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Dispersion of Phonons in Liquid He^4 †

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The long-wavelength dispersion of the phonon spectrum in liquid He^4 can be computed from first principles. Part of the dispersion coefficient enters through the Feynman spectrum $\epsilon_0(k)$, and the other part enters through corrections to $\epsilon_0(k)$. The latter has been investigated by Jackson, Feenberg, and Lee using a perturbation expansion in the free-phonon basis. We define here a more plausible ordering scheme for the expansion and carry out the computation to second order.

I. INTRODUCTION

Departure of the phonon spectrum $\epsilon(k)$ in liquid He^4 from the linear dependence $\hbar ck$ is of importance for phonon-phonon interactions.[1] At small k, this departure can be characterized by a dispersion coefficient γ defined by

$$\epsilon(k)=\hbar ck(1-\gamma k^2) \quad . \tag{1}$$

Due to the smallness of the effect, γ as yet cannot be determined from experimental data. A recent theoretical calculation[2] derives γ from the hydrodynamic Hamiltonian; however it involves a cutoff momentum which lies beyond the hydrodynamic limit.

A perturbative procedure developed by Jackson and Feenberg[3] employs as basis functions the set of "free-phonon" wave functions: density fluctuations operating on the ground-state eigenfunction. Using matrix elements interpreted as phonon-splitting or coalescing processes, they calculated the second-order correction to the Feynman spectrum $\epsilon_0(k)$:

$$\epsilon_0(k)=\hbar^2k^2/2mS(k) \quad , \tag{2}$$

where $S(k)$ is the liquid-structure function at $T=0$. Their results are consistent with those of Feynman and Cohen, and with experiment. Lee[4] extended the calculation to include some third-order terms, obtaining numerically the departure of the spectrum from $\epsilon_0(k)$ at long wavelengths. It is the purpose of this paper to define a more plausible ordering scheme for the perturbative procedure and to complete the Jackson-Feenberg-Lee calculation under the new scheme.

First we construct the free phonon functions

$$|\vec{k})=\rho_{\vec{k}}\Psi_0/[S(k)]^{1/2} \quad ,$$
$$|\vec{k}-\vec{h},\vec{h})=\rho_{\vec{k}-\vec{h}}\rho_{\vec{h}}\Psi_0/[S(|\vec{k}-\vec{h}|)S(h)]^{1/2} \quad ,$$
$$|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}')=\rho_{\vec{k}-\vec{h}-\vec{h}'}\rho_{\vec{h}}\rho_{\vec{h}'}\Psi_0 \times [S(|\vec{k}-\vec{h}-\vec{h}'|)S(h)S(h')]^{-1/2}, \text{ etc.}, \tag{3}$$

where Ψ_0 is the ground-state eigenfunction of liquid He^4, and

$$\rho_{\vec{k}}=(1/\sqrt{N})\sum_m e^{i\vec{k}\cdot\vec{r}_m} \quad ,$$
$$S(k)=1+\rho\int e^{i\vec{k}\cdot\vec{r}}[g(r)-1]\,d\vec{r} \quad , \tag{4}$$
$$g(r_{12})=\frac{N(N-1)}{\rho^2}\int\Psi_0^2\,d\vec{r}_{34\cdots N} \quad .$$

These functions are normalized[5] but not orthogonal. A convenient partial orthogonalization can be achieved by applying the Schmidt procedure to functions containing different numbers of free phonons. Thus,

$$|\vec{k}\rangle=|\vec{k}) \quad ,$$
$$|\vec{k}-\vec{h},\vec{h}\rangle=[|\vec{k}-\vec{h},\vec{h})-\sum_{\vec{l}}|\vec{l}\rangle\langle\vec{l}|\vec{k}-\vec{h},\vec{h}]/I \quad ,$$
$$|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}'\rangle=[|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}') -\sum_{\vec{l}}|\vec{l}\rangle\langle\vec{l}|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}')$$

$$-\sum_{\vec{l}\vec{m}}{}^{'}|\vec{l},\vec{m}\rangle\langle\vec{l},\vec{m}|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}')]/I', \text{ etc.}, \tag{5}$$

where the prime on the double summation reminds us to avoid double counting, and the denominators I and I' are normalization factors each differing from unity by a term of order $1/N$. It is in the set of basis functions (5) that we compute all matrix elements of the unity operator and the Hamiltonian, the latter being given by

$$H=\sum_{i=1}^{N}\frac{-\hbar^2}{2m}\nabla_i^2+\sum_{i<j=1}^{N}v(r_{ij}) \quad . \tag{6}$$

The matrix elements relevant to our present calculation are schematically represented in Fig. 1. The computation of these matrix elements is straightforward, though tedious, if proper transformations are made to take advantage of the fact that Ψ_0 satisfies the Schrödinger equation

$$H\Psi_0=E_0\Psi_0 \quad . \tag{7}$$

Thus the two-particle potential $v(r)$ becomes completely absorbed into the liquid-structure function $S(k)$, which is directly related to the two-particle distribution function. Where three- and four-particle distribution functions appear, one may employ the convolution approximation which expresses them as functionals of the two-particle distribution function. We shall mention briefly in Sec. II the computation of these matrix elements and refer to Ref. 4, where matrix elements of types (a) and (e) were obtained.

Using the nondiagonal matrix elements as perturbations, we can carry out a perturbation expansion for the energy $E_{\vec{k}}$ in the one-phonon state, taking $k \ll \rho^{1/3}$. Since the unperturbed energy is given by the diagonal matrix element

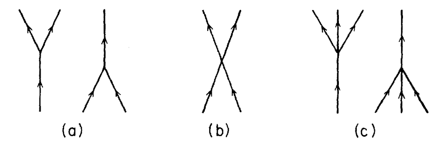

FIG. 1. Schematic representation of matrix elements calculated.

$$\langle\vec{k}|H|\vec{k}\rangle=E_0+\epsilon_0(k)\equiv E_{\vec{k}}^{(0)} \quad , \tag{8}$$

all terms in the perturbation series give rise to corrections to the Feynman spectrum $\epsilon_0(k)$. Here we call attention to the fact that the basis is not completely orthogonal, therefore each vertex actually contains the nondiagonal element of H as well as of 1, in the combination

$$\langle\vec{l},\ldots|H|\vec{m},\ldots\rangle-E_{\vec{k}}^{(0)}\langle\vec{l},\ldots|\vec{m},\ldots\rangle \equiv\langle\vec{l},\ldots|H'|\vec{m},\ldots\rangle \quad . \tag{9}$$

II. MATRIX ELEMENTS OF H'

The matrix elements of H' may be computed with the a d of Eqs. (9) and (5). We first express them in terms of the free-phonon functions, Eq. (3). To order $1/N$, we find[6]

$$\langle\vec{k}|\vec{k}-\vec{h},\vec{h}\rangle=0 \quad , \tag{10}$$

$$\langle\vec{k}|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}'\rangle=0 \quad , \tag{11}$$

$$\langle\vec{k}-\vec{h},\vec{h}|\vec{k}-\vec{h}',\vec{h}'\rangle=(\vec{k}-\vec{h},\vec{h}|\vec{k}-\vec{h}',\vec{h}')-(\vec{k}-\vec{h},\vec{h}|\vec{k})(\vec{k}|\vec{k}-\vec{h}',\vec{h}'), \text{ etc.}, \tag{12}$$

$$\langle\vec{k}|H|\vec{k}-\vec{h},\vec{h}\rangle=(\vec{k}|H|\vec{k}-\vec{h},\vec{h})-E_{\vec{k}}^{(0)}(\vec{k}|\vec{k}-\vec{h},\vec{h}), \tag{13}$$

$$\begin{aligned}\langle\vec{k}|H|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}'\rangle=&(\vec{k}|H|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}')-E_{\vec{k}}^{(0)}(\vec{k}|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}')-[(\vec{k}|H|\vec{k}-\vec{h},\vec{h})-E_{\vec{k}}^{(0)}(\vec{k}|\vec{k}-\vec{h},\vec{h})]\\&\times[(\vec{k}-\vec{h},\vec{h}|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}')-(\vec{k}-\vec{h},\vec{h}|\vec{k})(\vec{k}|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}')]\\&-[(\vec{k}|H|\vec{k}-\vec{h}',\vec{h}')-E_{\vec{k}}^{(0)}(\vec{k}|\vec{k}-\vec{h}',\vec{h}')][(\vec{k}-\vec{h}',\vec{h}'|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}')]-[(\vec{k}|H|\vec{k}-\vec{h}-\vec{h}',\vec{h}+\vec{h}')\\&-E_{\vec{k}}^{(0)}(\vec{k}|\vec{k}-\vec{h}-\vec{h}',\vec{h}+\vec{h}')][(\vec{k}-\vec{h}-\vec{h}',\vec{h}+\vec{h}'|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}')]-[(\vec{k}-\vec{h}-\vec{h}',\vec{h}+\vec{h}'|\vec{k})\\&\times(\vec{k}|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}')],\end{aligned} \tag{14}$$

$$\begin{aligned}\langle\vec{k}-\vec{h},\vec{h}|H|\vec{k}-\vec{h}',\vec{h}'\rangle=&(\vec{k}-\vec{h},\vec{h}|H|\vec{k}-\vec{h}',\vec{h}')-(\vec{k}-\vec{h},\vec{h}|H|\vec{k})\\&\times(\vec{k}|\vec{k}-\vec{h}',\vec{h}')-(\vec{k}-\vec{h},\vec{h}|\vec{k})(\vec{k}|H|\vec{k}-\vec{h}',\vec{h}')+E_{\vec{k}}^{(0)}(\vec{k}-\vec{h},\vec{h}|\vec{k})(\vec{k}|\vec{k}-\vec{h}',\vec{h}') \quad , \text{ etc.}\end{aligned} \tag{15}$$

Using the convolution approximation for three-, four-, and five-particle distribution functions, we find that the matrix elements on the right-hand sides of Eqs. (12)–(15) are given as

$$(\vec{k}|\vec{k}-\vec{h},\vec{h})=[N^{-1}S(k)S(|\vec{k}-\vec{h}|)S(h)]^{1/2} \quad , \tag{16}$$

$$(\vec{k}|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}')=N^{-1}[S(|\vec{k}-\vec{h}-\vec{h}'|)S(h)S(h')S(k)]^{1/2}[-2+S(|\vec{h}+\vec{h}'|)+S(|\vec{k}-\vec{h}|)+S(|\vec{k}-\vec{h}'|)], \quad (17)$$

$$(\vec{k}-\vec{h},\vec{h}|\vec{k}-\vec{h}',\vec{h}')=N^{-1}[S(|\vec{k}-\vec{h}|)S(h)\ S(|\vec{k}-\vec{h}'|)S(h')]^{1/2}\ [-2+S(k)+S(|\vec{h}-\vec{h}'|)+S(|\vec{k}-\vec{h}-\vec{h}'|)]\ , \quad (18)$$

$$(\vec{k}-\vec{h},\vec{h}|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}')=(\vec{k}-\vec{h}|\vec{k}-\vec{h}-\vec{h}',\vec{h}'),\ \text{etc.}, \quad (19)$$

$$(\vec{k}|H|\vec{k}-\vec{h},\vec{h})=E_{\vec{k}}^{(0)}(\vec{k}|\vec{k}-\vec{h},\vec{h})+(\hbar^2/2m)\,[NS(k)S(|\vec{k}-\vec{h}|)S(h)]^{-1/2}$$
$$[\vec{k}\cdot(\vec{k}-\vec{h})S(h)+\vec{k}\cdot\vec{h}S(|\vec{k}-\vec{h}|)\ -k^2S(|\vec{k}-\vec{h}|)S(h)]\ , \quad (20)$$

$$(\vec{k}|H|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}')=E_{\vec{k}}^{(0)}(\vec{k}|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}')+(\hbar^2/2mN)[S(|\vec{k}-\vec{h}-\vec{h}'|)\,S(h)S(h')S(k)]^{-1/2}$$
$$\times\{\vec{k}\cdot(\vec{k}-\vec{h}-\vec{h}')S(h)S(h')S(|\vec{h}+\vec{h}'|)+(\vec{k}\cdot\vec{h})S(|\vec{k}-\vec{h}-\vec{h}'|)S(|\vec{k}-\vec{h}|)\ S(h')+(\vec{k}\cdot\vec{h}')S(|\vec{k}-\vec{h}-\vec{h}'|)$$
$$\times S(|\vec{k}-\vec{h}'|)S(h)-[k^2/S(k)]\ S(|\vec{k}-\vec{h}-\vec{h}'|)S(h)S(h')S(k)[-2+S(|\vec{h}+\vec{h}'|)+S(|\vec{k}-\vec{h}'|)+S(|\vec{k}-\vec{h}|)]\}\ , \quad (21)$$

$$(\vec{k}-\vec{h},\vec{h}|H|\vec{k}-\vec{h}',\vec{h}')=E_{\vec{k}}^{(0)}(\vec{k}-\vec{h},\vec{h}|\vec{k}-\vec{h}',\vec{h}')+(\hbar^2/2mN)[S(|\vec{k}-\vec{h}|)S(h)S(|\vec{k}-\vec{h}'|)S(h')]^{-1/2}$$
$$\times\{(\vec{k}-\vec{h})\cdot(\vec{k}-\vec{h}')S(h)S(h')S(|\vec{h}-\vec{h}'|)+\vec{h}\cdot\vec{h}'S(|\vec{k}-\vec{h}|)S(|\vec{k}-\vec{h}'|)S(|\vec{h}-\vec{h}'|)+\vec{h}\cdot(\vec{k}-\vec{h}')S(|\vec{k}-\vec{h}|)S(h')$$
$$\times S(|\vec{k}-\vec{h}-\vec{h}'|)+\vec{h}'\cdot(\vec{k}-\vec{h})S(h)S(|\vec{k}-\vec{h}'|)S(|\vec{k}-\vec{h}-\vec{h}'|)-(k^2/S(k))S(|\vec{k}-\vec{h}|)S(h)S(|\vec{k}-\vec{h}'|)$$
$$\times S(h')[-2+S(k)+S(|\vec{h}-\vec{h}'|)+S(|\vec{k}-\vec{h}-\vec{h}'|)]\},\ \text{etc.} \quad (22)$$

Equations (16), (18), (20), and (22) have appeared previously in Ref. 4. Equations (17) and (21) are evaluated in an analogous manner. Equation (19) requires the knowledge of the five-particle distribution function, and is evaluated with the aid of the convolution approximation. Details are given in the Appendix. Substituting Eqs. (16)–(22) into Eqs. (13)–(15), we find in the limit of small $k(k \ll \rho^{1/3})$:

$$\langle\vec{k}|H|\vec{k}-\vec{h},\vec{h}\rangle=(-\hbar^2/2m)[NS(k)S^2(h)]^{-1/2}$$
$$\times\{k^2[S(h)-1]S(h)+[(\vec{k}\cdot\vec{h})^2/h]\,dS(h)/dh\}, \quad (23)$$

$$\langle\vec{h}|H|\vec{h}'-\vec{h},\vec{h}\rangle=(-\hbar^2/2m)[NS(h)S(h')S(|\vec{h}'-\vec{h}|)]^{-1/2}$$
$$\times\{h'^2[S(|\vec{h}'-\vec{h}|)-1]S(h) \quad (24)$$
$$+\vec{h}'\cdot\vec{h}[S(h)-S(|\vec{h}'-\vec{h}|)]\}\ ,$$

$$\langle\vec{k}|H|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}'\rangle=0\ , \quad (25)$$

$$\langle\vec{k}-\vec{h}',\vec{h}'|H|\vec{k}-\vec{h},\vec{h}\rangle=(\hbar^2/mN)\vec{h}\cdot\vec{h}'$$
$$\times[S(|\vec{h}-\vec{h}'|)-S(|\vec{h}+\vec{h}'|)]\ . \quad (26)$$

In these results we display only terms which are consequential in the perturbative calculation that follows. In particular, Eq. (25) does not indicate that $\langle\vec{k}|H|\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}'\rangle$ vanishes identically, it simply means that the perturbative correction to the phonon spectrum which arises from this matrix element is of an order higher than what we intend to retain.

III. PERTURBATIVE CORRECTIONS TO PHONON SPECTRUM

The knowledge of the matrix elements of H' permits us to evaluate perturbative corrections to the phonon spectrum. We define a coefficient γ' so that the perturbed phonon energy $\epsilon(k)$ may be expressed as

$$\epsilon(k)=\epsilon_0(k)[1+\gamma' k^2],\quad k\ll\rho^{1/3}\ . \quad (27)$$

Figure 2 shows schematically the correction terms we seek. Each diagram makes a contribution toward γ'. Even though we begin by writing down a Brillouin-Wigner perturbative expansion, it is clear that for small k we may replace $\epsilon(k)$ in the energy denominators by $\epsilon_0(k)$ without affecting the value of γ'. In essence we employ a Rayleigh-Schrödinger expansion. With the aid of Figs. 2 (a)–(g) and the usual diagrammatic prescription, we obtain[7] the following expressions, retaining only terms of order k^3 in a power series in k,

$$\frac{\epsilon_a(k)}{k^2\epsilon_0(k)}=\frac{-1}{8\pi^2\rho}\int_0^\infty\frac{dh}{S(h)}\left(S^2(h)[S(h)-1]^2\right.$$
$$\left.+\tfrac{2}{3}hS(h)[S(h)-1]S'(h)+\tfrac{1}{5}h^2[S'(h)]^2\right)\equiv\gamma'_a\ , \quad (28)$$

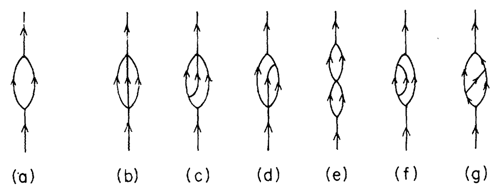

FIG. 2. Schematic representation of perturbative corrections to $\epsilon_0(k)$.

$$\frac{\epsilon_b(k)}{k^2\epsilon_0(k)} = \frac{\epsilon_c(k)}{k^2\epsilon_0(k)} = \frac{\epsilon_d(k)}{k^2\epsilon_0(k)} = 0 \ , \quad (29)$$

$$\frac{\epsilon_e(k)}{k^2\epsilon_0(k)} = \frac{1}{4(2\pi)^6\rho^2}\int\int d\vec{h}\,d\vec{h}' \times\{S(|\vec{h}-\vec{h}'|)[S(h)-1]\,[S(h')-1] \times S(h)S(h')\frac{\vec{h}\cdot\vec{h}'}{(hh')^2} + \frac{2\vec{h}\cdot\vec{h}'}{3h^2h'} \times S(|\vec{h}-\vec{h}'|)[S(h)-1]S(h)S'(h') + \tfrac{1}{15}\,[2\left(\frac{\vec{h}\cdot\vec{h}'}{hh'}\right)^3 + \frac{\vec{h}\cdot\vec{h}'}{hh'}] \times S(|\vec{h}-\vec{h}'|)S'(h)S'(h')\} \equiv \gamma_e' \ , \quad (30)$$

$$\frac{\epsilon_f(k)}{k^2\epsilon_0(k)} = \frac{-1}{8(2\pi)^6\rho^2}\int\int d\vec{h}\,d\vec{h}'\,\epsilon_0(h') \times\{h'^2S(h)[S(|\vec{h}-\vec{h}'|)-1] + \vec{h}\cdot\vec{h}' \times[S(h)-S(|\vec{h}-\vec{h}'|)]\}^2\{S^2(h')[S(h')-1]^2 + \tfrac{1}{5}h'^2[S'(h')]^2 + \tfrac{2}{3}h'\,[S(h')-1]S(h')S'(h')\} \times\{[\epsilon_0(h')+\epsilon_0(h)+\epsilon_0(|\vec{h}-\vec{h}'|)] \times h'^6S(h)S(|\vec{h}-\vec{h}'|)\}^{-1} \equiv \gamma_f' \ , \quad (31)$$

$$\frac{\epsilon_g(k)}{k^2\epsilon_0(k)} = \frac{-1}{4(2\pi)^6\rho^2}\int\int d\vec{h}\,d\vec{h}'\,\epsilon_0(h) \times\{[S(h')-1][S(h)-1]S(h')S(h) + \tfrac{1}{3}h'\,[S(h)-1]S(h)S'(h') + \tfrac{1}{3}h[S(h')-1]S(h')S'(h) + \frac{1}{15hh'} \times[2(\vec{h}\cdot\vec{h}')^2 + h^2h'^2]S'(h)S'(h')\} \times\{h'^2S(h)[S|\vec{h}-\vec{h}'|)-1] + \vec{h}\cdot\vec{h}'[S(h)-S(|\vec{h}-\vec{h}'|)]\} \times\{h^2S(h')[S(|\vec{h}-\vec{h}'|)-1] + \vec{h}\cdot\vec{h}'[S(h')-S(|\vec{h}-\vec{h}'|)]\} \times\{[\epsilon_0(h')+\epsilon_0(h)+\epsilon_0(|\vec{h}-\vec{h}'|)] \times h^4h'^2S(h')S(|\vec{h}-\vec{h}'|)\}^{-1} \equiv \gamma_g' \ . \quad (32)$$

Finally, $\gamma' = \gamma_a' + \gamma_b' + \cdots + \gamma_g' + \cdots$. (33)

A careful analysis of the matrix elements attests to the obvious conclusion that the increasingly more complicated vertices rapidly diminish in strength. There is no reason to expect the usual scheme of counting vertices to provide proper ordering of the correction series, and thus the proper grouping of the γ' contributions. The scheme which we adopt counts the number of rings formed by free-phonon lines[8]; thus, Fig. 2(a) corresponds to the only first-order contribution, and Figs. 2(b)–(g) constitute second-order contributions. Such a scheme proved successful in the study of collective excitations in a high-density charged Bose gas,[9] and will be accepted here on grounds to be given presently. Under this scheme, the Jackson-Feenberg calculation[3] is complete to first order, whereas Lee[4] selectively evaluated (e), one of six second-order diagrams. Alternately we could say that our diagrams are classified by the number of internal free-phonon momenta to be integrated over. A plausible and/or convincing argument begins with associating a smallness parameter α with the volume of the region in $\vec{k}$ space in which $[S(h)-1]$ and $S'(h) \equiv dS(h)/dh$ differ appreciably from zero: Both are short-range functions contributing significantly only when the argument is small. One must take care not to overcount. For example, the expression $[S(h)-1] \times[S(h')-1][S(|\vec{h}-\vec{h}'|)-1]$ should be regarded as of order α^2 and not α^3. Exactly the same line of reasoning was used in connection with a previous calculation[10] involving matrix elements for liquid He^3. In the microscopic theory of liquid helium, there is no transparent way of defining a numerical expansion parameter. We must devise from experience a plausible scheme and leave the final justification to numerical computation. Under the present scheme, then, the contributions to γ' should be grouped thus:

$$\gamma' = \gamma_a' + (\gamma_b' + \cdots + \gamma_g') + \cdots \ . \quad (34)$$

What we obtain in this calculation is the correction to $\epsilon_0(k)$, not to $\hbar ck$. The prescription for calculating the more interesting coefficient γ defined in Eq. (1) requires a determination of how $S(k)$ deviates from $\hbar k/2mc$ at small k. Conforming to the popular practice, we guess

$$S(k) = (\hbar k/2mc)(1+\gamma''k^2), \quad k \ll \rho^{1/3} \quad (35)$$

and find $\gamma = \gamma'' - \gamma'$. (36)

The calculation of γ'' by variational methods so far has not proved successful. However, the difficulty is a matter of practicality and not of principle.[11]

Table I shows the numerical values calculated for γ' using a theoretical $S(h)$ obtained by Campbell and Feenberg.[12] Our γ'_a differs from that of Ref. 4 on account of an integration error in that reference. The difference in γ'_e has resulted from the different $S(h)$ employed. Note that $|\gamma'_f| \gtrsim |\gamma'_e|$, and should not be omitted as done in Ref. 4. Also,

$$(\gamma'_b + \cdots + \gamma'_g)/\gamma'_a = 0.287 \qquad (37)$$

lends support to the ordering scheme and indicates satisfactory convergence.

TABLE I. Contributions toward the dispersion coefficient.

	γ'_a	γ'_e	γ'_f	γ'_g
Lee (Ref. 4)	−0.239	−0.0678	⋯	⋯
This calculation	−0.462	−0.0591	−0.0640	−0.0096

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APPENDIX: EVALUATION OF $(\vec{k}_1+\vec{k}_2, \vec{k}_3 | \vec{k}_1, \vec{k}_2, \vec{k}_3)$

$$(\vec{k}_1+\vec{k}_2, \vec{k}_3 | \vec{k}_1, \vec{k}_2, \vec{k}_3) = [S^2(k_3)S(|\vec{k}_1+\vec{k}_2|)S(k_1)S(k_2)]^{-1/2} \int \rho^*_{\vec{k}_1+\vec{k}_2}\rho^*_{\vec{k}_3}\rho_{\vec{k}_1}\rho_{\vec{k}_2}\rho_{\vec{k}_3}\Psi_0^2 \, d\vec{r}_{12\ldots N} , \qquad (A1)$$

where

$$\rho^*_{\vec{k}_1+\vec{k}_2}\rho^*_{\vec{k}_3}\rho_{\vec{k}_1}\rho_{\vec{k}_2}\rho_{\vec{k}_3} = \frac{1}{N^{5/2}} \sum_{ijklm=1}^{N} e^{i\vec{k}_1\cdot\vec{r}_{il} + i\vec{k}_2\cdot\vec{r}_{jl} + i\vec{k}_3\cdot\vec{r}_{km}} \to N^{-3/2}\{1 + (N-1)$$
$$\times [1 + 2(e^{i\vec{k}_1\cdot\vec{r}_{12}} + e^{i\vec{k}_2\cdot\vec{r}_{12}} + e^{i\vec{k}_3\cdot\vec{r}_{12}} + e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{r}_{12}}$$
$$+ e^{i(\vec{k}_1+\vec{k}_2+\vec{k}_3)\cdot\vec{R}_{12}} + e^{i(\vec{k}_1+\vec{k}_3)\cdot\vec{r}_{12}} + e^{i(\vec{k}_2+\vec{k}_3)\cdot\vec{r}_{12}} + e^{i(\vec{k}_1-\vec{k}_3)\cdot\vec{r}_{12}} + e^{i(\vec{k}_2-\vec{k}_3)\cdot\vec{r}_{12}}]$$
$$+ (N-1)(N-2)[e^{i\vec{k}_1\cdot\vec{r}_{12}} + e^{i\vec{k}_2\cdot\vec{r}_{13}} + e^{i\vec{k}_3\cdot\vec{r}_{12}} + e^{i(\vec{k}_1+\vec{k}_2+\vec{k}_3)\cdot\vec{R}_{12}} + 3e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_2\cdot\vec{r}_{32})}$$
$$+ e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_3\cdot\vec{r}_{23})} + e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_3\cdot\vec{r}_{32})} + e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_3\cdot\vec{r}_{13})} + e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_3\cdot\vec{r}_{31})}$$
$$+ e^{i(\vec{k}_2\cdot\vec{r}_{12}+\vec{k}_3\cdot\vec{r}_{23})} + e^{i(\vec{k}_2\cdot\vec{r}_{12}+\vec{k}_3\cdot\vec{r}_{13})} + e^{i(\vec{k}_2\cdot\vec{r}_{12}+\vec{k}_3\cdot\vec{r}_{31})}$$
$$+ e^{i(\vec{k}_2\cdot\vec{r}_{21}+\vec{k}_3\cdot\vec{r}_{31})} + e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{r}_{12}+i\vec{k}_3\cdot\vec{r}_{13}} + e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{r}_{12}+i\vec{k}_3\cdot\vec{r}_{31}} + e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{r}_{12}+i\vec{k}_3\cdot\vec{r}_{23}}$$
$$+ e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{r}_{12}+i\vec{k}_3\cdot\vec{r}_{32}} + e^{i\vec{k}\cdot\vec{r}_{12}+i(\vec{k}_2-\vec{k}_3)\cdot\vec{r}_{32}} + e^{i\vec{k}_1\cdot\vec{r}_{12}+i(\vec{k}_2+\vec{k}_3)\cdot\vec{r}_{32}} + e^{i(\vec{k}_1-\vec{k}_3)\cdot\vec{r}_{12}+i\vec{k}_2\cdot\vec{r}_{32}}$$
$$+ 2e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_2\cdot\vec{r}_{32}+\vec{k}_3\cdot\vec{r}_{13})} + e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_2\cdot\vec{r}_{32}+\vec{k}_3\cdot\vec{r}_{31})} + (N-1)(N-2)(N-3)$$
$$\times [e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_2\cdot\vec{r}_{32})} + e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_3\cdot\vec{r}_{34})} + e^{i(\vec{k}_2\cdot\vec{r}_{12}+\vec{k}_3\cdot\vec{r}_{34})} + e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{r}_{12}+i\vec{k}_3\cdot\vec{r}_{34}}$$
$$+ e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_2\cdot\vec{r}_{32}+\vec{k}_3\cdot\vec{r}_{34})} + e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_2\cdot\vec{r}_{32}+\vec{k}_3\cdot\vec{r}_{43})}$$
$$+ e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_2\cdot\vec{r}_{32}+\vec{k}_3\cdot\vec{r}_{24})} + e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_2\cdot\vec{r}_{32}+\vec{k}_3\cdot\vec{r}_{42})} + 2e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_2\cdot\vec{r}_{32}+\vec{k}_3\cdot\vec{r}_{14})}]$$
$$+ (N-1)(N-2)(N-3)(N-4)e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_2\cdot\vec{r}_{32}+\vec{k}_3\cdot\vec{r}_{45})}\} \qquad (A2)$$

Substituting Eq. (A2) into Eq. (A1), and making use of Eqs. (A3)–(A7) of Ref. 4, we find all the integrals in Eq. (A1), except

$$(N-1)(N-2)(N-3)(N-4) \times \int e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_2\cdot\vec{r}_{32}+\vec{k}_3\cdot\vec{r}_{45})}\Psi_0^2 \, d\vec{r}_{12\cdots N}$$
$$= (1/N)\int e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_2\cdot\vec{r}_{32}+\vec{k}_3\cdot\vec{r}_{45})} \times P(12345)\, d\vec{r}_{12345} , \qquad (A3)$$

which involves the five-particle distribution function $P(12345)$. We shall use the convolution approximation[13] for $P(12345)$, as shown in Fig. 3, where an open circle takes on a label selected nonrepetitively from 1, 2, 3, 4, and 5, a darkened circle denotes a particle i, $i>5$, with a volume integration over its coordinates, and each bond between particles i and j represents a factor $[g(r_{ij})-1]$, $g(r)$ being the radial distribution function defined by Eq. (4). The number above the summation sign informs us of the number of terms contained. Now, keeping terms of order N in (A3) as we did in evaluating all the other integrals,

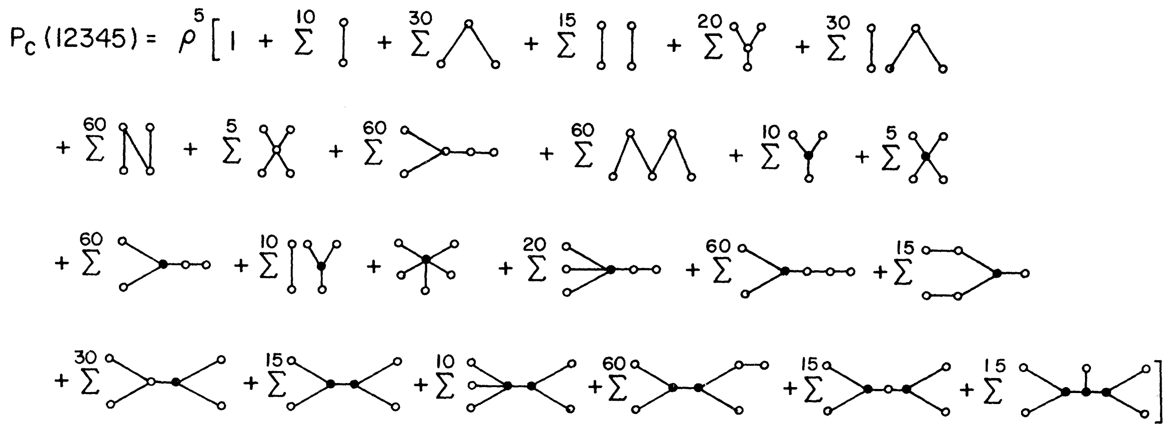

FIG. 3. Convolution approximation for the five-particle distribution function.

$$
\begin{aligned}
&(N-1)(N-2)\ (N-3)(N-4) \\
&\quad\times\int e^{i(\vec{k}_1\cdot\vec{r}_{12}+\vec{k}_2\cdot\vec{r}_{32}+\vec{k}_3\cdot\vec{r}_{45})}\,\Psi_0^2\,d\vec{r}_{12\cdots N} \\
&\quad\simeq N[S(k_1)S(k_2)S(k_3)S(|\vec{k}_1+\vec{k}_2|) \\
&\quad\quad -S(k_1)S(k_2)S(|\vec{k}_1+\vec{k}_2|)-S(|\vec{k}_1+\vec{k}_2|)S(k_3) \\
&\quad\quad -S(k_2)S(k_3)-S(k_1)S(k_3)+S(|\vec{k}_1+\vec{k}_2|) \qquad \text{(A4)} \\
&\quad\quad +S(k_2)+S(k_1)+S(k_3)+S(k_3)-2]\ .
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
(\vec{k}_1+\vec{k}_2,\vec{k}_3\,|\,\vec{k}_1,\vec{k}_2,\vec{k}_3) &= [N^{-1}S(k_1)S(k_2)S(|\vec{k}_1+\vec{k}_2|)]^{1/2} \\
&= (\vec{k}_1+\vec{k}_2\,|\,\vec{k}_1,\vec{k}_2)\ .
\end{aligned}
$$

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[5] Strictly speaking, these functions are normalized only if the wave vectors involved are all different. Otherwise there will be additional numerical factors in the denominators. The errors caused by the omissions of these factors lead to negligible errors when we later carry out summations over all states.

[6] In Eq. (14), we have neglected matrix elements of the type $\langle\vec{l},\vec{m}\,|\,\vec{k}-\vec{h}-\vec{h}',\vec{h},\vec{h}'\rangle$ where both $\vec{l}$ and $\vec{m}$ differ from $\vec{k}-\vec{h}-\vec{h}'$, $\vec{h}$, and $\vec{h}'$. These matrix elements would have contributed to the phonon spectrum in orders higher than what we retain in the present scheme of calculation. (See Sec. III for the ordering scheme.)

[7] In these integrations, we replace $(\vec{k}\cdot\vec{h})^2$, $(\vec{k}\cdot\vec{h})^2(\vec{k}\cdot\vec{h}')^2$ and $(\vec{k}\cdot\vec{h})^4$ in the integrands by their average values over the $\vec{k}$ space. This is possible because the contributions to $\epsilon(k)$ are independent of the orientation of $\vec{k}$.

[8] Following the notations of Ref. 9, we regard diagrams (b), (c), (d), (f), and (g) as consisting of two rings, and not three rings.

[9] C-W. Woo and S. Ma, Phys. Rev. 159, 176 (1967).

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[11] By forcing a hard-sphere model on liquid He^4, with the scattering length fixed by the experimental velocity of sound, γ'' can be estimated to be -0.05. This is clearly not a quantitatively meaningful procedure.

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