In conclusion, we have shown that it is possible to determine the form of the pair potential once the temperature dependence of the hard-core potential is known if one is making use of the BH relationship. We further point out that there is some computational advantage in using the LV form of the temperature dependence of the hard-sphere diameter, since this allows an analytical inverse Laplace transform together with an invertible $r(V)$-to-V relation: Furthermore, it is usually flexible enough to provide a good fitting. For more general cases, higher-degree polynomial ratios can be used, which are also Laplace transformable by elementary procedures. As a last point we mention the possibility of using this formulation to get information about the repulsive part of the effective interaction in liquid metals. In fact, temperature-dependent hard-sphere diameters have been frequently used in connection with liquid metals in the recent past,[8–10] and it seems plausible to test the consistency of these diameters with effective-pair interaction deduced by the pseudopotential theory of metals.[14]

---

[1]B. J. Alder and W. G. Hoover, in *Physics of Simple Liquids*, edited by H. N. V. Temperley *et al.* (Wiley, New York, 1968).

[2]J. A. Barker and D. Henderson, J. Chem. Phys. 47, 4714 (1967).

[3]D. Levesque and L. Verlet, Phys. Rev. 182, 307 (1969).

[4]The convergence of the BH expansion has not been established; however D. Levesque [J. Phys. Japan Suppl. 26, 270 (1969)] has recently pointed out that a comparison between the BH procedure and the exact Monte-Carlo computation for Lennard-Jones (6-12) potentials has shown that the BH approximation is very good, both at low temperature and high density and at higher temperature and lower density. The same findings are obtained in Ref. 3.

[5]J. Hirschfelder, D. Stevenson, and M. Eyring, J. Chem. Phys. 5, 896 (1937).

[6]H. Reiss, Advan. Chem. Phys. 9, 1 (1965).

[7]J. H. Dymond and B. J. Alder, J. Chem. Phys. 45, 2061 (1966).

[8]P. Ascarelli, Phys. Rev. 173, 271 (1968).

[9]N. W. Ashcroft and J. Lekner, Phys. Rev. 145, 83 (1966).

[10]P. Ascarelli and A. Paskin, Phys. Rev. 165, 223 (1968).

[11]A similar method has been applied to the second virial coefficient by J. B. Keller and B. Zumino, J. Chem. Phys. 30, 1351 (1959); and by H. L. Frisch and E. Helfand, *ibid.* 32, 269 (1960).

[12]H. Bateman, *Tales of Integral Transforms*, Vol. 1. (McGraw-Hill, New York, 1956).

[13]J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, *Molecular Theory of Gases and Liquids* (Wiley, New York, 1954).

[14]W. Harrison, *Pseudopotentials in the Theory of Metals* (Benjamin, New York, 1966).

PHYSICAL REVIEW A VOLUME 1, NUMBER 5 MAY 1970

Slowing Down of a Fast Test Particle in a Plasma*

A. A. Husseiny and H. K. Forsen
Department of Nuclear Engineering, The University of Wisconsin, Madison, Wisconsin 53706
(Received 7 January 1970)

The elastic scattering of fast charged particles in a background plasma is considered. Investigation is limited to scattering by Maxwellian particles whose most probable speed is much less than the velocity of the fast test particle. A shielded-potential elastic-scattering cross section is used to formulate the scattering kernel. By taking the moments of the kernel, it is possible to obtain expressions for the rate of energy transfer, the slowing-down time, the rate of the mean-square energy transfer, and the dispersion of the initial energy distribution of the fast particles released to the background.

I. INTRODUCTION

The scattering of energetic charged test particles by electrons and positive ions is of interest in the study of thermonuclear plasmas. In particular the problem is of interest when considering the interaction of energetic charged particles injected or released by fusion reactions with the background plasma.

Scattering of charged particles is often investigated through the derivation of Fokker-Planck coefficients as given by Spitzer[1] by analogy to the treatment by Chandrasekhar in stellar dynamics.[2,3] However, Allis and Buchsbaum[4] have criticized such treatment as liable to erroneous results due to two deficiencies in the course of the derivation. First, the two-body Rutherford formula is used with the Fokker-Planck equation which only treats the many small deflections and thus the correctness of the Fokker-Planck approach is lost. Second, in the treatment by Chandrasekhar, Spitzer, and others,[5] the logarithmic term, which appears as a function of the test-particle energy, is replaced with a constant term. Such a treatment can be dangerous and is shown here to yield incorrect results. Spitzer, on the other hand, points out that the diffusion coefficients derived by Chandrasekhar are inaccurate in the special case when the velocity of the test particles greatly exceeds that of the field particles. In this velocity range, it is found that the terms designated by Chandrasekhar as "nondominant" do in fact exceed those retained as "dominant." Moreover, the relaxation times of Spitzer are obtained for the energy range in which the speed of the test particles is less than or approximately equal to the thermal speeds of Maxwellian field particles. Butler and Buckingham[6] and Shkarofsky, Johnston, and Bachynski[7] derive a closed-form expression for the behavior of the mean energy of test particles as a function of time for essentially the whole range of interest. However, the use of a Rutherford scattering cross section with a classical cutoff angle again leads to an incorrect slowing-down time.

In the present paper, the nonrelativistic scattering of energetic charged test particles by charged background particles is studied in the range of $v_x > V_s$, where v_x is the velocity of the test particle and V_s is the velocity of the background field particles. The method of analysis does not require the approximations used in the previous calculations and appropriate quantum-mechanical effects are included. This is accomplished by the use of a Boltzmann-equation type of calculation which is more consistent with the two-body cross section for scattering and consequently more accurate results may be expected. Although the present treatment can yield closed expressions for diffusion coefficients and relaxation times, principal attention is given to obtaining the mean energy transfer rate, the energy spread around the mean and the slowing-down time. To demonstrate the accuracy of the results in the velocity range of interest, the rate of the mean-square energy loss is also calculated.

II. SCATTERING KERNEL

Consider a test particle (x) of mass m_x and charge q_x traveling with velocity v_x and encountering a field particle of mass M_s and charge q_s traveling with velocity $V_s < v_x$. The elastic-scattering cross section is estimated by a Born approximation calculation for a Debye length shielded potential and is given in the c.m. system by

$$\sigma(\chi)\,d\Omega(\chi) \simeq \left(\frac{q_x q_s}{4\pi\epsilon_0}\right)^2 \left(\frac{\operatorname{cosec}^2(\frac{1}{2}\chi)}{4\mu E_x/m_x}\right)^2 \times \left(\frac{\sin^2(\frac{1}{2}\chi)}{\frac{1}{4}\chi_0^2 + \sin^2(\frac{1}{2}\chi)}\right)^2 . \tag{1}$$

Here E_x is the energy of the test particle, μ is the reduced mass, χ is the c.m. deflection angle, $d\Omega(\chi)$ is the differential solid angle, and χ_0 is the minimum cutoff angle below which the Rutherford single potential scattering cross section is no longer applicable. χ_0 corresponds to the Debye screening length (λ_s) and is given by

$$\chi_0 = \frac{\hbar}{\mu v_x \lambda_s} = \frac{\hbar q_s}{\mu}\left(\frac{n_s m_x}{2\epsilon_0 k T_s E_x}\right)^{1/2} . \tag{2}$$

The probability distribution function $P(E_x \to E_x')$ is the probability that a test ion of energy E_x will emerge from an encounter with an energy between E_x' and $E_x' + dE_x'$. On the other hand, $W(\chi)\,d\Omega(\chi)$ is the probability that a test particle will be deflected through a solid angle between $\Omega(\chi)$ and $\Omega(\chi) + d\Omega(\chi)$. Since both probabilities are equal and $W(\chi)\,d\Omega(\chi)$ is explicitly given in the c.m. system as $-\sigma(\chi)\,d\Omega(\chi)/\sigma_T$, where σ_T is the total elastic-scattering cross section, then

$$P(E_x \to E_x')\,dE_x' = -[\sigma(\chi)/\sigma_T]\,d\Omega(\chi) . \tag{3}$$

The scattering kernel is denoted as $\sigma_T(E_x \to E_x')$ and is given by $\sigma_T P(E_x \to E_x')$. Using Eqs. (1) and (3), the scattering kernel is given for the case where $E_x > E_s$ as

$$\begin{aligned} \sigma_T(E_x \to E_x') &= 0, && E_x' > E_x \\ \sigma_T(E_x \to E_x') &= \frac{\sigma_T \hbar^2}{2M_s\lambda_s^2[E_x - E_x' + (\hbar^2/2M_s\lambda_s^2)]^2} , && E_{\min} \leq E_x' \leq E_x \\ \sigma_T(E_x \to E_x') &= 0 , && E_x' < E_{\min} \end{aligned} \tag{4}$$

where $E_{\min} = (m_x - M_s)^2 E_x/(m_x + M_s)^2$ or the minimum energy a test particle can emerge with after a single encounter. We know that a test particle whose mass is equal to that of the field particle can lose all its energy in one encounter, but a heavy-mass test particle loses only negligible energy in a single encounter with an electron.

It is observed that the contribution of small-angle scattering is included in the cross section of Eq. (1) and consequently in the scattering kernel. It should be recalled here that we have used the quantum value of the scattering angle [Eq. (2)]

rather than that used by others.[1-4] This is because we choose the largest minimum scattering angle, which for Coulomb collisions corresponds to $(q_s q_x/4\pi\epsilon_0 \hbar v_x) < 1$, and therefore the results are properly valid in this limit. This is also the limit of validity of the Born approximation.

III. MOMENTS OF THE SCATTERING KERNEL

Let us now consider the moments of the elastic-scattering kernel of Eq. (4). The νth moment is given by

$$\Pi_\nu = \int_0^\infty (E_x - E_x')^\nu \sigma_T(E_x \to E_x')\, dE_x' \quad . \tag{5}$$

For the first three moments this gives

$$\Pi_0 = \sigma_T(E_x)(1 - \tfrac{1}{4}\chi_0^2) \ , \tag{6}$$

$$\Pi_1 = \frac{\sigma_T(E_x)\hbar^2}{M_s \lambda_s^2}[\ln\Lambda(E_x) + 0.193] \ , \tag{7}$$

and
$$\Pi_2 = \frac{2\sigma_T(E_x)\hbar^2\mu^2 E_x}{M_s^2 m_x \lambda_s^2} \times \left(1 + \frac{\chi_0^2}{4+\chi_0^2} - \chi_0^2 \ln\Lambda(E_x) - 0.693\chi_0^2\right), \tag{8}$$

where $\Lambda = 1/\chi_0$. Throughout the course of deriving these results there is no need to neglect any terms or to make any simplifying assumption. At the same time it is entirely within the discretion of the investigator in the use of these moments to drop or retain terms of low order. For energetic test particles, i.e., the case under study, χ_0 is of the order of 10^{-10} and therefore all terms of the order of χ_0^2 may be dropped out without error. The logarithmic term that appears in Eqs. (7) and (8) corresponds to the well-known Coulomb logarithm. It can be seen that the energy dependence of this term has been retained and the integration has been evaluated without difficulty.

IV. RATE OF ENERGY TRANSFER

The mean energy loss by a test particle per encounter is

$$\langle E_x - E_x' \rangle = \Pi_1(E_x)/\Pi_0(E_x) \ , \tag{9}$$

while the number of encounters per unit time per unit volume is $n_s \sigma_T(E_x)(2E_x/m_x)^{1/2}$. Therefore, the rate of the mean energy transferred from the test particle to the field particles can be written

$$\frac{dE_x}{dt} \simeq -\frac{2\pi m_s}{M_s}\left(\frac{q_x q_s}{4\pi\epsilon_0}\right)^2 \left(\frac{2m_x}{E_x}\right)^{1/2} \ln\Lambda(E_x) \quad . \tag{10}$$

This equation is applicable for ions or electrons so long as $v_x > V_s$ and $E_x > E_s$. For ions as test particles and electrons as field particles the form of Eq. (10) is in agreement with a result obtained by Spitzer,[1] Butler and Buckingham,[6] and Shkarofsky, *et al.*[7] in the limit of $v_i > (2kT_e/m_e)^{1/2}$ where the subscripts e and i refer to electrons and ions, respectively, and the field electrons are considered Maxwellian at a temperature T_e. Some investigators[1,5] use the classical value for the Coulomb logarithm of Eq. (10). Moreover, they replace the test-particle speed by the electron speed. Generally, the results are insensitive to the choice of Λ; however, as we shall see later it does introduce important differences in the slowing-down time and energy dispersion.

The choice of the classical cutoff angle rather than the quantum-mechanical cutoff angle is in error in the case of energetic test particles because the rule is to choose the larger of the two angles. For energetic low-Z test particles the quantum cutoff angle is the larger. The choice of a classical value for $\Lambda(E_x)$ is only appropriate for the special case considered by Spitzer[1] and Chandrasekhar.[2] This case is limited to the energy range at which the test-particle velocity is of the same order of that of the field particles. The replacement of v_x by the thermal speed of the ions is unnecessary and leads to considerable errors as indicated in Sec V.

V. SLOWING-DOWN TIME

Our definition of the slowing-down time is the time necessary for a fast test particle to transfer $(E_0 - E_x)/E_0$ of its energy to the background particles. Here E_0 is the energy at which the test particle is released in the volume containing the background particles. Rearranging Eq. (10) and integrating over time from an initial time zero to τ_s and the corresponding energy integral between E_0 and E_x, the slowing-down time is given as

$$\tau_s = \frac{4\pi q_s \epsilon_0{}^{1/2} M_s m_x \hbar^3 n_s{}^{1/2}}{q_x^2 \mu^3 (kT_s)^{3/2}} \times \left\{ \mathrm{li}\left[\frac{M_s}{\chi_0 \mu}\left(\frac{E_0}{E_x}\right)^{1/2}\right]^3 - \mathrm{li}\left[\frac{M_s}{\chi_0 \mu}\right]^3 \right\} \ , \tag{11}$$

where $\mathrm{li}(x)$ is the logarithmic integral. The Spitzer expression for the slowing-down time is given by

$$\tau_s = 8\pi\epsilon_0^2 \mu E_0^{3/2}/n_s q_x^2 q_s^2 (2m_x)^{1/2} \ln\Lambda \quad , \tag{12}$$

where v_x is replaced by $2kT_e/m_e$ in the expression for Λ which is chosen as the classical value. Although the logarithm is not sensitive to the choice of its argument, it is obvious that the above replacement leads to errors in the result of the integration. An incorrect choice in $\ln\Lambda$ will not affect the rate of energy transfer but it will result in underestimation of the slowing-down time by $\ln\Lambda$ as can be seen by comparing Eqs. (11) and (12). For test α particles released at $E_0 = 3.5$ MeV in a background plasma consisting of equal fractions of deuterium and tritium with total ion density of 10^{21} m^{-3}, the slowing-down time on plasma ions or time to reach $20kT_i$ is 2 sec as calculated from Eq. (11). The corresponding time from Eq. (12) is 10^{-1} sec. If the background particles are electrons of the same density and if the velocity of the α particles much exceeds the thermal velocity of the electrons, Eq. (11) yields a slowing-down time of about 0.4 msec. It should be pointed out, however, that this corresponds to rather cool electrons and to solve this example properly one must consider the energy-transfer rates to both background species in the general case with target speed $V_s \gtrsim v_x$, especially for electrons.

VI. RATE OF MEAN-SQUARE ENERGY LOSS

The mean-square energy loss per encounter is Π_2/Π_0. By multiplying this value by the encounter rate, the rate of mean-square energy loss is

$$\frac{d}{dt}\langle E_x^2\rangle = 4\pi n_s \left(\frac{q_x q_s}{4\pi\epsilon_0}\right)^2 \frac{\mu^2}{M_s^2}\left(\frac{2E_x}{m_x}\right)^{1/2} . \tag{13}$$

This result is in agreement with an expression given by Spitzer and Scott[8] for elastic collisions of a fast test electron with thermal electrons in the limit of $E_e \gg kT_e$. Their results are obtained from Chandrasekhar's detailed computations by considering the "nondominant" terms. The use of Chandrasekhar's result considering only the "dominant" terms gives

$$\frac{d}{dt}\langle E_x^2\rangle = 4\pi n_s \left(\frac{q_x q_s}{4\pi\epsilon_0}\right)^2 \frac{2kT_s}{M_s}\left(\frac{m_x}{2E_x}\right)^{1/2} \ln\Lambda \tag{14}$$

in the limit of $v_x > V_s$ where $\ln\Lambda$ is the Coulomb logarithm. The test-particle energy dependence of Eq. (13) is the inverse of that in Eq. (14) and yet Eq. (14) has been adapted by many authors in plasma physics especially in evaluating the energy relaxation time and diffusion-in-velocity coefficient. The disagreement between Eqs. (13) and (14) is expected because Eq. (14) is not valid in the energy range of interest as pointed out in the Introduction. Although the results of Chandrasekhar were derived for a general energy range, the fact that users of these results retain only the terms proportional to $\ln\Lambda$ leads to serious errors. As seen from Eq. (8) such terms are negligible in the present case and are dropped out in our calculations.

VII. ENERGY DISPERSION

The energy dispersion is defined as

$$D = (\Pi_2 - \Pi_1^2)^{1/2}/\Pi_1 , \tag{15}$$

and D is approximately equal to $1/[\chi_0 \ln\Lambda(E_x)]$ which is larger than unity. This indicates that a pulse $\delta(E_x - E_0)$ is broadened during the slowing-down process. Moreover, the dispersion of an initial distribution of test particles decreases as the energy decreases. Again Eq. (15) contradicts the value of D which can be calculated using Spitzer's results[1] because of the presence of the second moment.

For completeness we may investigate the diffusion of energies around the average. In order to do this we must obtain the difference $[\langle (E_x - E_x')^2\rangle - \langle E_x - E_x'\rangle^2]$ for each encounter, but this is simply $(\Pi_2 - \Pi_1^2)/\Pi_0$. Multiplying the result by the number of encounters in a short time δt, the mean-square deviation around the mean energy during this time is

$$\delta t \, 4\pi \left(\frac{q_x q_s}{4\pi\epsilon_0}\right)^2 n_s \frac{\mu^2}{M_s^2}\left(\frac{2E_x}{m_x}\right)^{1/2} . \tag{16}$$

Equation (16) may be written in terms of dE_x/dt, which is the rate of change of the mean energy. Using the results of Eq. (10) we get for the mean-square deviation

$$-\frac{2\mu^2 E_x}{m_x M_s \ln\Lambda(E_x)} \frac{dE_x}{dt}\delta t . \tag{17}$$

The total mean-square deviation in energy after a time t, say $\epsilon^2(t)$, can be obtained by integrating Eq. (17). Thus

$$\epsilon^2(t) = -\frac{q_s^4 n_s^2 m_x \hbar^4}{\epsilon_0^2 \mu^2 k^2 T_s^2 M_s}\left[\mathrm{li}(\gamma E_0^2) - \mathrm{li}(\gamma \overline{E}_x^2)\right] , \tag{18}$$

where $\overline{E}_x$ is the mean energy at some time t and

$$\gamma = 4\mu^4 \epsilon_0^2 k^2 T_s^2 / q_s^4 n_s^2 \hbar^4 m_x^2 .$$

It is observed that $\epsilon^2(t)$ depends on the mass and energy of the background particles. The energy of the test particle will be given by $\overline{E}_x$ to within $\pm\epsilon(t)$, i.e.,

$$E_x = \overline{E}_x \pm \left(\frac{4\mu^2\gamma}{m_x M_s}\left[\mathrm{li}(\gamma E_0^2) - \mathrm{li}(\gamma \overline{E}_x^2)\right]\right)^{1/2} . \tag{19}$$

These results are different from those obtained by Butler and Buckingham.[6] Although they carefully emphasized the case of fast test particles, their result concerning the diffusion around the mean is subject to error because of their attempts to generalize the results for too wide a range of energy. Such an attempt makes the result liable to the same errors as if the result of Eq. (14) had been adapted in an energy range where it is no longer correct.

VIII. CONCLUSION

In this paper we seek to demonstrate that it is possible to derive closed expressions for the scattering parameters in the energy range $v_x > V_s$ without the previously used approximations. The results obtained should therefore be more accurate in the energy range specified than results currently used in the literature.

The treatment of scattering in the energy range where $v_x \lesssim V_s$ can be handled in the same manner.[9] However, attempts to derive general expressions which cover the whole energy range from zero to E_0 are shown to lead to unacceptable errors. It is thus recommended that the energy scale be divided into two ranges: the slowing-down range in which the test particle does not feel the thermal motion of the field particles and the thermalization range in which neither the test particle nor the field particles can be regarded as fixed centers. The results obtained here are applicable for the slowing-down range whereas the results of Spitzer[1] and Chandrasekhar[3] are believed to apply only in the thermalization range within the limitations of the assumptions involved.

---

*Work supported in part by the U.S. Atomic Energy Commission and the Faculty of Engineering, University of Alexandria, Alexandria, UAR.

[1]L. Spitzer, *Physics of Fully Ionized Gases* (Interscience, New York, 1967), 2nd ed.

[2]S. Chandrasekhar, Astrophys. J. 93, 285 (1941).

[3]S. Chandrasekhar, *Principles of Stellar Dynamics* (Dover, New York, 1942).

[4]*Electrons, Ions and Waves, Selected Works of William Phelps Allis*, edited by S. C. Brown (The M. I. T. Press, Cambridge, Mass., 1967).

[5]D. J. Rose and M. Clark, Jr., *Plasmas and Controlled Fusion* (The M.I.T. Press, Cambridge, Mass., 1961).

[6]S. T. Butler and M. J. Buckingham, Phys. Rev. 126, 1 (1962).

[7]I. P. Shkarofsky, T. W. Johnston, and M. P. Bachynski, *The Particle Kinetics of Plasmas* (Addison-Wesley, Reading, Mass., 1966).

[8]L. Spitzer, Jr., and E. H. Scott, Astrophys. J. 158, 161 (1969).

[9]A. A. Husseiny, Ph.D. thesis, University of Wisconsin, Report No. PLP 323, 1969 (unpublished).