Propagation of Small-Area Pulses of Coherent Light through a Resonant Medium*

M. D. Crisp†
Columbia Radiation Laboratory, Department of Physics, Columbia University, New York, New York 10027
(Received 9 December 1969)

This article presents a study of the propagation of small-area pulses of coherent light through matter. It is found that the small-area pulses exactly obey an area theorem which, in the case of an attenuating medium, requires that the pulse area decay to zero exponentially with increasing distance of propagation. This, however, does not necessarily imply that the pulse energy decays exponentially. Instead, it is found that pulses of duration comparable to or shorter than the transverse relaxation time of the medium T_2 (including both homogeneous and inhomogeneous broadening) propagate with low energy loss. The results are explained in terms of simple physical arguments which indicate that the pulse envelope should oscillate between positive and negative values, causing the area to decrease without a comparable decrease in the pulse energy. Analytic solutions are presented for the case of a pulse whose envelope varies as $t^k e^{-t}$ for $t > 0$, a rectangular pulse, an ultrashort pulse (i.e., pulse width much less than T_2), and the leading portion of a step-function pulse. The propagation of a small-area Gaussian pulse is studied numerically.

INTRODUCTION

Recent investigations[1-3] into the propagation of coherent light pulses through a resonant medium have revealed several interesting effects. Although most of the phenomena observed so far are unique to large-area pulses [the area of a pulse with envelope $\mathcal{E}(z,t)$ is usually defined to be equal to $\int_{-\infty}^{\infty} \mu \mathcal{E}(z,t)\, dt/\hbar$], there still are interesting pulse-shaping effects which occur for small-area pulses. Furthermore, the propagation of small-area pulses may be treated analytically to a much greater degree than has been found possible for large-area pulses. The analytic solutions obtained for small-area pulses may serve as guides for interpreting and checking the accuracy of machine calculations for pulses of area up to $\frac{1}{4}\pi$.

This article is devoted to the study of the propagation of small-area coherent pulses of light through a resonant medium. It will be shown that these pulses exactly obey an area theorem which, in the case of an attenuating medium, requires that the pulse area drop to zero exponentially with increasing distance of propagation into the medium. But this does not necessarily imply that the pulses lose their energy exponentially. In fact, for pulses short compared with the transverse relaxation time T_2 (including both homogeneous and inhomogeneous broadening), the electric-field envelope oscillates between positive and negative values in just such a way that the area theorem is satisfied but the pulses lose little energy after their initial reshaping. It will be shown that the formation of these "zero-degree pulses" gives rise to significant deviations from Beer's law.[4]

The physical basis for the formation of the zero-degree pulses which are described below can be simply understood. As a small-area pulse enters the resonant medium, its leading edge excites a macroscopic polarization in the thin slice of medium located at the surface. This polarization radiates 180° out of phase with respect to the input pulse for a time of the order of T_2 after the pulse has passed. If the trailing edge of the pulse drops off faster than the decay of the macroscopic polarization, then the envelope of the pulse leaving this slice will go through zero and become negative. The next slice of resonant material now sees a field envelope whose trailing edge drops off faster than before and then becomes negative. As a result, the polarization induced in the second slice by the positive lobe of the pulse envelope radiates 180° out of phase with respect to that field and adds to the negative lobe. In this way, a pulse can develop a field envelope which has negative-area regions that subtract from the total-pulse area. Thus, the total-pulse area can go to zero while the pulse energy remains finite.

Many of the results to follow can be understood on the basis of another argument. A pulse of duration comparable to or less than the transverse relaxation time T_2 will have a spectral content that is comparable to or broader than the absorption line of the attenuating medium. As such a pulse propagates, the absorbing medium eats a hole in the pulse spectrum. In this way, the pulse spectrum develops a dip and begins to resemble the spectrum one would get from the superposition of two quasimonochromatic beams which differ slightly in central frequency. Such a light beam would develop temporal beats. As the pulse propagates in the medium, the hole in the pulse spectrum becomes deeper. This causes the two apparent central frequencies to become further apart so that the temporal beats become more rapid.

In terms of this spectral argument, the anomalously low absorption can be simply understood as the result of small absorption of those Fourier components which are far off resonance.

FORMULATION

The basic equations which describe the propagation of coherent pulses in a resonant medium have been derived elsewhere in the literature[5,6] and for this reason a cursory treatment seems appropriate here.

The resonant medium will be assumed to consist of N two-level atoms per unit volume embedded in a homogeneous dielectric that may be characterized by an index of refraction η which is constant over the spectrum of the pulse. Any state of one of the two-level atoms may be written

$$\Psi(\vec{x}, t) = a(t)\Psi_a(\vec{x}) + b(t)\Psi_b(\vec{x}) \quad , \tag{1}$$

where Ψ_a and Ψ_b are eigenfunctions of the unperturbed atomic Hamiltonian which correspond to the eigenvalues $\frac{1}{2}\hbar\Omega$ and $-\frac{1}{2}\hbar\Omega$, respectively. Alternatively, the state of an atom may be represented by the real variables X, Y, and Z, which are defined according to [7,8]

$$(X - iY)e^{-i\omega(t-\eta z/c)+\varphi(z,t)} \equiv 2ab^* \, , \tag{2a}$$

$$Z \equiv aa^* - bb^* \quad , \tag{2b}$$

where $\varphi(z, t)$ is the slowly varying phase of the electric field as defined in Eq. (5). These variables satisfy $X^2+Y^2+Z^2=1$ when $aa^*+bb^*=1$. The phases Ψ_a and Ψ_b can be chosen so that the electric dipole moment matrix element between the two states μ is real. The expectation of the dipole moment operator for an atom in the state given by Eq. (1) becomes

$$\langle \mu_{op} \rangle = \mu \, \mathrm{Re}[(X - iY)e^{-i\omega(t-\eta z/c)+i\varphi(z,t)}] \tag{3}$$

in terms of the new variables. It will be assumed that there is a distribution of atomic frequencies Ω which may be described by a normalized distribution function $g(\Delta)$, where $\Delta \equiv \Omega - \omega$ is the amount of resonance of a particular atom. For such an inhomogeneously broadened medium, the polarization induced by the applied electric field would be

$$P(z, t) = N\mu \, \mathrm{Re}\{\int_{-\infty}^{\infty} [X(z, t, \Delta) - iY(z, t, \Delta)] \times g(\Delta)\, d\Delta \exp[-i\omega(t-\eta z/c) + i\varphi(z, t)]\} \, , \tag{4}$$

where $X(z, t, \Delta)$ and $Y(z, t, \Delta)$ are the variables, defined in Eq. (2a), which refer to an atom with transition frequency $\Omega = \omega + \Delta$.

The coherent light pulse will be considered to be linearly polarized along the x direction and to propagate in the positive z direction so it can be written

$$E(z, t) = \mathcal{E}(z, t)\cos[\omega(t - \eta z/c) - \varphi(z, t)] \quad . \tag{5}$$

In general, one would describe the evolution of the plane-wave light pulse of Eq. (5) by the second-order wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\eta^2}{c^2}\frac{\partial^2}{\partial t^2}\right) E(z, t) = \frac{4\pi}{c^2}\frac{\partial^2 P}{\partial t^2}(z, t) \quad . \tag{6}$$

But for the pulses that are used in most experiments, $\mathcal{E}(z, t)$ and $\varphi(z, t)$ vary little during an optical period or over the distance of an optical wavelength. Under these conditions, one can replace Eq. (6) with the reduced wave equation[5,6,9]

$$\left(\frac{\partial}{\partial z} + \frac{\eta}{c}\frac{\partial}{\partial t}\right)\mathcal{E}(z, t)e^{i\varphi(z,t)} = \frac{2\pi N\mu\omega}{\eta c} i\int_{-\infty}^{\infty}[X(z, t, \Delta) - iY(z, t, \Delta)]g(\Delta)\, d\Delta\, e^{i\varphi(z,t)} \quad . \tag{7}$$

The response of an atomic system which has a transition frequency $\Omega = \omega + \Delta$ to an electric field of the form of Eq. (5) is described by

$$\dot{X}(z, t, \Delta) = -[\Delta + \dot{\varphi}(z, t)]Y(z, t, \Delta) - X(z, t, \Delta)/T_2' \, , \tag{8a}$$

$$\dot{Y}(z, t, \Delta) = [\Delta + \dot{\varphi}(z; t)]X(z, t, \Delta) + [\mu\mathcal{E}(z, t)/\hbar]Z(z, t, \Delta) - Y(z, t, \Delta)/T_2' \, , \tag{8b}$$

$$\dot{Z}(z, t, \Delta) = -[\mu\mathcal{E}(z, t,)/\hbar]Y(z, t, \Delta) - [Z(z, t, \Delta) - Z_0]/T_1 \quad . \tag{8c}$$

Relaxation effects have been introduced phenomenologically by means of a longitudinal relaxation time T_1 and by a homogeneous transverse relaxation time T_2'. The description of resonant pulse propagation is now obtained by solving Eq. (7) and Eqs. (8) simultaneously. The boundary conditions

$$X(z, t_0, \Delta) = Y(z, t_0, \Delta) = 0 \, ,$$

$$Z(z, t_0, \Delta) = Z(0) \, ,$$

together with the time dependence of $\mathcal{E}(z, t)$ and $\varphi(z, t)$ at the boundary of the media $z = 0$ are sufficient to describe most experimental situations. The time t_0 is chosen to be immediately before the entrance of the pulse into the medium. The study of solutions of the equations is facilitated if one defines the pulse area according to[1]

$$\theta(z) = \int_{-\infty}^{\infty} \frac{\mu\mathcal{E}(z, t)}{\hbar}\, dt \quad . \tag{9}$$

The pulse area so defined may be interpreted as the angle through which the vector $[X(z, t, 0), Y(z, t, 0), Z(z, t, 0)]$ is turned during the pulse.

Equations (7) and (8) have been studied numerically for both the attenuator case[10] $Z(0)<0$ and the amplifier case[11,12] $Z(0)>0$. Previous studies have emphasized large-area input pulses $[\theta(0)\gtrsim\pi/4]$ since several interesting pulse-shaping effects occur in this region. In this paper, it will be shown that small-area imput pulses $[\theta(0)\lesssim\pi/32]$, which are of short duration compared to the transverse relaxation time T_2, undergo reshaping by the medium and lose significantly less energy than predicted by Beer's law.

SMALL-AREA APPROXIMATION

In Appendix A, it is shown that for field amplitudes $\mathcal{E}(z,t)$ and times t such that

$$\left|\int_{-\infty}^{t}\frac{\mu\mathcal{E}(z,t')}{\hbar}dt'\right|^2\ll 1 \quad , \tag{10}$$

the energy of an atom does not change significantly, and one can replace the variable $Z(z,t,\Delta)$ by $Z(z,t_0,\Delta)=Z(0)$ in Eqs. (8). This "small-area" approximation linearizes the equations, and, as shown in Appendix A, the propagation of the field can then be described by a single equation

$$\left[\frac{\partial}{\partial z}+\frac{\eta}{c}\frac{\partial}{\partial t}\right]\mathcal{E}(z,t)e^{i\varphi(z,t)}$$
$$=-\alpha_0\int_0^\infty G(x)\mathcal{E}(z,t-x)e^{i\varphi(z,t-x)}dx \quad . \tag{11}$$

The quantity $G(x)$ and the constant α_0 are defined according to

$$G(x)\equiv e^{-x/T_2'}\int_{-\infty}^{\infty}g(\Delta)e^{-i\Delta x}d\Delta \tag{12}$$

and $\alpha_0\equiv-(2\pi N\mu^2\omega/\hbar\eta c)Z(0)$. (13)

In the case of a resonant medium consisting of pink ruby and with $Z(0)\approx-1$, α_0 is of the order of 10^{12} $\mathrm{cm^{-1}\,sec^{-1}}$. If one introduces the Fourier transform

$$\epsilon(z,\nu)=\int_{-\infty}^{\infty}\mathcal{E}(z,t)e^{i\varphi(z,t)+i\nu t}dt \quad , \tag{14}$$

Eq. (11) may be rewritten

$$\left(\frac{\partial}{\partial z}-\frac{i\eta}{c}\nu+A(\nu)\right)\epsilon(z,\nu)=0 \quad , \tag{15}$$

where $A(\nu)$ is defined according to

$$A(\nu)=\alpha_0\int_0^\infty G(x)e^{i\nu x}d\nu=\alpha_0\int_{-\infty}^{\infty}\frac{g(\Delta)d\Delta}{1/T_2'+i(\Delta-\nu)} \quad . \tag{16}$$

Equation (15) can be integrated as

$$\epsilon(z,\nu)=\epsilon(0,\nu)\exp[(i\eta/c)\nu z-A(\nu)z] \quad . \tag{17}$$

If one now takes the inverse Fourier transform, the resulting expression for the complex pulse envelope is

$$\mathcal{E}(z,t)e^{i\varphi(z,t)}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\epsilon(0,\nu)$$
$$\times\exp\{-i\nu[t-(\eta/c)z]-A(\nu)z\}d\nu \quad . \tag{18}$$

The expression for $A(\nu)$ given by Eq. (16) takes a particularly simple form when it is assumed that the distribution of atomic frequencies is Lorentzian:

$$g(\Delta)=(T_2^*/\pi)[1+(\Delta T_2^*)^2]^{-1} \quad . \tag{19}$$

In this case, one obtains

$$A(\nu)=\frac{i\alpha_0}{i/T_2+\nu} \quad , \tag{20}$$

where the total transverse relaxation time T_2 is given by $1/T_2=1/T_2'+1/T_2^*$. With this specialization, Eq. (18) becomes

$$\mathcal{E}(z,t)e^{i\varphi(z,t)}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\epsilon(0,\nu)$$
$$\times\exp\left(-i\nu(t-\eta z/c)-\frac{i\alpha_0 z}{i/T_2+\nu}\right)d\nu \quad . \tag{21}$$

Equation (21) describes the evolution of the envelope and phase of a small-area pulse in terms of the Fourier transform of the envelope and phase of the input pulse. The remainder of this paper will be devoted to the study of Eq. (21) for various input pulses. Equation (21) is reminiscent of results from classical dispersion theory.[13] It should be pointed out that the usual formulation of classical dispersion theory results in an expression similar to Eq. (21) but containing $E(z,t)$ which oscillates at optical frequencies, rather than $\mathcal{E}(z,t)\times e^{i\varphi(z,t)}$ which oscillates at frequencies much less than optical frequencies. On the assumption that the host medium is characterized by an index of refraction η and that $\mathcal{E}(z,t)$ and $\varphi(z,t)$ vary little in the time $2\pi/\omega$, the interesting phenomenon of precursors has been neglected. However, it will be shown that there are still other interesting propagational effects which can be described by Eq. (21).

BEER'S LAW LIMIT

Suppose that $\epsilon(0,\nu)$ is negligible for frequencies $|\nu|>1/\tau$. This would be true, for example, in the case of a Gaussian pulse of width τ. Now if τ is large compared with the transverse relaxation time T_2, then

$$\frac{i\alpha_0 z}{i/T_2+\nu}\approx\alpha_0 T_2 z+i\alpha_0 T_2^2\nu z$$

in the region where $\epsilon(0,\nu)$ differs from zero. Substituting this expression into Eq. (21), it is seen that the pulses are exponentially damped in the case $\alpha_0>0$ or exponentially amplified in the case $\alpha_0<0$ according to

$$\mathcal{E}(z,t)\,e^{i\varphi(z,t)}=e^{-\alpha_0 T_2 z}\,\mathcal{E}[0,t-(\eta/c-\alpha_0T_2^2)z]$$
$$\times\,e^{i\varphi[0,t-(\eta/c-\alpha_0T_2^2)z]} \quad . \tag{22}$$

The energy of the pulse is proportional to $\int_{-\infty}^{\infty} \mathcal{E}^2(z,t) \times dt$ and is attenuated or amplified exponentially with a Beer's law coefficient $\alpha = 2\alpha_0 T_2$. In addition, Eq. (22) indicates that a pulse is speeded up in an attenuator ($\alpha_0 > 0$) and delayed in an amplifier ($\alpha_0 < 0$) by an amount $(\alpha_0 T_2{}^2)^{-1}$. In the attenuator case, the apparent speeding up is due to the fact that the resonant dipoles absorb more energy from the trailing half of the pulse than from the leading half. The apparent slowing down of a small-area amplified pulse results because the resonant dipoles contribute more energy to the trailing half of the pulse than to the front.

AREA THEOREM

It is easy to deduce an area theorem from Eq. (21) [or more generally from Eq. (18)] by multiplying by $\mu/\hbar$ and integrating over all time; thus,

$$\theta(z) \equiv \int_{-\infty}^{\infty} \mu\, \mathcal{E}(z,t)\, e^{i\varphi(z,t)}\, dt/\hbar = \theta(0) e^{-\alpha_0 T_2 z} \quad , (23)$$

where Eq. (14) has been used to conclude $\theta(0) = \mu\epsilon(0,0)/\hbar$. This area theorem is quite similar to the low-area limit of the theorem derived by McCall and Hahn[10] with the exception that now the effects of phase $\varphi(z,t)$ and both homogeneous and inhomogeneous broadening are taken into account. The area theorem expressed in Eq. (23) was derived from Eq. (21) without approximation. In the case of an attenuator it requires that the pulse area go to zero exponentially with increasing z. However, this does not necessarily imply that the pulses lose energy exponentially. In the following section it will be shown that, for short pulses, the electric-field envelope oscillates between positive and negative values in just such a way that the area theorem is satisfied, but the pulses lose little energy after an initial reshaping.

SOLUTIONS FOR AN ATTENUATOR

The medium will be referred to as an attenuator if it is prepared so that $\alpha_0 > 0$. In the case where the total area of the pulses is small, the requirement of Eq. (10) may be satisfied for all times t.[14] For other pulses, the small-area approximation will be valid for only limited time t, and hence the analysis of the preceding section can only be applied to a portion of the leading edge of the pulse.

In Appendix B it is shown that the response to an input pulse of the form

$$\mathcal{E}(0,t)\, e^{i\varphi(0,t)} = \mathcal{E}_0 (t/\tau)^k e^{-t/\tau} U(t) \quad , \qquad (24)$$

where $U(t)$ is the unit step function, is given by[15]

$$\mathcal{E}(z,t) = \mathcal{E}_0 U(t-\eta z/c) \exp - \frac{(t-\eta z/c)}{T_2}$$
$$\times \left\{ \frac{[(t-\eta z/c)/\alpha_0 z]^{k/2}}{\tau^k} \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} \left(\frac{(t-\eta z/c)}{\alpha_0 z}\right)^{n/2} \right.$$
$$\left. \times \left(\frac{1}{T_2} - \frac{1}{\tau}\right)^n J_{n+k}\{2[\alpha_0 z(t-\eta z/c)]^{1/2}\} \right\}. \qquad (25)$$

For the case of a pulse whose width is nearly equal to the transverse relaxation time $1/\tau \approx 1/T_2$ just the first term in the series of Eq. (25) is significant, and one concludes that

$$\mathcal{E}(z,t) = k!\exp - \frac{(t-\eta z/c)}{T_2} \left(\frac{(t-\eta z/c)}{\alpha_0 z}\right)^{k/2} \tau^k$$
$$\times U(t-\eta z/c) J_k\{2[\alpha_0 z(t-\eta z/c)]^{1/2}\}. \qquad (26)$$

From the known behavior of the Bessel function, it is seen that such a pulse envelope will oscillate between positive and negative values. These oscillations occur in just such a way that the area goes to zero according to the exact result of Eq. (23). Since the argument of the Bessel function is $[\alpha_0 z(t-\eta z/c)]^{1/2}$, the rapidity of the oscillation increases with increasing $\alpha_0 z$. (See Fig. 1.) It is interesting to note that the zeros of the field envelope depend only on α_0 and hence the prediction of Eq. (26) along with the definition of Eq. (13)

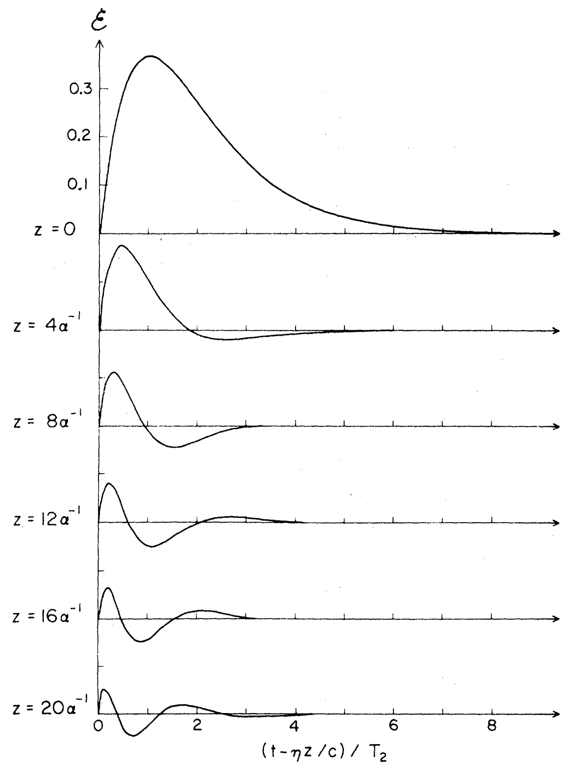

FIG. 1. Propagation of a pulse which is initially described by Eq. (24) with $k=1$. Time is measured in units of T_2 and distance z is measured in Beer's law absorption lengths α^{-1}. The different graphs correspond to different depths in the resonant medium.

suggests a method for measuring the dipole moment matrix element which does not require knowledge of T_2. Similarly, the other solutions presented in this section suggest an experimental determination of α_0.

The energy per cm^2 of a pulse described by Eq. (5) is given by

$$U(z)=\frac{1}{8\pi}\int_{-\infty}^{\infty}\mathcal{E}^2(z,t)dt. \quad (27)$$

The integral in Eq. (27) can be easily evaluated with $\mathcal{E}(z,t)$ given by Eq. (26) with $k=0$. In this case, the energy may be expressed in terms of a modified Bessel function

$$U(z)=U(0)e^{-\alpha_0 T_2 z}\,I_0(\alpha_0 T_2 z). \quad (28)$$

This pulse starts off by decaying exponentially with a decay constant equal to one-half the Beer's law constant. For large $\alpha_0 T_2 z$, the decay is even slower, decreasing as $(\alpha_0 T_2 z)^{-1/2}$.

Another interesting limit can be obtained from Eq. (25); when $k=0$ and $\tau\to\infty$, one obtains

$$\mathcal{E}(z,t)=\mathcal{E}_0 U(t-\eta z/c)\times\exp\left(-\frac{(t-\eta z/c)}{T_2}\right)\sum_{l=0}^{\infty}\left[\frac{1}{T_2}\left(\frac{t-\eta z/c}{\alpha_0 z}\right)^{1/2}\right]^l \times J_l\{2[\alpha_0 z(t-\eta z/c)]^{1/2}\}. \quad (29)$$

It can be seen from Eq. (24) that this limit is the response of the resonant medium to a step function. As indicated above, this small-area solution will be able to describe only the front part of the propagating pulse. Figure 2 displays the result of a computer solution of the full nonlinear problem of solving Eq. (7) and Eqs. (8) for the case of a step-function input pulse with $T_2=\infty$. Here the leading part of the pulse develops positive and negative oscillations which serve to keep the left-hand side of Eq. (10) small, and as a result the small-area solution would be valid for longer times than one would have anticipated. The leading part of the pulse shown in Fig. 2 can be described by the limit of Eq. (29) as $T_2\to\infty$, i.e.,

$$\mathcal{E}(z,t)=\mathcal{E}_0 U(t-\eta z/c)J_0\{2[\alpha_0 z(t-\eta z/c)]^{1/2}\}. \quad (30)$$

The integral of Eq. (21) has been evaluated for the case of a rectangular pulse of the form

$$\mathcal{E}(0,t)e^{i\varphi(0,t)}=\mathcal{E}_0[U(t)-U(t-\tau)]. \quad (31)$$

Such a pulse propagates according to

$$\mathcal{E}(z,q)=\mathcal{E}_0 U(q)e^{-q/T_2}\sum_{l=0}^{\infty}\left[\frac{1}{T_2}\left(\frac{q}{\alpha_0 z}\right)^{1/2}\right]^l \times J_l[2(\alpha_0 zq)^{1/2}]-\mathcal{E}_0 U(q-\tau)e^{-(q-\tau)/T_2}\times\sum_{l=0}^{\infty}\left[\frac{1}{T_2}\left(\frac{q-\tau}{\alpha_0 z}\right)^{1/2}\right]^l J_l\{2[\alpha_0 z(q-\tau)]^{1/2}\}, \quad (32)$$

where $q=(t-\eta z/c)$. As a result of the linearity of Eq. (21), this is the same as the response to two step functions, one of which is delayed by a time τ and differs in phase by 180° with respect to the other.

If one takes the limit as τ approaches zero in such a way that the area of the input pulse $\theta(0)=\mathcal{E}_0\tau$ remains constant, Eq. (31) becomes a delta function. From Eq. (32) the response to the delta function is[16]

$$\mu\mathcal{E}(z,q)/\hbar=\theta(0)\delta(q)-\theta(0)U(q)e^{-q/T_2}\times[\alpha_0 z/q]^{1/2}J_1[2(\alpha_0 zq)^{1/2}]. \quad (33)$$

The area under the first term of Eq. (33) remains $\theta(0)$, while integration reveals that the area under the second term is $-\theta(0)[1-e^{-\alpha_0 T_2 z}]$. Thus the second term makes a negative contribution of just the right magnitude to satisfy the area theorem of Eq. (23).

The integral of Eq. (21) has been evaluated numerically for Gaussian input pulses of the form

$$\mathcal{E}(0,t)e^{i\varphi(0,t)}=\mathcal{E}_0 e^{-4t^2/\tau^2}. \quad (34)$$

Figures 3 and 4 show the results of such a calcula-

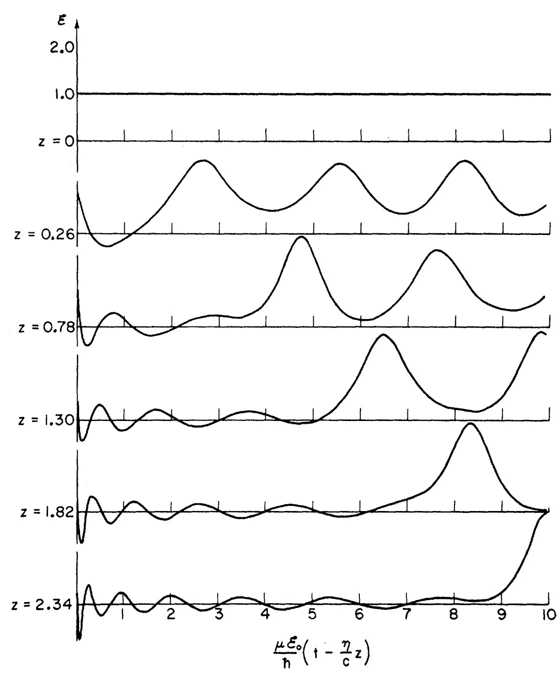

FIG. 2. Propagation of a step-function input pulse in an attenuating medium for the case $T_2=\infty$. The depth z is measured in units of $0.8\mu\mathcal{E}_0/\hbar\alpha_0$ and the field $\mathcal{E}$ is measured in units of $\mathcal{E}_0$.

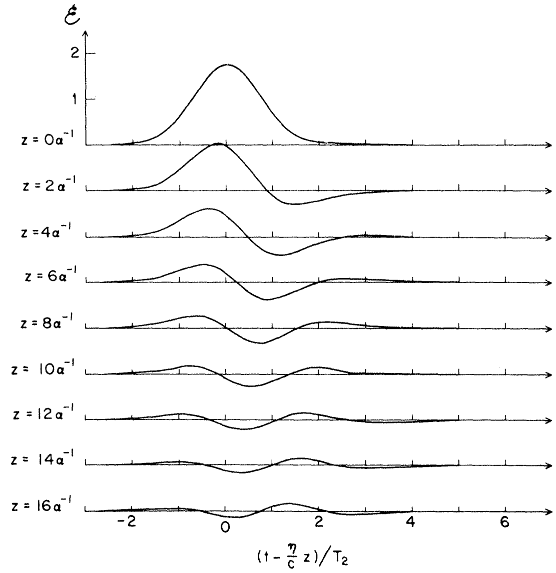

FIG. 3. Propagation of a Gaussian pulse of width $\tau = 2T_2$. The depth z is measured in Beer's law absorption lengths α^{-1}.

tion for pulses of width comparable to T_2. It is seen that the pulse envelope develops a negative component and propagates many absorption lengths without vanishing. The energy of the pulses was calculated from Eq. (27), and the results are displayed in Fig. 5. There are large deviations from

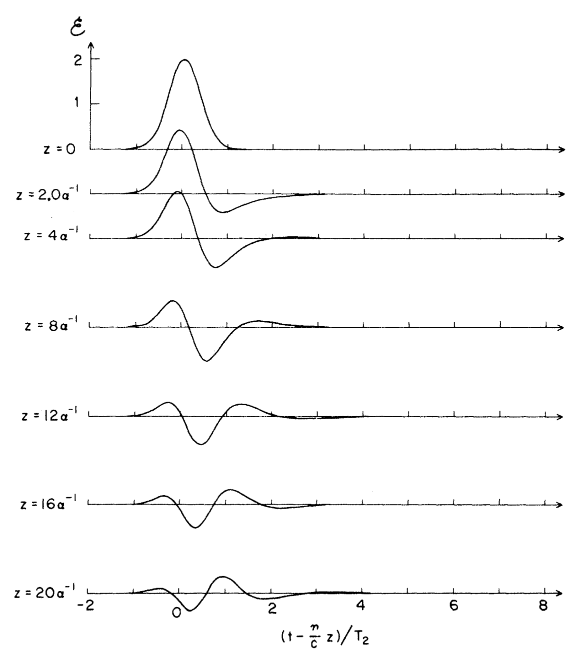

FIG. 4. Propagation of a Gaussian pulse of width $\tau = T_2$.

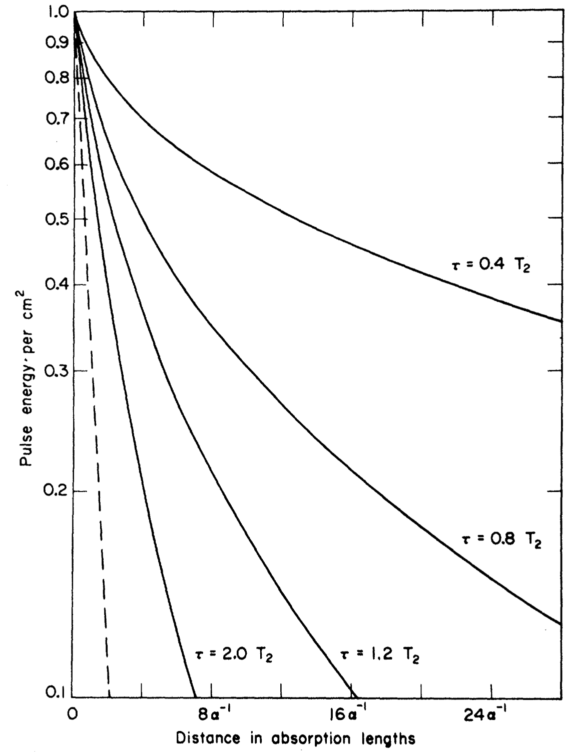

FIG. 5. Variation of pulse energy per cm^2 with distance for Gaussian pulses of various widths.

Beer's law, which is indicated by the dashed line, for pulses of width comparable to or less than T_2.

SOLUTIONS FOR AN AMPLIFIER

If $\alpha_0 < 0$, the previous solutions are still valid provided one uses the identity

$$J_l(2[-|\alpha_0| z(t-\eta z/c)]^{1/2}) = (i)^l I_l(2[|\alpha_0| z(t-\eta z/c)]^{1/2}).$$

For example, the solution of Eq. (26) would become

$$\mathcal{E}(z,t) = k!\,\mathcal{E}_0 e^{-(t-\eta z/c)/T_2}\, U(t-\eta z/c) \times \left(\frac{(t-\eta z/c)}{|\alpha_0| z}\right)^{k/2} \tau^{-k} I_k\{2[|\alpha_0| z(t-\eta z/c)]^{1/2}\} \tag{35}$$

in the case of an amplifying medium. The amplification of a very short pulse would be described by

$$\mu\mathcal{E}(z,q)/\hbar = \theta(0)\delta(q) + \theta(0)\,U(q)\,e^{-q/T_2}[|\alpha_0| z/q]^{1/2} \times I_1[2(|\alpha_0| zq)^{1/2}]\,, \tag{36}$$

as derived from Eq. (33).

A study of the amplification of a short small-area Gaussian pulse of the form given in Eq. (34)

was made using Eq. (21) with $\alpha_0 = -|\alpha_0|$, and the result is displayed in Fig. 6. The short pulse is broadened because the macroscopic polarization radiates more energy into the tail of the pulse than into the front.

The study of amplification of pulses using the small-area approximation is of limited usefulness because the amplified pulses soon violate the condition of Eq. (10).

ACKNOWLEDGMENT

I would like to thank Professor S. R. Hartmann for several discussions which stimulated my interest in the problem.

APPENDIX A

Adding Eq. (8a) to $-i$ times Eq. (8b), one obtains

$$\mathring{X} - i\mathring{Y} = -\left(\frac{1}{T_2'} + i(\Delta + \dot{\varphi})\right)(X - iY) - i\frac{\mu\mathcal{E}}{\hbar} Z \,. \quad \text{(A1)}$$

This equation can be rewritten as an integral equation as

$$[X(z,t,\Delta) - iY(z,t,\Delta)]e^{i\varphi(z,t)} = -i\int_{-\infty}^{t} \frac{\mu\mathcal{E}(z,t')}{\hbar} e^{i\varphi(z,t')} Z(z,t',\Delta) \times \exp\left[-\left(\frac{1}{T_2'} + i\Delta\right)(t-t')\right]dt' \,. \quad \text{(A2)}$$

Substituting this into Eq. (7), one obtains

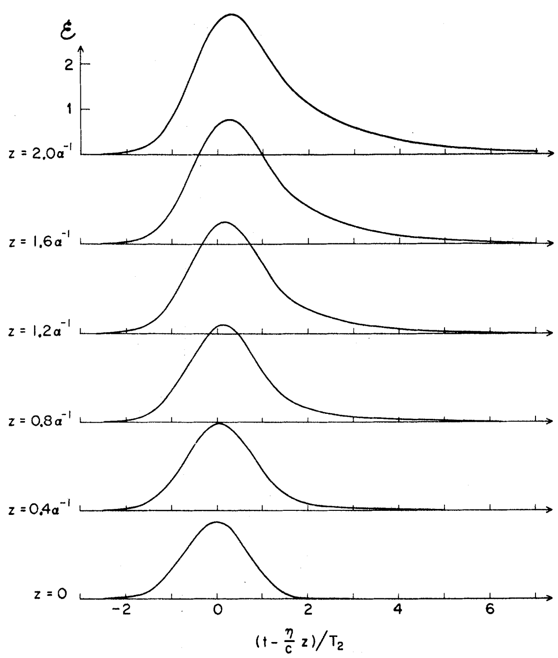

FIG. 6. Amplification of a small-area Gaussian pulse whose initial width is $\tau = 2T_2$.

$$\left(\frac{\partial}{\partial z} + \frac{\eta}{c}\frac{\partial}{\partial t}\right)\mathcal{E}(z,t)e^{i\varphi(z,t)} = \frac{2\pi N\mu^2\omega}{\hbar\eta c}\int_{-\infty}^{t}\mathcal{E}(z,t')e^{i\varphi(z,t')}e^{-(t-t')/T_2'} \times \int_{-\infty}^{\infty} g(\Delta)e^{-i\Delta(t-t')}Z(z,t',\Delta)\,d\Delta\,dt' \,. \quad \text{(A3)}$$

Now change to the variable $x = t - t'$ in Eq. (A3). The result is

$$\left(\frac{\partial}{\partial z} + \frac{\eta}{c}\frac{\partial}{\partial t}\right)\mathcal{E}(z,t)e^{i\varphi(z,t)} = \frac{2\pi N\mu^2\omega}{\hbar\eta c}\int_{0}^{\infty}\mathcal{E}(z,t-x)e^{i\varphi(z,t-x)} \times e^{-x/T_2'}\int_{-\infty}^{\infty} g(\Delta)e^{-i\Delta x}Z(z,t-x,\Delta)\,d\Delta\,dx. \quad \text{(A4)}$$

In a similar manner, Eq. (8c) can be converted into an integral equation[17]

$$Z(z,t,\Delta) = Z(0) - \int_{-\infty}^{t}\frac{\mu\mathcal{E}(z,t')}{\hbar} \times Y(z,t',\Delta)\exp[-(t-t')/T_1]\,dt' \,. \quad \text{(A5)}$$

Since $|Z(z,t,\Delta)| \leq 1$, it follows from Eqs. (A2) and (A5) that

$$|Z(z,t,\Delta) - Z(0)| \leq \left|\int_{-\infty}^{t}\frac{\mu\mathcal{E}(z,t')}{\hbar}dt'\right|^2. \quad \text{(A6)}$$

Thus, when the right-hand side of Eq. (A6) is small, it is a good approximation to replace $Z(z,t,\Delta)$ by $Z(0)$ in Eq. (A4), and the resulting expression appears in Eq. (11).

APPENDIX B

An example of how the integral in Eq. (21) may be evaluated for a specific pulse shape will be presented here. Consider an input pulse of the form

$$\mathcal{E}(0,t)e^{i\varphi(0,t)} = \mathcal{E}_0(t/\tau)^k e^{-t/\tau}U(t) \,, \quad \text{(B1)}$$

where $U(t)$ is the unit step function and k is zero or a positive integer. The Fourier transform of this input pulse shape is

$$\epsilon(0,\nu) = [\epsilon_0 k!\, i^{k+1}/\tau^k(\nu + i/\tau)^{k+1}] \quad \text{(B2)}$$

in the notation of Eq. (14). Substitution of Eq. (B2) into Eq. (21) shows that the evolution of the pulse is described by

$$\mathcal{E}(z,q)e^{i\varphi(z,q)} = \frac{\mathcal{E}_0 k!\, i^{k+1}}{2\pi\,\tau^k} \times \int_{-\infty}^{\infty}\frac{\exp[-i\nu q - i\alpha_0 z/(i/T_2 + \nu)]}{(\nu + i/\tau)^{k+1}}d\nu \,, \quad \text{(B3)}$$

where $q=(t-\eta z/c)$.

The integral has two singularities in the lower half of the ν plane. If $q<0$, the contour of integration may be closed in the upper half-plane and the integral is zero. If $q>0$, the contour may be closed in the lower half-plane and deformed into a circle centered at $\nu=-i/T_2$ and having a radius R large enough to include the singularity at $-i/\tau$. Thus, the integral Eq. (B3) becomes

$$\mathcal{E}(z,q)\,e^{i\varphi(z,q)}=\frac{\mathcal{E}_0 e^{-q/T_2}\, i^{k+1}U(q)k!}{2\pi\tau^k}\times\oint_{|x|=R}\frac{\exp[-ixq-i\alpha_0 z/x]}{[i(1/\tau-1/T_2)+x]^{k+1}}\,dx\,, \tag{B4}$$

in terms of the variable $x=i/T_2+\nu$. The fact that $|x|>|1/\tau-1/T_2|$ on the contour used in Eq. (B4) allows one to expand

$$[x+i(1/\tau-1/T_2)]^{-k-1}=\sum_{n=0}^{\infty}\frac{(n+k)!}{n!\,k!}\;\frac{[i(1/T_2-1/\tau)]^n}{x^{n+k+1}}\;. \tag{B5}$$

Furthermore, the exponential term in the integral of Eq. (B4) may be written in terms of the generating function for the Bessel function as

$$\exp[-ixq-i\alpha_0 z/x]=\sum_{l=-\infty}^{\infty}\left[-i\left(\frac{q}{\alpha_0 z}\right)^{1/2}x\right]^l J_l[2(\alpha_0 zq)^{1/2}]\,. \tag{B6}$$

Substituting Eqs. (B5) and (B6) into (B4) and using the fact that

$$\oint_{|x|=R} x^{l-n-k-1}dx=-2\pi i\delta_{l,n+k}\;,$$

one obtains

$$\mathcal{E}(z,q)e^{i\varphi(z,q)}=\mathcal{E}_0 e^{-q/T_2}U(q)\times\frac{(q/\alpha_0 z)^{k/2}}{\tau^k}\sum_{n=0}^{\infty}\frac{(n+k)!}{n!}\,(q/\alpha_0 z)^{n/2}\times(1/T_2-1/\tau)^n J_{n+k}[2(\alpha_0 zq)^{1/2}]\,. \tag{B7}$$

---

*Work supported wholly by the Joint Services Electronics Program (U.S. Army, Navy, and Air Force), under Contracts DA-28-043 AMC-00099(E) and DAAB-07-69-C-0383.

†Present address: Fundamental Research, Owens-Illinois Technical Center, Toledo, Ohio 43601.

[1]S. L. McCall and E. L. Hahn, Phys. Rev. Letters 18, 908 (1967).

[2]G. B. Hocker and C. L. Tang, Phys. Rev. Letters 21, 591 (1968).

[3]C. K. Rhodes and A. Szöke, Phys. Rev. 184, 25 (1969).

[4]A. Beer, Ann. Chem. Phys. 86, 78 (1852).

[5]F. T. Arrechi and R. Bonifacio, IEEE J. Quant. Electron. QE-1, 169 (1965).

[6]C. L. Tang and B. D. Silverman, *Physics of Quantum Electronics*, edited by P. L. Kelley, B. Lax, and P. E. Tannenwald (McGraw-Hill, New York, 1966), p. 280.

[7]R. Feynman, F. Vernon, and R. Hellwarth, J. Appl. Phys. 28, 49 (1957).

[8]Notation used in this article was introduced in M. D. Crisp, Phys. Rev. Letters 22, 820 (1969).

[9]M. D. Crisp, Opt. Commun. 1, 59 (1969).

[10]S. L. McCall and E. L. Hahn, Phys. Rev. 183, 457 (1969).

[11]F. A. Hopf and M. O. Scully, Phys. Rev. 179, 399 (1969).

[12]A. Icsevgi and W. E. Lamb, Jr., Phys. Rev. 185, 517 (1969).

[13]L. Brillouin, *Wave Propagation and Group Velocity*, (Academic, New York, 1960).

[14]It is possible that a pulse may have an oscillating envelope so that Eq. (10) is violated for a time t_1 but satisfied for a later time t_2. Of course, the small-area analysis presented above cannot be applied to such a pulse.

[15]If there is no phase variation in the input pulse $\varphi(0,t)=0$ then, according to Eq. (21), there will not be any phase variation in the output pulse.

[16]D. C. Burnham and R. Y. Chiao, [Phys. Rev. 188, 667 (1969)] predict the development of negative oscillations of the field envelope for a thin sample of resonant medium with all the resonant atoms having the same frequency ($T_2=\infty$). These predictions follow from Eq. (32) and Eq. (33) when one takes the limit as T_2 goes to infinity.

[17]If $Z(0)\neq Z_0$, the derivation of Eq. (A5) requires the additional assumption that T_1 is long compared to the time it takes the pulse to traverse the medium.