Energy and Motion of Vortex Rings in Liquid Helium II in the Presence of Various Plane Obstacles

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A classical calculation is given for the kinetic energy of a vortex ring in an inviscid fluid (He II) in the presence of an infinitely extended plane, a coaxial circular disk, and an infinitely extended plane with a coaxial circular aperture. The energy is obtained as an explicit function of the ring position relative to the obstacle being considered. Formulas are given for the velocity components of the vortex-ring motion, the force exerted on an obstacle by a stationary vortex ring, and the impulse of the vortex ring in the presence of the obstacle. In the case of the circular aperture, there is found to be a critical energy (or ring size) beyond which a vortex ring cannot pass through the aperture. The path of the vortex ring near the different obstacles is obtained by numerical computation, departing from the explicit energy expression. The calculation method is easily extended to other axisymmetric configurations and consists of Fourier and Hankel integral-transform techniques in combination with results from the theory of dual integral equations.

I. INTRODUCTION

Since the appearance of a paper by Feynman[1] attributing the breakdown of superfluid helium flow to the action of vortex lines or rings, there has been an increasing interest in the behavior of vortex rings in He II.

Feynman considered the creation of vortex rings with a quantized circulation:

$$\oint (\vec{v}_s \cdot d\vec{l}) = h/m,$$

where h is Planck's constant, and m is the mass of a ^{4}He atom. By Landau's criterion,[2] the creation of these vortex rings might then be expected at superfluid velocities v_S equal to or in excess of the critical velocity v_C, given as

$$v_c = \text{minimum value of } E/P,$$

where E and P are the energy and momentum of the vortex ring.

In order to predict the value of v_C in capillary-flow experiments, various authors have presented calculations of E and P for a vortex ring contained inside a circular cylinder.[3-5] Recently, in a paper by the present author and co-authors,[6] a review was given of these calculations, together with a different method of solution for the associated potential problem, confirming the result for E given by Raja Gopal.[5] In addition to this, it was shown that P is identical to zero for an enclosed vortex ring, with the result that Landau's criterion cannot be applied to vortex-ring excitations inside a cylinder.

The calculation of E and P was affected by expressing the velocity field inside the cylinder in terms of a vector potential $\vec{A}$, after which the partial differential equation for $\vec{A}$ was solved by the use of Fourier-Hankel integral-transform techniques.

The present paper is an extension of this method to the case where the motion of a vortex ring in an inviscid fluid is obstructed by one of the following obstacles: (i) An infinitely extended plane in parallel with the plane of the ring, (ii) a circular disk, coaxial with the ring, or (iii) an infinitely extended plane in parallel with the plane of the ring, containing a coaxial circular aperture. In all three cases an explicit expression is obtained for the energy of the vortex ring. This expression is of the form

$$E = E_0 - E_1 .$$

Here, E_0 is the kinetic energy associated with the unobstructed ring, while E_1 denotes the interaction energy between the unobstructed vortex velocity field and the perturbation field due to the obstacle.

Under the influence of the obstacle, the path taken by the vortex ring can now be described by the condition

$$E = \text{const},$$

which is a direct consequence of the dissipationless character of the superfluid and the negligible interaction with rotons at sufficiently low temperature ($T < 0.5\,°\text{K}$). It has been found that E_1 is a positive quantity which increases during the approach of the ring towards the obstacle. As a result, E_0 and the size of the vortex ring must increase likewise if the total energy is to remain constant.

A calculation of E_0 is given in Sec. II, after which the interaction energies for the different obstacles are derived in Secs. III–V, together with the path taken by the vortex ring. The same method of solution is employed in all three cases. Although case (i) could also be solved by the method of images, it serves as a useful introduction to the method employed here. Cases (ii) and (iii) are related through the complementary character of their boundary conditions and are solved by means of recent results from the theory of dual integral equations.[7,8]

II. ENERGY OF AN UNOBSTRUCTED VORTEX RING

The velocity field of the superfluid will be denoted by the vector $\vec{V}$, while the normal component of the fluid is assumed to be at rest. Due to the incompressibility of the fluid as a whole, it follows that the superfluid flow is divergence-free and can be described in terms of a vector potential $\vec{A}$ having zero divergence:

$$\vec{V} = \nabla \times \vec{A}, \quad \nabla \cdot \vec{A} = 0. \tag{1}$$

The vorticity of $\vec{V}$ will be denoted by the vector $\vec{\kappa} = \nabla \times \vec{V}$, with the result that $\vec{A}$ and $\vec{\kappa}$ are related as

$$\nabla \times \nabla \times \vec{A} = \vec{\kappa}. \tag{2}$$

It will be assumed that the vortex ring has the form of a hollow toroid with a circular cross section, carrying on its surface a distributed vorticity which induces the encircling fluid flow around the core. For a vortex ring with aperture $2r_0$ and core diameter $2a$, the surface of the core will be given by the equation

$$z^2 + (r - r_0)^2 = a^2 .$$

Here, circular cylinder coordinates have been used with unit vectors $\vec{e}_r$, $\vec{e}_\varphi$, and $\vec{e}_z$. The ring axis of symmetry is taken along $\vec{e}_z$, while the center of the ring is the origin of the coordinate frame. As the core vorticity is φ directed and independent of φ, both $\vec{A}$ and $\vec{\kappa}$ are of the form

$$\vec{A} = A(r, z)\vec{e}_\varphi, \quad \vec{\kappa} = \kappa(r, z)\vec{e}_\varphi .$$

The corresponding radial and axial components of $\vec{V}$ are

$$V_r = -\frac{\partial A}{\partial z}, \quad V_z = \frac{1}{r}\frac{\partial (rA)}{\partial r} . \tag{3}$$

In practice, a being of the order of a few angstroms, the core cross section is extremely small when compared with the ring apertures contemplated in the literature. Under these circumstances, the core vorticity can be taken to be uniformly distributed over the core surface, while the field outside the core can be calculated from an equivalent vorticity concentrated along the core center:

$$\kappa(r, z) \to \kappa\delta(r - r_0)\delta(z) .$$

As a result, the differential equation for $A(r, z)$ can be written

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right) A(r, z) = -\kappa\delta(r - r_0)\delta(z), \tag{4}$$

where κ is a constant equal to h/m. This differential equation will be solved by the repeated use of integral transforms. The variable r is transformed by means of the first-order Hankel transform defined by the functional relations

$$\bar{f}(s) = \int_0^\infty r J_1(rs) f(r)\, dr,$$

$$f(r) = \int_0^\infty s J_1(rs) \bar{f}(s)\, ds ,$$

where $J_1(rs)$ denotes the ordinary Bessel function of the first order. This transform has the advantage that, provided

$$\lim_{r \to 0} r\frac{\partial}{\partial r}[J_1(rs) f(r)] = 0,$$

$$\lim_{r \to \infty} r\frac{\partial}{\partial r}[J_1(rs) f(r)] = 0 , \tag{5}$$

the Bessel operator in Eq. (4) is transformed according to

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) f(r) \to -s^2 \bar{f}(s). \tag{6}$$

Also, the variable z is transformed with the complex Fourier transform defined by

$$\bar{g}(p) = \int_{-\infty}^{\infty} e^{-ipz} g(z)\, dz ,$$

$$g(z) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{ipz} \bar{g}(p)\, dp .$$

Application of both integral transforms to the two sides of Eq. (4) and division by $-(s^2 + p^2)$ yields the doubly transformed vector potential

$$\bar{\bar{A}}(s, p) = \kappa r_0 (s^2 + p^2)^{-1} J_1(s r_0) . \tag{7}$$

Since the inverse Fourier transform of $(s^2 + p^2)^{-1}$ is given by $(1/2s)\exp(-s|z|)$, $\mathrm{Re}(s) > 0$, Fourier inversion of Eq. (7) results in

$$\bar{A}(s,z)=(\kappa r_0/2s)J_1(r_0 s)e^{-s|z|}. \qquad (8)$$

$A(r,z)$ is now obtained by the application of the inverse Hankel transform to Eq. (8), after which the components of $\vec{V}$ can be found with Eq. (3). This results in the following set of equations:

$$A(r,z) = \tfrac{1}{2}\kappa r_0 \int_0^\infty J_1(r_0 s)J_1(rs)e^{-s|z|}ds, \qquad (9a)$$

$$V_r(r,z)=\tfrac{1}{2}\kappa r_0 \operatorname{sgn}(z)\int_0^\infty sJ_1(r_0 s) \times J_1(rs)e^{-s|z|}ds, \qquad (9b)$$

$$V_z(r,z)=\tfrac{1}{2}\kappa r_0\int_0^\infty sJ_1(r_0 s)J_0(rs)e^{-s|z|}ds. \qquad (9c)$$

The vector potential $A(r,z)$ is finite for all values of r and z, except at the points $(r,z)=(r_0,0)$, where it exhibits a logarithmic singularity. For large values of r, A goes to zero as r^{-2}, in accordance with the well-known behavior of the vector potential of a stationary electric-current distribution of finite extent.[9] Thus, the conditions (5) necessary for the transformation of the Bessel operator are satisfied.

The kinetic energy of the vortex ring is given by the volume integral

$$E_0=\tfrac{1}{2}\rho_S\iiint|\vec{V}|^2dv, \qquad (10)$$

where ρ_S denotes the mass density of the superfluid. The integral should be extended over the whole region of the vortex velocity field. With the assumption that $\rho_S=0$ inside the core, this region will be taken as the volume between the core surface and the surface of the infinite sphere $r^2+z^2=R^2$, as $R\to\infty$. The integral can also be written as an integral over the boundary surfaces by means of the Gauss theorem and the following vector identity:

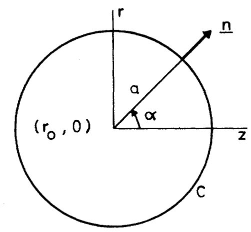

FIG. 1. Contour of the core cross section.

$$\nabla\cdot(\vec{A}\times\vec{V})=\vec{V}\cdot(\nabla\times\vec{A})-\vec{A}\cdot(\nabla\times\vec{V}). \qquad (11)$$

Since $\nabla\times\vec{V}=0$ and $\nabla\times\vec{A}=\vec{V}$ inside the integration volume, the integrand in Eq. (10) may be replaced with the left-hand side of Eq. (11). Application of the divergence theorem then yields the result

$$E_0=-\tfrac{1}{2}\rho_S\iint_{S_0}(\vec{A}\times\vec{V})\cdot\vec{n}\,dS. \qquad (12)$$

Here S_0 denotes the core surface, the contribution over the infinite sphere tending to zero with $R\to\infty$. Also, $\vec{n}$ denotes the unit normal on S_0, pointing outside the core. As the integrand in Eq. (12) is independent of φ, the integral can be further reduced to a line integral along the core circumference. Figure 1 depicts this contour (C) and shows the angle α, used as the integration variable. The energy can now be written as

$$E_0=-2\pi r_0 a\rho_S\int_{-\pi/2}^{\pi/2}A(V_z\sin\alpha-V_r\cos\alpha)d\alpha. \qquad (13)$$

Due to the singular behavior of A and V at the core center, E_0 diverges logarithmically for $a\to 0$. For small values of a/r_0, the integral expressions for A, V_r, and V_z can be expanded in powers of a/r_0 with coefficients depending on α:

$$A=\frac{\kappa}{2\pi}\left[\left(1-\frac{a\sin\alpha}{2r_0}+\frac{3a^2}{16r_0^2}(3\sin^2\alpha+\cos^2\alpha)\right)\ln\left(\frac{8r_0}{a}\right)-2+\frac{3a}{2r_0}\sin\alpha+\cdots\right], \qquad (14a)$$

$$V_r=\frac{\kappa\cos\alpha}{2\pi a}\left[1-\frac{3a^2}{8r_0^2}\ln\left(\frac{8r_0}{a}\right)-\frac{a\sin\alpha}{2r_0}+\frac{a^2}{16r_0^2}(11\sin^2\alpha+5\cos^2\alpha)+\cdots\right], \qquad (14b)$$

$$V_z=\frac{\kappa r_0}{4\pi a^2}\left[\frac{a^2}{r_0^2}\left(1-\frac{3a}{4r_0}\sin\alpha\right)\ln\left(\frac{8r_0}{a}\right)-\frac{2a}{r_0}\sin\alpha-\frac{a^2}{r_0^2}\cos\alpha+\frac{a^3}{8r_0^3}(5\sin^2\alpha+11\cos^2\alpha)+\cdots\right]. \qquad (14c)$$

After substitution of these expansions in Eq. (13), the resulting expansion for E_0 may be replaced by its dominant term

$$E_0=\tfrac{1}{2}\rho_S\kappa^2 r_0[\ln(8r_0/a)-2], \qquad (15)$$

the error being of the order $(a/r_0)^2$.

III. ENERGY OF VORTEX RING NEAR INFINITELY EXTENDED PLANE

The obstacle will be taken as an infinitely thin plane coincident with the plane $z=0$, acting as a barrier to the superfluid. A vortex ring of circulation κ and aperture $2r_0$ is situated parallel with and at a distance z_0 to the left of the plane $z=0$ (see Fig. 2). It is desired to find the kinetic energy of the velocity field $\vec{V}$ associated with this configuration and the motion of the ring towards the plane $z=0$.

As before, the velocity $\vec{V}$ will be derived from a vector potential $\vec{A}$ arising from the vorticities of $\vec{V}$. Due to the presence of the obstacle, there will be a distributed surface vorticity $f(r)\vec{e}_\varphi$ on the plane $z=0$, which appears as a second source term in the differential equation for $A(r,z)$:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{\partial^2}{\partial z^2}\right)A(r,z)$$

$$=-\kappa\delta(r-r_0)\delta(z+z_0)-f(r)\delta(z)\ . \qquad (16)$$

This equation is identical with the equation for the vector potential of an electric-current loop of strength κ facing a superconducting wall. Here $f(r)$ would have been the surface density of the eddy currents induced on the obstacle. In the electromagnetic case, $f(r)$ follows from the boundary condition at $z=0$, which prescribes that the normal component of the magnetic field strength must vanish on the superconducting surface. Likewise, in the hydrodynamical problem considered here, it is the component V_z which must vanish at $z=0$. By integration of the expression for V_z given in Eq. (3), it is seen that A must be of the form c/r on $z=0$, where c is a constant. Since A must remain finite on $z=0$ (the vector potential passing continuously through a surface distribution of vorticity), c must be zero, and the boundary condition for A is found to be $A(r,0)=0$.

Application of the Hankel-Fourier transforms to each side of Eq. (16) now yields the result

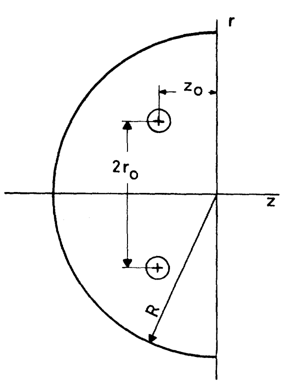

FIG. 2. Cross sectional view of a vortex ring opposite a plane wall.

$$\bar{\bar{A}}(s,p)=[\kappa r_0 J_1(r_0 s)e^{ipz_0}+\bar{f}(s)](s^2+p^2)^{-1},$$

where $\bar{f}(s)$ denotes the first-order Hankel transform of $f(r)$. After this, the use of the inverse Fourier transform gives $\bar{A}\ (s,z)$ as

$$\bar{A}(s,z)=\tfrac{1}{2}\kappa r_0 s^{-1}J_1(r_0 s)e^{-s|z+z_0|}$$

$$+\tfrac{1}{2}s^{-1}\bar{f}(s)e^{-s|z|}\ . \qquad (17)$$

Substituting $z=0$ in this equation, the left-hand side becomes zero by virtue of the boundary condition for A, and $\bar{f}(s)$ is found to be

$$\bar{f}(s)=-\kappa r_0 J_1(r_0 s)e^{-s|z_0|}\ . \qquad (18)$$

By substituting this result in Eq. (17) and using the inverse Hankel transform, $A(r,z)$ is obtained. It is found that $A(r,z)$ can be regarded as the sum of the potentials $A_0(r,z)$ and $A_1(r,z)$, which are the respective potentials of an unobstructed vortex ring at $(r_0,-z_0)$ and of the induced boundary vorticity $f(r)$, namely,

$$A_0(r,z)=\tfrac{1}{2}\kappa r_0\int_0^\infty J_1(r_0 s)J_1(rs)$$

$$\times e^{-s|z+z_0|}\,ds\ , \qquad (19a)$$

$$A_1(r,z)=-\tfrac{1}{2}\kappa r_0\int_0^\infty J_1(r_0 s)J_1(rs)$$

$$\times e^{-(s|z|+s|z_0|)}\,ds\ . \qquad (19b)$$

It is seen that $A_0+A_1=0$ for $z=0$, in agreement with the boundary condition. Also, A and V are zero for $z>0$, confirming the fact that there can be no flow beyond the barrier.

The integrals in Eqs. (19a) and (19b) can be evaluated in terms of the complete elliptic integrals $\mathrm{K}(k)$ and $\mathrm{E}(k)$.[10] In particular, the potential A_1, which will be needed for the calculation of the interaction energy E_1, is found to be

$$A_1(r,z)=-\tfrac{1}{2}(\kappa/\pi k)(r_0/r)^{1/2}[(2-k^2)$$

$$\times\mathrm{K}(k)-2\mathrm{E}(k)]\ ,$$

with $$k^2=4rr_0[(|z|+|z_0|)^2+(r+r_0)^2]^{-1}\ . \qquad (20)$$

Denoting the velocities arising from the ring and boundary vorticities by $\vec{V}_0=\nabla\times\vec{A}_0$ and $\vec{V}_1=\nabla\times\vec{A}_1$, the kinetic energy of the field $\vec{V}=\vec{V}_0+\vec{V}_1$ will be given by

$$E=\tfrac{1}{2}\rho_s\int\int\int|\vec{V}_0+\vec{V}_1|^2dv\ .$$

The integration volume now corresponds with the region outside the core surface S_0 and inside the infinite hemisphere $r^2+z^2=R^2$, $z<0$ with $R\to\infty$ (see Fig. 2). Using Eq. (11) and the divergence theorem, this integral can again be written as a surface integral over S_0, the contributions over the plane $z=0$ and the hemisphere being zero by virtue of the boundary condition for A and the behavior of A for $R\to\infty$:

$$E=-\tfrac{1}{2}\rho_S\iint_{S_0}(\vec{A}_0+\vec{A}_1)\times(\vec{V}_0+\vec{V}_1)\cdot\vec{n}\,dS\,. \qquad (21)$$

For small values of a/r_0, V_0 and A_0 on S_0 are of the order r_0/a and $\ln(r_0/a)$, respectively, while A_1 and V_1 remain finite throughout the core region. Thus, only the products $\vec{A}_0\times\vec{V}_0$ and $\vec{A}_1\times\vec{V}_0$, implicit in the integrand of Eq. (21), can yield significant contributions to E. The first of these products yields the energy E_0 of the unobstructed vortex ring, while the second product represents the interaction energy E_1. The surface integral can be reduced to a contour integral along the circumference C of the core cross section. In this way, the interaction energy for an infinitely thin vortex ring near a plane wall is found to be

$$E=\pi r_0\rho_S\lim_{a\to0}\oint_C(\vec{A}_1\times\vec{V}_0)\cdot\vec{n}\,dC$$

$$=-\pi r_0\rho_S\kappa A_1(r_0,-z_0)\,. \qquad (22)$$

If a/r_0 is sufficiently small, this result together with Eq. (15) can be used to approximate the energy E:

$$E=E_0-E_1=\tfrac{1}{2}\rho_S\kappa^2r_0[\ln(8r_0/a)-2]$$

$$+\pi r_0\rho_S\kappa A_1(r_0,-z_0)\,. \qquad (23)$$

This formula can be used for the kinetic energy of a vortex ring near a plane wall, provided that a/r_0 is small and the distance between the core and the wall remains large compared with a. With these restrictions, the energy E is found to be

$$E(r_0,z_0)=\tfrac{1}{2}\rho_S\kappa^2r_0[\ln(8r_0/a)-2-k^{-1}(2-k^2)$$

$$\times K(k)+2k^{-1}E(k)],$$

with $$k^2=r_0^2(r_0^2+z_0^2)^{-1}\,. \qquad (24)$$

It is seen that E is a function of the core position, denoted by the coordinate pair r_0, $-z_0$. At sufficiently low temperatures, the motion of the core through the fluid will be without dissipation, and the path taken by the vortex ring is simply given by the implicit equation $E(r_0,z_0)=\text{const}$.

For given values of E and a, the path can be calculated by taking r_0 as the independent variable and solving for z_0. It is found that r_0 increases with decreasing values of z_0, in accordance with earlier considerations based on the theory of images.[11] In the present treatment of the problem, this behavior is understood to be caused by the increase of E_1 when the ring approaches the obstacle, necessitating a corresponding increase of E_0 if the total energy E is to remain constant. Conversely, $E_1\to0$ for $z_0\to\infty$, and with large values of z_0 the radius of the ring tends to the limiting value r_∞. With this limit, the path of the ring can be written in the following form:

$$(r_\infty/r_0)[\ln(8r_\infty/a)-2]=\ln(8r_0/a)-2-k^{-1}(2-k^2)$$

$$\times K(k)+2k^{-1}E(k)\,. \qquad (25)$$

Figure 3 shows the path of the ring, as calculated from Eq. (25), in terms of the normalized parameters r_0/r_∞, z_0/r_∞, and a/r_∞. The curves were computed and plotted with a Philips Electrologica EL-X8 computer, the elliptic integral procedures of Bulirsch[12] being used.

The energy $E(r_0,z_0)$, associated with a given core position, can be seen as the work done against a force acting on the core. Per unit length of the core, this force will then be given by

$$\vec{F}_1=-(2\pi r_0)^{-1}\nabla E\,. \qquad (26)$$

Physically, this force is caused by the Magnus effect and arises from the fact that the core vorticity is subjected to the velocity fields of the obstacle and the core itself. Since the ring must remain in mechanical equilibrium, the core tends to move with a velocity $\vec{V}_c$ (not to be confused with the critical velocity $\vec{v}_c$), resulting in a second Magnus force $\vec{F}_2$ which balances $\vec{F}_1$. By the Kutta-

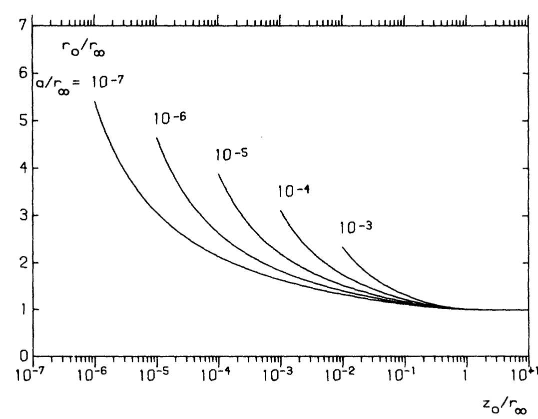

FIG. 3. Vortex-ring paths towards a plane wall. The coordinates r_0 and z_0 of the core and the core radius a are given in units of r_∞, which is the value of the ring radius for $z_0\to\infty$.

Joukowski theorem, $\vec{F}_2$ (per unit core length) is given by

$$\vec{F}_2 = \rho_s(\vec{\kappa} \times \vec{V}_c) . \tag{27}$$

With the equilibrium condition $\vec{F}_1 + \vec{F}_2 = 0$, $\vec{V}_c$ can be obtained from Eqs. (26) and (27). The components of $\vec{V}_c$ are found to be

$$V_{cr} = -(2\pi r_0 \rho_s \kappa)^{-1} \frac{\partial E}{\partial z_0} ,$$
$$V_{cz} = (2\pi r_0 \rho_s \kappa)^{-1} \frac{\partial E}{\partial r_0} . \tag{28}$$

Using the expression for E provided by Eq. (24), V_{cr} and V_{cz} can also be expressed in terms of the complete elliptic integrals $\mathrm{E}(k)$ and $\mathrm{K}(k)$. This yields the following formulas:

$$V_{cr} = -\frac{\kappa k z_0}{4\pi r_0^2}\left[2\mathrm{K}(k) - \left(\frac{2-k^2}{1-k^2}\right)\mathrm{E}(k)\right] , \tag{29a}$$

$$V_{cz} = \frac{\kappa}{4\pi r_0}\left[\ln(8r_0/a) - 1 - \frac{\mathrm{K}(k) - \mathrm{E}(k)}{k}\right] , \tag{29b}$$

with $k^2 = r_0^2(r_0^2 + z_0^2)^{-1}$.

Together with Eq. (25), these formulas represent the solution for the problem of a vortex ring

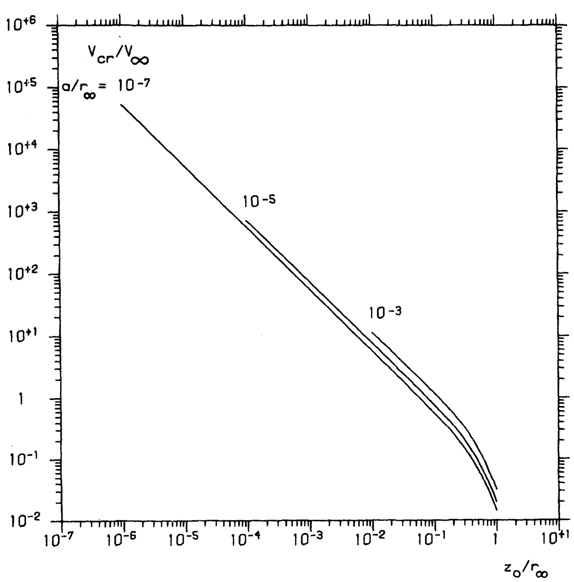

FIG. 4. Radial core velocity of a vortex ring at a distance z_0 from a plane wall. The velocity is given in units of V_∞ (the axial core velocity for $z_0 \to \infty$), while the core radius a and the distance z_0 are in units of r_∞ (the value of the ring radius for $z_0 \to \infty$).

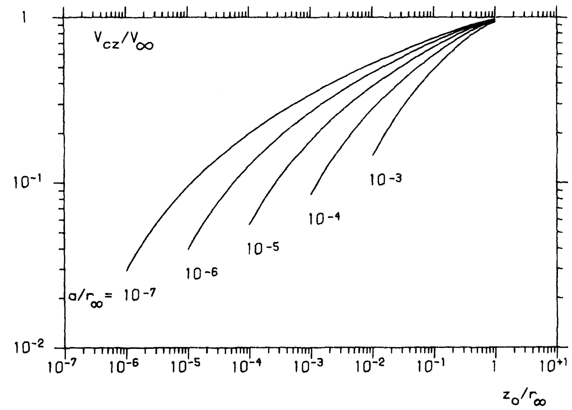

FIG. 5. Axial core velocity of a vortex ring at a distance z_0 from a plane wall. The velocity is given in units of V_∞ (the axial core velocity for $z_0 \to \infty$), while the core radius a and the distance z_0 are in units of r_∞ (the ring radius for $z_0 \to \infty$).

moving towards an infinitely extended flat plane. At large distances from the obstacle, the ring moves with a constant axial velocity given by

$$V_\infty = \lim_{z_0 \to \infty} V_{cz} = \kappa(4\pi r_\infty)^{-1}[\ln(8r_\infty/a) - 1] . \tag{30}$$

Figures 4 and 5 show the behavior of the normalized velocities V_{cr}/V_∞ and V_{cz}/V_∞. It is seen that, while the axial velocity drops slowly to zero, the radial velocity increases explosively when the ring approaches the obstacle.

The above analysis indicates that, once $\bar{f}(s)$ has been obtained from the boundary condition, the problem of finding E and $\vec{V}_c$ reduces to the Hankel inversion of $\bar{A}_1(s,z)$. This inversion can be avoided in the calculation of certain other quantities, examples of which are the force exerted on the obstacle by a stationary vortex ring or the impulse associated with the boundary vorticity $f(r)$. In these cases the calculations can be carried out in the s domain, use being made of the following properties of the Hankel transformation. If $\bar{f}(s)$ is the first-order Hankel transform of $f(r)$, the following theorem holds:

$$\lim_{s\to 0} 2\frac{\bar{f}(s)}{s} = \lim_{s\to 0} 2\int_0^\infty \frac{J_1(rs)}{s} r f(r)\, dr$$
$$= \int_0^\infty r^2 f(r)\, dr . \tag{31}$$

Thus (provided the limit exists) the second moment of $f(r)$ can be found from a simple limit operation in the s domain.

Also, if $\bar{f}(s)$ and $\bar{g}(s)$ are the νth-order Hankel transforms of $f(r)$ and $g(r)$, there is a relation of the Parseval type[13] which states that, provided $\nu \geq -\frac{1}{2}$,

$$\int_0^\infty rf(r)g(r)\,dr = \int_0^\infty s\bar{f}(s)\bar{g}(s)\,ds. \qquad (32)$$

The theorem given by Eq. (31) can be used to find the impulse associated with $f(r)$. The concept of the impulse of a vortex ring was introduced by Kelvin. This has the dimension of momentum and is given by Lamb[14]

$$P_0 = \pi\kappa\rho_s r_0^{\,2} .$$

The boundary vorticity can be considered as the vorticity of an ensemble of elementary vortex rings of impulse

$$dP_0 = \kappa\rho_s r^2 f(r)\,dr.$$

Integration over the whole surface of the obstacle yields the total impulse of this vortex sheet:

$$P_0 = \pi\rho_s \int_0^\infty r^2 f(r)\,dr.$$

Using the expression for $\bar{f}(s)$ from Eq. (18) and the theorem of Eq. (31) yields the following result:

$$P_0 = -\pi\rho_s\kappa r_0 \lim_{s\to 0} s^{-1}J_1(r_0 s)e^{-s|z_0|} = -\pi\rho_s\kappa r_0^{\,2}.$$

Thus, the impulse of $f(r)$ is equal and opposite to the impulse of the inducing vortex ring. In the analogous electromagnetic problem, this corresponds to the statement that the magnetic moment of the electric-current loop is compensated by the magnetic moment of the induced eddy currents on the superconducting surface.

If the vortex ring is kept fixed, there will be a mutually attractive force between the ring and the obstacle. This force can be regarded as a Magnus force, which is caused by the radial components of $\vec{V}_0$ and $\vec{V}_1$ acting, respectively, on the boundary and core vorticities. Integrating over the plane $z=0$, the net attractive force on the obstacle is given by

$$\vec{F} = 2\pi\rho_s \vec{e}_z \int_0^\infty rf(r)V_{0r}(r,0)\,dr$$

$$= 2\pi\rho_s \vec{e}_z \int_0^\infty s\bar{f}(s)\bar{\vec{V}}_{0r}(s,0)\,ds.$$

Here, Eq. (32) has been used to express $\vec{F}$ by an integral in the s domain. The radial component $\bar{V}_{0r}(s,0)$, appearing in the integrand, can be found by differentiation of $\bar{A}_0(s,z)$. After substitution of $\bar{V}_{0r}(s,0)$ and $\bar{f}(s)$, $\vec{F}$ is found to be

$$\vec{F} = -\pi\rho_s\kappa^2 r_0^{\,2}\vec{e}_z \int_0^\infty sJ_1^{\,2}(r_0 s)e^{-2s|z_0|}\,ds$$

$$= \frac{\rho_s\kappa^2 k z_0}{2r_0}\vec{e}_z\left[2\mathrm{K}(k) - \left(\frac{2-k^2}{1-k^2}\right)\mathrm{E}(k)\right],$$

with $k^2 = r_0^{\,2}(r_0^{\,2}+z_0^{\,2})^{-1}$.

Comparing this result with the formula for V_{cr}, as given by Eq. (29a), it is seen that $\vec{F}$ satisfies the relation

$$\vec{F} = -2\pi r_0\rho_s(\vec{V}_{cr}\times\vec{\kappa}) .$$

Thus, $\vec{F}$ is seen to be equal but opposite to the net axial Magnus force acting on the ring. This result can also be understood from the classical law of vortex motion, which requires the ring to move with the velocity of the surrounding fluid. Should the vortex ring be kept fixed, then its core will be subjected to a relative velocity $-\vec{V}_c$. By the Kutta-Joukowski theorem, this results in a Magnus force with a net axial component as given above.

IV. ENERGY OF VORTEX RING NEAR COAXIAL DISK

The obstacle to be considered in this section is an infinitely thin disk occupying the region $r \leq r_d$ of the z plane. A vortex ring of circulation κ and aperture $2r_0$ is assumed to be situated coaxially with and at a distance z_0 to the left of the disk. It is desired to find the kinetic energy of the resulting fluid flow and the motion of the ring in the vicinity of the disk.

As before, the velocity will be derived from a vector potential $\vec{A} = \vec{A}_0 + \vec{A}_1$, which satisfies Eq. (16) and vanishes on the surface of the obstacle. As a result, $A_0(r,0) + A_1(r,0) = 0$ on the disk surface, and writing $A_1(r,0)$ as an integral over $\bar{f}(s)$ yields the integral equation

$$\tfrac{1}{2}\int_0^\infty \bar{f}(s)J_1(rs)\,ds = -A_0(r,0), \qquad (r<r_d) . \qquad (33a)$$

Conversely, $f(r)$ must become zero over the remainder of the z plane, this resulting in a second integral equation for $\bar{f}(s)$:

$$\int_0^\infty s\bar{f}(s)J_1(rs)\,ds = 0, \qquad (r > r_d)\;. \tag{33b}$$

Equations (33a) and (33b) constitute a pair of "dual" integral equations of a type investigated by Sneddon,[8] who gives the following solution for $s\bar{f}(s)$ (with $r_d = 1$):

$$\bar{f}(s) = -2\int_0^1 \left(\frac{2s}{\pi t}\right)^{1/2} J_{1/2}(st)\,\frac{d}{dt}\left(t^2\int_0^1 \frac{\zeta^2 A_0(\zeta t, 0)}{(1-\zeta^2)^{1/2}}\,ds\right) dt. \tag{34}$$

Substituting A_0 from Eq. (19a) in Eq. (34) results in a triple integral for $\bar{f}(s)$:

$$\bar{f}(s) = -\kappa r_0\int_0^1 \left(\frac{2s}{\pi t}\right)^{1/2} J_{1/2}(st)\,dt\int_0^\infty e^{-y|z_0|}J_1(r_0 y)\,dy\,\frac{d}{dt}\int_0^1 \frac{t^2\zeta^2 J_1(\zeta t y)}{(1-\zeta^2)^{1/2}}\,d\zeta\;.$$

The integration with respect to ζ can be carried out to give[15]

$$\int_0^1 \frac{\zeta^2 J_1(\zeta y t)}{(1-\zeta^2)^{1/2}}\,d\zeta = \left(\frac{\pi}{2yt}\right)^{1/2} J_{3/2}(yt) = -\frac{d}{d(yt)}\left(\frac{\sin(yt)}{(yt)}\right).$$

Using this result and the well-known relation $\sin(x) = \frac{1}{2}\pi x^{1/2}J_{1/2}(x)$, $\bar{f}(s)$ can be represented by the double integral

$$\bar{f}(s) = -2\kappa r_0\pi^{-1}\int_0^{r_d} dt\int_0^\infty e^{-y|z_0|}J_1(r_0 y)\sin(st)\sin(yt)\,dy, \tag{35}$$

where r_d has been reinserted in the upper limit of the outer integral. Multiplication of Eq. (35) by the factor $(1/2s)\exp(-s|z|)$, and application of the inverse Hankel transform, results in

$$A_1(r, z) = -\kappa r_0\pi^{-1}\int_0^{r_d} dt\int_0^\infty e^{-s|z|}\sin(st)J_1(rs)\,ds\int_0^\infty e^{-y|z_0|}\sin(yt)J_1(r_0 y)\,dy\;. \tag{36}$$

As before, the total energy E can be expressed by an integral over the boundary surfaces of the velocity field, i.e., the infinite sphere about the origin, the disk surface, and the core surface S_0. With the same arguments as used in Sec. III, only S_0 is found to yield a significant contribution to E, which is again given by Eq. (23). Thus, the problem of finding E reduces to one of determining the quantity $A_1(r_0, -z_0)$, appearing in the formula for the interaction energy.

Referring to Eq. (36), it is seen that the integrals in s and y become identical for $(r, z) = (r_0, -z_0)$. A further reduction occurs if $r_0 > r_d$ and $z = z_0 = 0$ (vortex ring surrounding the disk in the plane $z = 0$). In this case, the exponential factors vanish from the integrand in Eq. (36), and the integrals in s and y reduce to a special form of the Weber-Schafheitlin integral,[16] as

$$\int_0^\infty J_1(r_0 x)\sin(tx)\,dx = (t/r_0)(r_0^2 - t^2)^{-1/2}, \qquad (t < r_0)$$

$$\int_0^\infty J_1(r_0 x)\sin(tx)\,dx = 0\;, \qquad (t > r_0)\;.$$

With this result the integral in Eq. (36) becomes elementary, and $A_1(r_0, 0)$ is given by (with $r_0 > r_d$)

$$A_1(r_0, 0) = \frac{\kappa}{\pi r_0}\left(r_d + \tfrac{1}{2}r_0\ln\left|\frac{r_0 - r_d}{r_0 + r_d}\right|\right).$$

Substituting in Eq. (23), the normalized energy $E/(\rho_s\kappa^2 r_d)$ is found to be

$$\frac{E(r_0, 0)}{\rho_s\kappa^2 r_d} = \frac{r_0}{2r_d}\left[\ln\left(\frac{8r_0}{a}\right) - 2\right] + 1 + \frac{r_0}{2r_d}\ln\left|\frac{r_0 - r_d}{r_0 + r_d}\right|\;. \tag{37}$$

Figure 6 shows the behavior of the normalized energy as a function of $\Delta = (r_0/r_d) - 1$, which is the distance between the rim of the disk and the core center, normalized to r_d. For $\Delta < 0.1$, the normalized en-

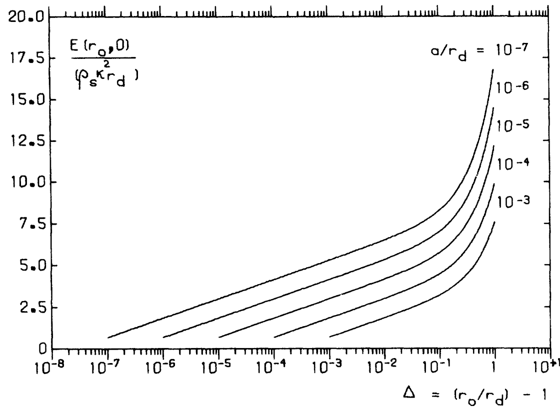

FIG. 6. Energy, in units of $\rho_s\kappa^2 r_d$, of a vortex ring of strength κ surrounding a coplanar disk of radius r_d. The energy is given as a function of the distance Δ from the core center to the rim of the disk for various core radii a (Δ and a in units of r_d).

ergy behaves approximately as $\frac{1}{2}\ln(4\Delta r_d/a)$, indicating a strong interaction between the disk and the vortex ring. For larger values of Δ, the energy rapidly approaches the energy of an unobstructed ring of the same size. Conversely, the energy approaches the limiting value ln(2) if the core is allowed to touch the rim of the disk ($\Delta = a/r_d$, and $a \to 0$). Of course, this situation is no longer covered by the assumptions made at the outset of the analysis, since the distribution of κ over the core surface will certainly become nonuniform if the core is allowed to approach the obstacle as close as this.

If $z_0 \neq 0$ (vortex ring not in the plane $z=0$), $A_1(r_0, -z_0)$ can be evaluated by taking $p = |z_0| - it$. The integrals in s and y, appearing in Eq. (36), are then found as the imaginary part of the known Laplace transform

$$\int_0^\infty e^{-px} J_1(r_0 x)\,dx = r_0^{-1}[1 - p(p^2 + r_0^2)^{-1/2}] \,. \tag{38}$$

The remaining integral in t is of algebraic type and can be reduced to elementary functions and incomplete elliptic integrals. This reduction is a standard, if lengthy procedure, and will be omitted here. Ultimately, the normalized energy is given by

$$\frac{E(r_0, z_0)}{\rho_s \kappa^2 r_d} = \frac{r_0}{2r_d}\left[\ln\left(\frac{8r_0}{a}\right) - 2\right] + \tfrac{1}{2}[1 - (1 - k^2\sin^2\varphi)^{1/2}] + \frac{k}{8}\mathrm{cotan}(\tfrac{1}{2}\varphi)\ln\left(\frac{1 - k\sin\varphi}{1 + k\sin\varphi}\right)$$
$$+ \tfrac{1}{2}\mathrm{cotan}(\tfrac{1}{2}\varphi)[\mathrm{E}(\varphi, k) - (1 - \tfrac{1}{2}k^2)\mathrm{F}(\varphi, k)] \,, \tag{39}$$

with $\quad k^2 = r_0(r_0^2 + z_0^2)^{-1}, \quad \tan(\tfrac{1}{2}\varphi) = r_d(r_0^2 + z_0^2)^{-1/2} \,,$

where $\mathrm{F}(\varphi, k)$ and $\mathrm{E}(\varphi, k)$ are used to denote the incomplete elliptic integrals of the first and second kind.[10]

Using the relation $E(r_0, z_0) = \text{const}$ and the result of Eq. (39), it is possible to calculate the path of the vortex ring. Figure 7 shows the computed paths for $a/r_d = 10^{-4}$ and various values of Δ. The associated values of the ring energy E can be taken from Fig. 6. The paths are seen to be symmetric about the disk, the vortex ring attaining its largest size in passing the plane $z=0$.

V. ENERGY OF A VORTEX RING NEAR A PLANE WITH A CIRCULAR APERTURE

The obstacle will be taken as an infinitely thin plane occupying the z plane with the exclusion of the circular region $r < r_a$. As before, a vortex ring of aperture $2r_0$ and circulation κ is placed coaxially with and at a distance z_0 to the left of the obstacle. Also, the resulting velocity field $\vec{V}$ will be taken as the curl of the vector potential $\vec{A} = \vec{A}_0 + \vec{A}_1$, which satisfies Eq. (16) and is zero on the obstacle surface. In the present situation, the boundary condition $A(r, 0) = 0$ for $r > r_a$ can no longer be justified with the argument that $A(0, 0)$ should remain finite (see Sec. III), since the origin is no longer on the obstacle surface. However, it can be shown that the same condition follows *a posteriori* if the flux of $\vec{V}$ through the obstacle aperture is required to be zero.

Under the stated conditions, the transformed boundary vorticity $\bar{f}(s)$ must satisfy the dual integral equations

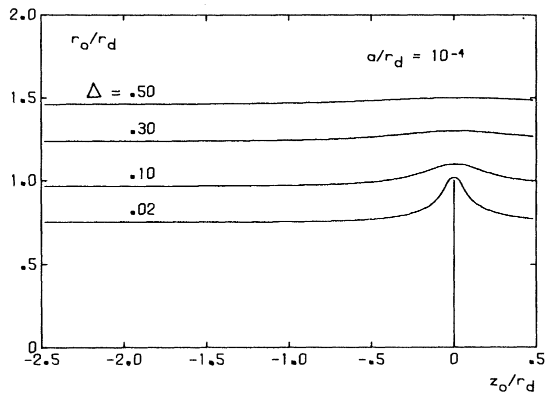

FIG. 7. Vortex-ring paths past a coaxial disk for various values of the crossing distance Δ between the ring and the disk. The coordinates r_0 and z_0 of the core Δ and the core radius ω are given in units of the disk radius r_d.

$$\int_0^\infty s\bar{f}(s)J_1(rs)\,ds = 0, \qquad (r < r_a) \tag{40a}$$

$$\frac{1}{2}\int_0^\infty \bar{f}(s)J_1(rs)\,ds = -A_0(r,0), \qquad (r_a < r < \infty). \tag{40b}$$

The solution of this system, as given by Sneddon,[8] is found to be

$$\bar{f}(s) = 2\int_0^\infty t\left(\frac{2st}{\pi}\right)^{1/2} J_1(st)\frac{d}{dt}\left(\int_\tau^\infty \frac{A_0(\tau,0)}{(\tau^2-t^2)^{1/2}}\,d\tau\right)dt\,. \tag{41}$$

Substituting $A_0(\tau, 0)$ and performing the integration with respect to τ, $\bar{f}(s)$ and, subsequently, $A_1(r,z)$ are given by the formulas

$$\bar{f}(s) = -\kappa r_0\int_{r_a}^\infty ts^{1/2}J_{3/2}(st)\,dt\int_0^\infty y^{1/2}J_1(yt)J_1(r_0y)e^{-y|z_0|}\,dy\,, \tag{42}$$

$$A_1(r,z) = -\tfrac{1}{2}\kappa r_0\int_{r_a}^\infty t\,dt\int_0^\infty s^{1/2}J_{3/2}(st)J_1(rs)e^{-s|z|}\,ds\int_0^\infty y^{1/2}J_{3/2}(yt)J_1(r_0y)e^{-y|z_0|}\,dy. \tag{43}$$

As in Sec. IV, it is readily seen that the kinetic energy is again given by Eq. (23) and can be obtained explicitly once $A_1(r_0, -z_0)$ is known. It is found instructive to consider first the case $z = z_0 = 0$ (vortex ring in the obstacle aperture). Referring to Eq. (43), the integrals in s and y are found to reduce to another Weber-Schafheitlin integral, namely,

$$\int_0^\infty x^{1/2}J_1(rx)J_{3/2}(x)\,dx = (r/t)[\tfrac{1}{2}\pi t(t^2-r^2)]^{-1/2}\,, \qquad (t > r)$$

$$\int_0^\infty x^{1/2}J_1(rx)J_{3/2}(x)\,dx = 0\,, \qquad (t < r).$$

Substituting this result in Eq. (43) and taking $r = r_0$, the remaining integral in t can be performed to give

$$A_0(r_0, 0) = \frac{\kappa\pi^{-1}}{r_a}\left(r_0 + \frac{r_a}{2}\ln\left|\frac{r_a - r_0}{r_a + r_0}\right|\right), \qquad (r_0 < r_a)\,.$$

Using Eq. (23), the normalized energy $E/(\rho_s\kappa^2 r_a)$ is now found to be

$$\frac{E(r)_0, 0)}{\rho_s\kappa^2 r_a} = \frac{r_0}{2r_a}\left(\ln\frac{8r_0}{a-2}\right) + \left(\frac{r_0}{r_a}\right)^2 + \frac{r_0}{2r_a}\ln\left|\frac{r_a - r_0}{r_a + r_0}\right|\,. \tag{44}$$

Figure 8 shows the behavior of the normalized energy as a function of $\delta = 1 - (r_0/r_a)$, being the normalized distance between the aperture edge and the core center. For $\delta < 0.01$, the energy is seen to behave in a way analogous to the case of a vortex ring surrounding a disk. Beyond this value, a maximum occurs

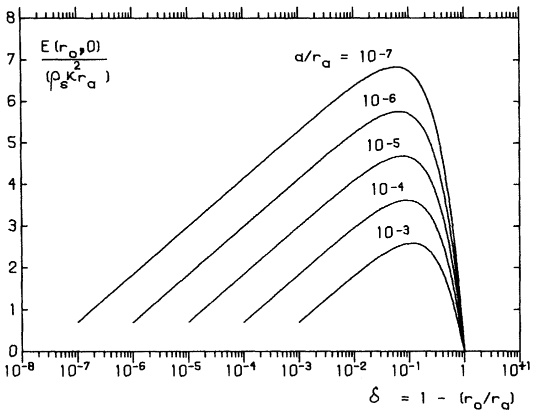

FIG. 8. Energy, in units of $\rho_s\kappa^2 r_a$, of a vortex ring of strength κ in a coaxial aperture of radius r_a. The energy is given as a function of the distance δ from the core center to the aperture edge for various core radii a (δ and a in units of r_a).

in the vicinity of $\delta = 0.1$, after which the energy drops rapidly to zero for $\delta \to 1$. Denoting the maximum value of $E(r_0, 0)$ and the associated value of r_0, by E_{max} and r_{max}, some interesting conclusions can be drawn: (i) Since E_{max} is the maximum possible energy in the aperture, vortex rings with an energy in excess of this critical value will be unable to pass through; (ii) there are two different values of r_0, situated on either side of r_{max}, which can be associated with a given $E < E_{max}$; (iii) vortex rings which are situated on opposite sides of r_{max}, and have the same sense of circulation, will pass through the aperture in opposite directions; and (iv) a vortex ring, situated in the aperture at $r_0 = r_{max}$, will be at rest.

Conclusions (i) and (ii) follow directly from the functional behavior of $E(r_0, 0)$, as given by Eq. (44) and shown in Fig. 8. Conclusions (iii) and (iv) follow from Eq. (28), giving the components of the core velocity $\vec{V}_c$. Since $E(r_0, z_0)$ is an even function of z_0, V_{cr} (being proportional to $\partial E/\partial z_0$) is zero in the aperture. During its passage through the aperture, V_c becomes equal to V_{cz}, which in turn is proportional to $\partial E/\partial r_0$. As the derivative $\partial E/\partial r_0$ changes sign at $r_0 = r_{max}$, vortex rings on opposite sides of r_{max} must move in opposite directions. For $r_0 = r_{max}$, both derivatives are zero, and the ring will be at rest.

Since the position $r_0 = r_{max}$ is seen to divide the vortex rings in the aperture into two distinctly different groups, it is of interest to investigate how this division is continued outside the aperture. To this end, it is necessary to find the dividing locus defined by the implicit equation

$$E(r_0, z_0) = E_{max}\,. \tag{45}$$

The calculation of $A_1(r_0, z_0)$, necessary for the determination of $E(r_0, z_0)$, can be affected in the following way. Referring to Eq. (43), it is found that the integrals in s and y become identical for $(r, z) = (r_0, z_0)$. Writing the Bessel function $J_{3/2}(x)$ in trigonometrical form, both integrals are modified as shown below:

$$\int_0^\infty x^{1/2} J_1(r_0 x) J_{3/2}(tx)\, e^{-|z_0|x}\, dx = \left(\frac{2}{\pi t}\right)^{1/2} \int_0^\infty J_1(r_0 x)\left(\frac{\sin(tx)}{(tx)} - \cos(tx)\right) e^{-|z_0|x} dx\,.$$

After this, the constituent integrals with the factors $\cos(xt)$ and $\sin(xt)$ can be evaluated by setting $p = |z_0| - it$ and by taking the real or imaginary part of the Laplace transforms given in Eq. (38) and below:

$$\int_0^\infty e^{-px} x^{-1} J_1(r_0 x)\, dx = r_0^{-1}[(p^2 + r_0^2)^{1/2} - p]\,. \tag{46}$$

As in the case of a vortex ring near a disk, the remaining integral in t can be reduced to elliptic integrals. The ultimate result for $E(r_0, z_0)$ is found to be

$$\frac{E(r_0, z_0)}{\rho_s \kappa^2 r_a} = \frac{r_0}{2r_a}\left[\ln\left(\frac{8r_0}{a}\right) - 2\right] + \tfrac{1}{2}\tan^2 \tfrac{1}{2}\psi\ \left[1 - (1 - k^2 \sin^2\psi)^{1/2}\right] + \tfrac{1}{8} k \tan(\tfrac{1}{2}\psi) \ln\left(\frac{1 - k\sin\psi}{1 + k\sin\psi}\right)$$

$$+ \tfrac{1}{2}\tan(\tfrac{1}{2}\psi)[\mathrm{E}(\psi, k) - (1 - \tfrac{1}{2}k^2)\mathrm{F}(\psi, k)]\,, \tag{47}$$

with $\quad k^2 = r_0^2(r_0^2 + z_0^2)^{-1/2}, \quad \cot(\tfrac{1}{2}\psi) = r_a(r_0^2 + z_0^2)^{-1/2}\,.$

As before, $\mathrm{F}(\psi, k)$ and $\mathrm{E}(\psi, k)$ are the incomplete elliptic integrals of the first and second kind. There is a strong resemblance between the above result and the corresponding expression given in Eq. (39). In fact, if $r_a = r_d$, the angles φ and ψ in Eqs. (39) and (47) are found to be supplementary, resulting in the functional relation

$$E_{1a}(\psi, k) = \tan^2(\tfrac{1}{2}\psi) E_{1d}(\psi, k) \ , \tag{48}$$

where E_{1d} and E_{1a} are used to denote the interaction energies for a disk or for an aperture.

The dividing locus, defined by Eq. (45), can be computed from Eqs. (44) and (47). The result of these computations is given in Fig. 9, which shows the behavior of the locus in the vicinity of the aperture edge. Outside the aperture, the locus is found to split into two branches, these dividing the rz plane into the following three regions: (a) A hill-shaped region A, containing the vortex rings with $E < E_{max}$, which pass through the aperture in the positive z direction; (b) a drop-shaped region B, containing the rings of the same energy class as A, but which go through the aperture in the opposite direction; these rings remain close to the obstacle in their motion towards and away from the aperture; and (c) the remainder of the rz plane C, containing the vortex rings with $E > E_{max}$. The three regions meet in the common point $(r_{max}, 0)$, where grad E becomes zero.

Equation (47) allows the computation of the paths taken by the different types of vortex rings. Figure 10 gives examples of paths in the regions A and C. As expected, it is found that the ring in C is unable to go through the aperture and is forced to run up against the obstacle surface. Also, the rings in A are found to grow in diameter while approaching the obstacle, obtaining their maximum size in the aperture. It is seen from Fig. 10 that the relative increase of the ring diameter remains small, even for the limiting path $E = E_{max}$.

These results, describing vortex-ring behavior near an aperture in an infinite plane, may be used to give a qualitative description of vortex-ring motion in the apparatus of Fig. 11. This consists of a closed cylindrical container of superfluid, which is divided into two equal chambers by means of a central partition with a connecting aperture. Near the aperture edge, the dividing locus can be expected to approximate the shape given in Fig. 9, while its branches will parallel the wall and the central axis of the container at larger distances from the partition. Since both branches represent the same energy (E_{max}), they must close upon themselves near the ends of the container.

In this way, the locus is seen to divide the fluid into the regions C_1, C_2 (where $E > E_{max}$), and AB (where $E < E_{max}$). Consider a vortex ring in the left chamber and situated in the central part of AB. This ring will go through the aperture and traverse the right chamber to the end of the container. After this, it will go back to the left chamber along the container wall, passing through the aperture in the region B of Fig. 9. Eventually the ring returns to its starting position and will continue to shuttle back and forth between the two chambers. Conversely, a ring in C_1 or C_2 will keep circulating in the same chamber.

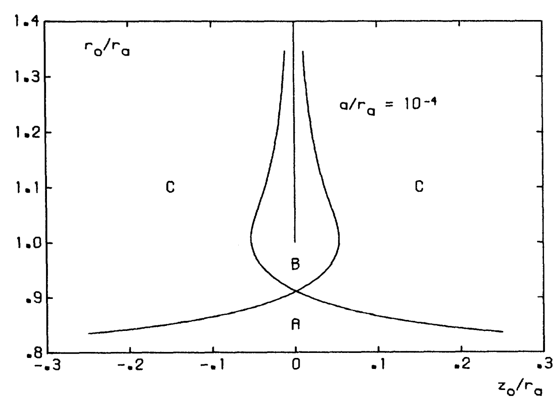

FIG. 9. The dividing locus in the vicinity of the aperture edge. The locus divides the rz plane into the regions A and B, containing the vortex rings which pass through the aperture, and the region C containing the rings which cannot pass through.

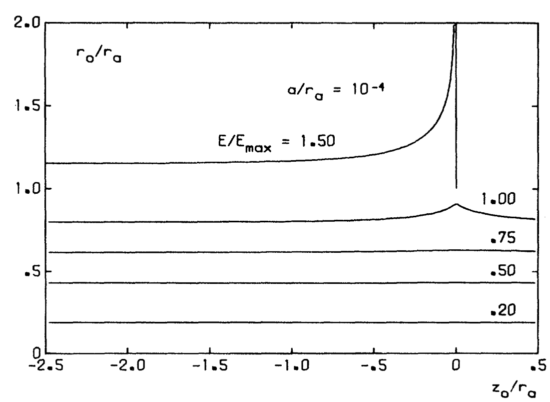

FIG. 10. Vortex-ring paths near a coaxial circular aperture for various values of the energy E. The coordinates r_0 and z_0 of the core and the core radius a are given in units of the aperture radius r_a, while E is normalized to the maximum possible energy E_{max} inside the aperture.

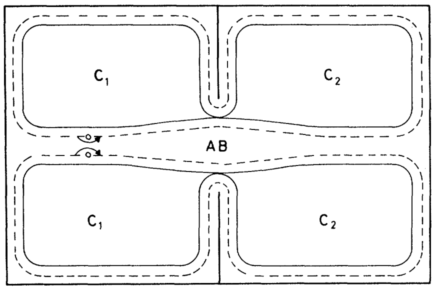

FIG. 11. Cross sectional view of a superfluid container with two chambers connected by a circular aperture in the central partition. The dividing locus (continuous line, not to scale) divides the interior in the regions C_1, C_2, and AB. The dashed line indicates the path taken by a vortex ring in AB.

VI. CONCLUSION

A calculation has been given for the kinetic energy of a vortex ring in a superfluid in the presence of various plane obstacles. The calculation method can be extended to other axisymmetric obstacles. For instance, the case of a vortex ring near a coaxial annular slit results in a system of triple-integral equation, which may be solved in a similar way.

As regards the case of the circular disk or aperture, here the associated potential problem could also be solved through the use of oblate spheroidal coordinates, where these obstacles coincide with the coordinate surfaces. In this case, the solution is obtained in terms of associated Legendre functions, which appear to be less amenable to numerical computation than the elliptic integrals employed here.

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