$\begin{pmatrix}\vec{v}_\perp\\ \vec{E}_\perp\\ \vec{B}_\perp\end{pmatrix}=\left(\sum_{m=1}^{3}\epsilon^m\underline{\psi}_{\perp m}^1(\xi,\eta,\tau)\right)e^{i(kx-\omega t)}, \tag{7}$$

and for the longitudinal variables

$$\begin{pmatrix}v_x\\ E_x\\ B_x\\ n\end{pmatrix}=\begin{pmatrix}0\\0\\B_0\\n_0\end{pmatrix}+\epsilon^2\begin{pmatrix}v_{x2}^0\\ E_{x2}^0\\ B_{x2}^0\\ n_2^0\end{pmatrix}+\epsilon^2\begin{pmatrix}v_{x2}^1\\ E_{x2}^1\\ B_{x2}^1\\ n_2^1\end{pmatrix}e^{i(kx-\omega t)}, \tag{8}$$

where $\vec{E}$, $\vec{B}$, $\vec{v}$ are the electric field, the magnetic field, and the ion and electron velocities. On the right-hand side of Eq. (8), the first term shows the dc magnetic field and plasma density, and the second term shows the slowly varying field quantities produced by the second-order nonlinearity of the transverse variables and represents the low frequency mode. The third term contains the high frequency longitudinal variables produced by the coupling to the longitudinal mode through the η dependency of the transverse variables.

We use the equation of motion and Maxwell's equations for high-frequency variables:

$$\frac{d\vec{v}_i}{dt}=\frac{e}{m_i}(\vec{E}+\vec{v}_i\times\vec{B}), \tag{9}$$

$$\frac{d\vec{v}_e}{dt}=-\frac{e}{m_e}(\vec{E}+\vec{v}_e\times\vec{B}),$$

$$\vec{\nabla}\times\vec{B}=e\mu_0 n(\vec{v}_i-\vec{v}_e)+\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}, \tag{10}$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}. \tag{11}$$

From the equations of order ϵ, we obtain

$$\underline{\psi}_{\perp1}^1=\varphi_\perp(\xi,\eta,\tau)\underline{r}, \tag{12}$$

where $\underline{r}$ is a column vector satisfying $\underline{Wr}=0$ and

$$|\underline{W}|=0 \tag{13}$$

is the dispersion relation of the cyclotron waves. Using a technique similar to that developed by Taniuti and Washimi [3] (details are shown in Appendix), we obtain a Schrödinger equation for ϕ that corresponds to Eq. (5b):

$$i\frac{\partial\varphi_1}{\partial\tau}+\frac{1}{2}\frac{dv_g}{dk}\frac{\partial^2\varphi_1}{\partial\xi^2}+\frac{v_g}{4k}\left(1+\frac{k^2c^2}{\omega^2-\omega_p^2}\right)\frac{\partial^2\varphi_1}{\partial\eta^2}$$
$$=kv_g\left(\frac{\omega}{kv_g}\frac{B_{x2}^0}{B_0}+\frac{v_{x2}^0}{v_g}-\frac{n_2^0}{2n_0}\right)\varphi_1, \qquad (14)$$

where c and ω_p are the speed of light and the electron plasma frequency. Equation (14), for appropriate $v_g(\omega)$ and $k(\omega)$, holds for both ion and electron cyclotron waves.

The slowly varying longitudinal variables B_{x2}^0, v_{x2}^0, and n_2^0 are determined by the equations describing the low-frequency mode. We use here MHD equations. For simplicity, we assume a plane perturbation for φ_1 as well as for the low-frequency mode, that propagates with an angle θ with respect to the x axis. The transverse variables couple through the $\vec{J}_\perp\times\vec{B}_\perp$ term ($\vec{J}$ is the current density) in the MHD equations. Then we can obtain for the longitudinal variables, corresponding to Eq. (4), the following set of equations:

$$\frac{\partial^2}{\partial\xi'^2}\left[(v_g^2-c_s^2)\cos^2\theta\, v_{x2}^0-c_s^2\sin\theta\cos\theta\, v_{y2}^0\right]$$
$$=v_A^2v_g\cos^2\theta\frac{\partial^2}{\partial\xi'^2}\left|\tfrac{1}{2}\varphi_1\right|^2,$$

$$\frac{\partial^2}{\partial\xi'^2}\left[c_s^2\sin\theta\cos\theta\, v_{x2}^0\right.$$
$$\left.+(c_s^2\sin^2\theta+v_A^2-v_g^2\cos^2\theta)\,v_{y2}^0\right]=0, \qquad (15)$$

$$\frac{\partial}{\partial\xi'}\left(\frac{n_2^0}{n_0}-\frac{v_{x2}^0}{v_g}-\frac{v_{y2}^0}{v_g}\tan\theta\right)=0,$$

$$\frac{\partial}{\partial\xi'}\left(\frac{B_{x2}^0}{B_0}-\frac{v_{y2}^0}{v_g}\tan\theta\right)=0,$$

where $\xi'=\xi\cos\theta+\eta\sin\theta$ is the coordinate along the direction of propagation of the MHD mode and $\varphi_1=B_{11}^1/B_0$ and v_A and c_s are Alfvén and sound speeds, respectively. If we eliminate B_{x2}^0, v_{x2}^0, and n_2^0 from Eqs. (14) and (15), the coupling term in Eq. (14) (the right-hand side) finally becomes

$$\frac{kv_g}{4}\frac{v_A^2\cos^2\theta\,(v_g^2\cos^2\theta-v_m^2)}{(v_g^2\cos^2\theta-v_s^2)(v_g^2\cos^2\theta-v_f^2)}\left(|\varphi_1|^2-|\varphi_0|^2\right)\varphi_1,$$

where v_f and v_s are the phase velocities of the fast (Alfvén) and slow (acoustic) MHD waves,

$$\begin{Bmatrix}v_f^2\\v_s^2\end{Bmatrix}=\frac{1}{2}\left\{v_A^2+c_s^2\pm\left[(v_A^2+c_s^2)^2-4v_A^2c_s^2\cos^2\theta\right]^{1/2}\right\}, \qquad (16)$$

φ_0 is the value of φ_1 at $\xi'\to-\infty$, and

$$v_m^2=v_A^2-\left(2\frac{\omega}{kv_g}-1\right)c_s^2\sin\theta. \qquad (17)$$

From Eqs. (16) and (17), v_m can be shown to exist between v_f and v_s. The expression derived above for the coupling term indeed possesses poles corresponding to the Cherenkov condition.

The ion cyclotron wave, whose group velocity is always smaller than the Alfvén velocity, may resonantly interact with the slow wave. The electron cyclotron wave, whose group velocity can exceed the Alfvén velocity, may interact with the fast wave.

III. CONCLUSION

It has been shown that the nonlinear self-action term $\partial\omega/\partial|\varphi|^2$, whose behavior is essential in considering self-focusing or self-trapping of a modulated wave, becomes proportional to $(v_g\cos\theta-v_p)^{-1}$ and hence is significantly modified in the presence of a low-frequency mode that can propagate in the same medium. The mathematical expression for the interaction is identical with that between quantum particles (modulated wave) and the self-generated field (low-frequency mode). When the resonant condition is exactly met, the process may be interpreted as a decay instability of the modulated wave.

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APPENDIX

For the purpose of illustrating the process leading to Eq. (14), we consider an electron cyclotron wave only and ignore ion dynamics. Eliminating the transverse magnetic field from Maxwell's equation, we find

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E}-\frac{\omega_p^2}{c^2\omega_c}\frac{\partial}{\partial t}(n\vec{v}_e)-\frac{1}{c^2}\vec{e}_x\frac{\partial^2E_x}{\partial t^2}=0, \qquad (A1)$$

where E, n, $\vec{v}_e$ are normalized by cB_0, n_0, and c; and the transverse displacement current is ignored for simplicity. The equation of motion must also be normalized in the same way. Then, corresponding to Eq. (7), we take the expansion of a column vector,

$$\begin{pmatrix}E_y\\E_z\\v_y\\v_z\end{pmatrix}\equiv\underline{\psi}_\perp=\sum_{m=1}^{3}\epsilon^m\underline{\psi}_{\perp m}(\xi,\eta,\tau). \qquad (A2)$$

Using Eq. (A1) and the normalized equation of motion, we can write an equation for $\underline{\psi}_\perp$ as

$$\epsilon \underline{W}_1\underline{\psi}_{\perp 1} + \epsilon^2\left(\underline{W}_1\underline{\psi}_{\perp 2} + \underline{W}_2\frac{\partial\underline{\psi}_{\perp 1}}{\partial\xi} + \underline{W}_3\frac{\partial\underline{\psi}_{\perp 1}}{\partial\tau'}\right) + \epsilon^3\left(\underline{W}_1\underline{\psi}_{\perp 3} + \underline{W}_2\frac{\partial\underline{\psi}_{\perp 2}}{\partial\xi} + \underline{W}_3\frac{\partial\underline{\psi}_{\perp 2}}{\partial\tau'} + \underline{W}_4\underline{\psi}_{\perp 1} + \underline{W}_5\frac{\partial\underline{\psi}_{\perp 1}}{\partial\xi^2} + \underline{W}_6\frac{\partial\underline{\psi}_{\perp 1}}{\partial\eta^2} + \underline{W}_7\frac{\partial\underline{\psi}_{\perp 1}}{\partial\tau}\right) = 0 \quad , \tag{A3}$$

where

$$\underline{W}_1 = \begin{pmatrix} k^2 & 0 & i\frac{\omega\omega_p^2}{c^2\omega_c} & 0 \\ 0 & k^2 & 0 & i\frac{\omega\omega_p^2}{c^2\omega_c} \\ \omega_c & 0 & -i\omega & \omega_c \\ 0 & \omega_c & -\omega_c & -i\omega \end{pmatrix},$$

$$\underline{W}_2 = \begin{pmatrix} -2ik & 0 & 0 & 0 \\ 0 & -2ik & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\underline{W}_3 = \begin{pmatrix} 0 & 0 & -\frac{\omega_p^2}{c^2\omega_c} & 0 \\ 0 & 0 & 0 & -\frac{\omega_p^2}{c^2\omega_c} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\underline{W}_4 = \begin{pmatrix} 0 & 0 & \frac{i\omega\omega_p^2 n_2^0}{c^2\omega_c} & 0 \\ 0 & 0 & 0 & \frac{i\omega\omega_p^2 n_2^0}{c^2\omega_c} \\ -\frac{ck\omega_c v_{x2}^0}{\omega} & 0 & ikcv_{x2}^0 & B_{x2}^0\omega_c \\ 0 & -\frac{ck\omega_c v_{x2}^0}{\omega} & -\omega_c B_{x2}^0 & ikcv_{x2}^0 \end{pmatrix},$$

$$\underline{W}_5 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\underline{W}_6 = \begin{pmatrix} \frac{-c^2k^2}{\omega^2-\omega_p^2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\underline{W}_7 = \begin{pmatrix} 0 & 0 & -\frac{\omega_p^2}{c^2\omega_c} & 0 \\ 0 & 0 & 0 & -\frac{\omega_p^2}{c^2\omega_c} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\tau' = \epsilon t$. Hence,

$$\xi = \epsilon x - v_g \tau' .$$

From order ϵ, we get

$$\underline{\psi}_{\perp 1} = \underline{r}\varphi_1(\xi, \eta, \tau), \tag{A4}$$

where $\underline{r}$ is a column vector that satisfies

$$\underline{W}_1\underline{r} = 0 \quad \text{or} \quad \underline{r} = \begin{pmatrix} 1 \\ i \\ i c^2k^2\omega_c/\omega_p^2\omega \\ -c^2k^2\omega_c/\omega_p^2\omega \end{pmatrix} \tag{A5}$$

while $|\underline{W}_1| = 0$ is the dispersion relation of the electron cyclotron wave which under our assumption takes the following form:

$$k^2 = -\frac{\omega_p^2}{c^2}\,\frac{\omega}{\omega - \omega_c} \quad . \tag{A6}$$

Let l now be the left eigenvector of $\underline{W}_1$ such that

$$\underline{l}\,\underline{W}_1 = 0 \quad \text{or} \quad \underline{l} = (1, -i, -k^2/\omega_c, ik^2/\omega_c). \tag{A7}$$

We multiply Eq. (A3) (to order ϵ^2) by $\underline{l}$ on the left, and substitute (A4) into it:

$$\underline{l}\,\underline{W}_2\underline{r}\frac{\partial\varphi_1}{\partial\xi} + \underline{l}\,\underline{W}_3\underline{r}\frac{\partial\varphi_1}{\partial\tau'} = 0,$$

$$\text{or} \qquad \frac{\partial\varphi_1}{\partial\xi} + \frac{c^2k^4\omega_c}{2k\omega_p^2\omega^2}\frac{\partial\varphi_1}{\partial\tau'} = 0. \tag{A8}$$

Using the dispersion relation (A6), we can easily show that the coefficient of the second term in Eq. (A8) is equal to $\partial k/\partial\omega = 1/v_g$. Hence, because of the definitions of ξ and τ', Eq. (A8) is identically satisfied. This verifies the use of the group velocity in the coordinate transformation shown in Eq. (6).

As the second-order solution of $\underline{\psi}_\perp$, we then choose

$$\underline{\psi}_{\perp 2} = \underline{r}\varphi_2 + \underline{s}\frac{\partial\varphi_1}{\partial\xi} \quad , \tag{A9}$$

where $\underline{s}$ is a column vector that satisfies

$$\underline{W}_1\underline{s} + (\underline{W}_2 - v_g\underline{W}_3)\,\underline{r} = 0$$

or $$\underline{s} = \begin{pmatrix} 0 \\ 0 \\ \frac{2kc^2}{\omega_p^2} \\ \frac{i2kc^2}{\omega_p^2} \end{pmatrix}. \tag{A10}$$

Now we go to order ϵ^3. Substituting Eqs. (A4) and (A9) into (A3) and multiplying the resulting equations by $\underline{l}$ from left we obtain

$$\underline{l}(\underline{W}_2 - v_g \underline{W}_3)\underline{s}\frac{\partial^2\varphi_1}{\partial\xi^2} + \underline{l}\,\underline{W}_4\underline{r}\varphi_1 + \underline{l}\,\underline{W}_5\underline{r}\frac{\partial^2\varphi_1}{\partial\xi^2} + \underline{l}\,\underline{W}_6\underline{r}\frac{\partial^2\varphi_1}{\partial\eta^2} + \underline{l}\,\underline{W}_7\underline{r}\frac{\partial\varphi_1}{\partial\tau} = 0,$$

which, after some manipulation, reduces to the expression shown as Eq. (14) in the text.

Although the ion dynamics is ignored here it is not difficult to include it. In this case, the dispersion relation changes, but it can be shown that if a suitable group velocity and wave number are used, the final expression [Eq. (14)] is still valid.

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Sound Attenuation and Dispersion near the Liquid-Gas Critical Point

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The attenuation and dispersion of sound near the gas-liquid critical point are studied theoretically using the author's extended mode-mode coupling theory. The results differ in the different regions of the sound-wave frequency f expressed in a dimensionless unit and of ϵ, the dimensionless temperature distance from the critical point. The attenuation behaves as $f^2\epsilon^{-3\nu-\alpha/2}$ for $0 \leq f \ll \epsilon^{3\nu}$, and as $f^{2-2p/3}\epsilon^{-3\alpha/2}$ for $\epsilon^{3\nu} \ll f \ll \epsilon^{\nu}$, where p is the exponent which appears in the wave-number ($\vec{k}$)-dependent correlation of the order parameter expressed as $A_1k^{-2+\eta} + A_2\epsilon^{1-\alpha}k^{-2+\eta-p}$, when k is much greater than the inverse correlation range of critical fluctuations. The relative sound-velocity change with f behaves as $f^{3/2}\epsilon^{-9\nu/2}$ for $0 \leq f \ll \epsilon^{3\nu}$, as $f^{1-2p/3}\epsilon^{-\alpha}$ if $p \leq \frac{3}{2}$, and as $f^0\epsilon^0$ if $p > \frac{3}{2}$ for $\epsilon^{3\nu} \ll f \ll \epsilon^{\nu+\alpha/2}$. The explicit expressions for the attenuation and dispersion are given for $f \sim \epsilon^{3\nu}$.

I. INTRODUCTION

In recent years the sound attenuation and dispersion near the critical points have attracted an increasing amount of attention as a means of studying dynamics of critical fluctuations.[1,2] In particular, the first successful theoretical study of sound attenuation and dispersion near the liquid-gas critical points was carried out in 1965 by Botch and Fixman.[3]

After 1965, there has been a considerable progress in our understanding of the dynamics of critical fluctuations.[4] In particular, Kadanoff and Swift's brilliant application[5] of the mode-mode coupling theory[6] to the liquid-gas transition yielded valuable information on the divergences of various transport coefficients near the transition. This progress made it necessary to reconsider the problem of the sound attenuation and dispersion near the liquid-gas transition. Thus, it is the purpose of the present paper to study this problem in some detail using the extended version of the