for the $3s$-$3p$ transition. Since the multiplet strength for both transitions will be proportional to Z^{-2}, the transiti n probability will be proportional to Z^4 for the $3p$-$4s$ transition and to Z for the $3s$-$3p$ transition. Taking the screening into account a calculation based upon the Z dependence for the mean life of the $3p$ levels in Ne I, Na II, and Mg III shows that the ratio between the mean lives for the $4s$ levels in Ne I and Na II should be 30–40, in reasonable agreement with the observed values.

ACKNOWLEDGMENT

One of us (JD) wishes to express his appreciation for the hospitality of the University of Aarhus beam-foil spectroscopy group.

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Analytically Solvable Problems in Radiative Transfer. II

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(Received 6 October 1969)

In the previous paper, the Biberman-Holstein integral equation was solved for a slab and for all line shapes of interest in the limit of high optical depth. The eigenfunctions and eigenvalues obtained are used for the general calculation of the stationary number density in the excited state, when collisional excitation and deexcitation takes place, when excitation takes place by absorption of external radiation, or by both combined. General expressions are given for the line shape of a spectral line emitted by an optically dense slab, showing typical broadening and self-reversal.

I. INTRODUCTION

In many plasmas, the assumption of (local) thermodynamic equilibrium for the number densities of atoms in low-lying levels is invalid. It is therefore necessary to solve the rate equations directly. For most levels, it can be assumed that the plasma is optically thin so that the radiation escapes without being absorbed. Resonance lines are severely absorbed, however, and allowance for this effect must be included. Up to now this has been done by fairly rough approximations[1] or by numerical calculations.[2] In a preceding paper[3] (hereafter referred to as I), we solved the transfer equation, when the optical depth was large, for a number of line shapes, Doppler profiles with and without hfs, and Voigt and Lorentz profiles. The geometry in that paper was taken to be a slab. These results are applied in order to obtain accurate solutions to the problem mentioned above for a realistic model.

In this paper, we shall consider the rate equations for the resonance states in the following approximation: We have two-level atoms consisting of a ground state and an excited (resonance) state. In Sec. II, it is assumed that the atoms are excited by collisions with electrons or absorption of resonance radiation emitted somewhere else in the volume. There is no external source of radiation. They are deexcited by radiative decay or by collisions with electrons. The system is in an enclosure (which we shall assume to be a slab) so that it can lose energy by radiation. This approximation contains the essential part of such systems in practice. It can easily be extended to apply to real atoms. This point will be discussed at the end of Sec. II, in connection with the scheme given by Bates, Kingston, and McWhirter.[4]

The fundamental solutions of the transfer equation already obtained by us are used to find the solution to this stationary problem. It will be shown that if we split up the rate equation for a level with self-absorption into rate equations for so-called quasilevels, the resulting expressions are formally the same as the expression for a level without self-absorption. They therefore fit into a scheme such as that given by Bates, Kingston, and McWhirter for an optically *thin* plasma.

In Sec. III, we apply the solutions obtained in I to the calculation of the density of excited atoms when the excitation mechanism acts by absorption of outward radiation. The formulas found in both Secs. II and III are cast in such a form that they can easily be applied to calculate the line shape of the spectral line emitted by an optically dense slab in every practical situation. This point will be discussed in Sec. IV. In particular, the formulas exhibit a characteristic broadening of the line and the well-known self-reversal.

II. STATIONARY PROBLEM IN RADIATIVE TRANSFER: COLLISIONAL EXCITATION

A. Restatement of Previous Results

The transient solution to some problems in radiative transfer has been studied in I. Our aim is now to discuss the stationary problem. Since the results obtained in I will often be referred to in the following sections, we shall restate them briefly here.

Suppose that at $t=0$ there exists in a volume V a certain density of atoms excited in the resonance state. It has been shown by Biberman[5] and Holstein[6] that the equation describing the decay is given by

$$\frac{\partial n}{\partial t} = -\gamma n + \gamma \int_V K(|\vec{r}-\vec{r}'|)\, n(\vec{r}')\, d\vec{r}' , \qquad (1)$$

where $n(\vec{r}, t)$ is the density of excited atoms and γ^{-1} the natural lifetime of the excited atoms. The integration is over the volume V. The integral kernel K is

$$K(|\vec{r}-\vec{r}'|) = \int_0^\infty d\nu\, \mathfrak{L}(\nu) k(\nu)\, \frac{\exp[-k(\nu)|\vec{r}-\vec{r}'|]}{4\pi|\vec{r}-\vec{r}'|^2} . \qquad (2)$$

$\mathfrak{L}(\nu)$ is the line shape of the spectral line; $k(\nu)$ is the absorption coefficient. The crucial assumption in Eqs. (1) and (2) is that the emission profile is proportional to the absorption profile or $\mathfrak{L}(\nu) \propto k(\nu)$. This assumption is fulfilled for a Lorentz profile when the broadening is due to pressure broadening but not when there is natural broadening. For a Doppler profile it is fulfilled to satisfactory approximation.[7] If we assume in Eq. (1) an exponential decay

$$n(\vec{r}, t) = n(\vec{r}) \exp[-\beta t] ,$$

we obtain

$$(1-\beta/\gamma)\, n(\vec{r}) = \int_V K(|\vec{r}-\vec{r}'|)\, n(\vec{r}')\, d\vec{r}' . \qquad (3)$$

This is a homogeneous eigenvalue problem. In I, the eigenvalues β_j and the eigenfunctions ψ_j were determined for a slab with thickness L and for large optical depth. For a Doppler profile

$$\mathfrak{L}(\nu)\, d\nu = \mathfrak{L}(u)\, du \equiv \frac{e^{-u^2}}{\sqrt{\pi}}\, du , \qquad u = 2\left(\frac{\nu-\nu_0}{\Delta\nu_D}\right)(\ln 2)^{1/2} ,$$

and absorption coefficient

$$k(\nu) = k(u) = k_0 \mathfrak{L}(u) , \qquad k_0 = \frac{2\pi e^2}{mc}\, \frac{Nf}{\Delta\nu_D} (\ln 2)^{1/2} ,$$

we have the eigenvalues

$$\frac{\beta_j}{\gamma}\, \frac{k_0 L}{\sqrt{\pi}} \left(\ln \frac{k_0 L}{2\sqrt{\pi}}\right)^{1/2} \sim \mu_j , \quad k_0 L \to \infty \qquad (4)$$

and the eigenfunctions (independent of possible hfs)

$$\psi_j \sim (1-\xi^2)^{1/2} \sum_{m=0}^{\infty} a_{m,j}\, U_m(\xi), \quad \xi = 2x/L . \qquad (4')$$

The numbers μ_j have been given in Table I of I and the coefficients $a_{m,j}$ in the Tables III and IV of I. For a Lorentz profile

$$\mathfrak{L}(\nu)\,d\nu = \mathfrak{L}(u)\,du \equiv \frac{1}{\pi}\,\frac{du}{u^2+1}\,, \qquad u = 2\left(\frac{\nu-\nu_0}{\Delta\nu_L}\right),$$

and absorption coefficient

$$k(\nu) = k(u) = k_0\mathfrak{L}(u)\,, \quad k_0 = \frac{2\pi e^2}{mc}\,\frac{Nf}{\Delta\nu_L}\,,$$

the eigenvalues are

$$(\beta_j/\gamma)(k_0L)^{1/2} \sim \mu_j', \quad k_0L \to \infty\,, \tag{5}$$

and the eigenfunctions are

$$\psi_j(\xi) \sim (1-\xi^2)^{1/4}\sum_{m=0}^{\infty} b_{m,j}U_m(\xi)\,. \tag{5'}$$

The numbers μ_j' have been given in Table II of I and the coefficients $b_{m,j}$ in Tables V and VI of I. The eigenfunctions constitute a complete orthonormal system. Every function can be expanded into a series of those eigenfunctions which is convergent in mean square. Integrals in which the eigenfunctions ψ_j appear are readily evaluated in many cases by using the relation between the Tschebyscheff polynomials $U_m(\xi)$ and trigonometric functions[8]

$$U_m(\cos\varphi) = \sin(m+1)\varphi/\sin\varphi\,. \tag{6}$$

In I, we discussed the extension of the theory to more complicated line shapes, Voigt profiles, and Doppler profiles with hfs. For a Voigt profile, the results for a Lorentz profile apply provided that the requirement stated in the beginning of Appendix A of I is fulfilled. This will always be assumed. The asymptotic eigenfunctions and eigenvalues are all independent of possible hfs of the spectral line, except the eigenvalues for a Doppler profile. Here we have Eq. (26) of I. Therefore, only in this case the results need sometimes be modified. They will simply be stated since the proof of it is always straightforward.

The collective behavior of the excited atoms in Eq. (2) can be accurately described with the solutions of Eqs. (3)–(5). The solutions can equally well be used to solve the stationary problem.

B. General Solution

Let us consider two-level atoms consisting of a ground state and an excited (resonance) state. When, for instance, electrons excite and deexcite atoms and the medium is optically dense so that atoms are also excited by absorption of resonance radiation emitted somewhere else in the volume, we have

$$A(2,1)\,n(2) + n_e\,n(2)K(2,1) = n_e\,n(1)K(1,2)$$
$$+A(2,1)\int K(|\vec{r}-\vec{r}'|)\,n(2)\,d\vec{r}'\,. \tag{7}$$

We have changed our notation in accordance with the convention adopted by Bates, Kingston, and McWhirter[1,4]: $n(1)$ (independent of place) and $n(2)$ (dependent on place) are the number densities in the levels 1 and 2 [ground state and first excited (or resonance) state]; n_e is the electron density; $K(1,2)$ and $K(2,1)$ are the rate coefficients ($\mathrm{cm}^3/\mathrm{sec}$) for collisional excitation and deexcitation by electrons. Finally, $A(2,1)(=\gamma)$ is the radiative decay constant. The coefficient $K(1,2)$ and $K(2,1)$ are dependent on the electron temperature T_e. For convenience we shall take T_e independently of place in the following discussion. This requirement is often fulfilled due to the high thermal conductivity of the electrons.[9] However, it is unnecessary to make this assumption. The formulas can easily be generalized to cover this case as well.

Since the eigenfunctions of the homogeneous problem Eq. (3) constitute a complete set, we can expand the number density $n(2)$ in Eq. (7), as

$$n(2)/n(1) = \sum_{i=1}^{\infty}\alpha_i\psi_i(\xi)\,. \tag{8}$$

The eigenfunctions ψ_j are selected according to the broadening mechanism. If the line shape is determined by Doppler broadening with or without hfs, we should use Eq. (4). If there is pressure broadening and the line shape is a Lorentz or Voigt ($a\neq 0$) profile, Eq. (5) is taken.

By substituting Eq. (8) in Eq. (7), the term with the Biberman-Holstein integral kernel is immediately resolved, because of Eq. (3). We now take the inner product with an eigenfunction ψ_j. Since the eigenfunctions constitute an orthonormal set, we find (for $j = 1, 2, \ldots$)

$$\tilde{A}_j(2,1)\alpha_j + K(2,1)\sum_{i=1}^{\infty}\alpha_i\int_{-1}^{+1}\psi_j n_e\psi_i\,d\xi$$
$$= K(1,2)\int_{-1}^{+1} n_e\,\psi_j(\xi)\,d\xi\,, \tag{9}$$

$$\xi = 2x/L\,,$$

where L is the breadth of the slab. $\tilde{A}_j(2,1)$ is our former β_j. The letter β has already been used by

Bates, Kingston, and McWhirter[1,4] for the rate coefficient for radiative recombination. Equation (9) constitutes a set of linear equations for the coefficients α_j of the type

$$(A+B)\vec{a}=\vec{b}\ , \tag{10}$$

with $A=[\tilde{A}_j(2,1)\delta_{ij}],\quad B=[K(2,1)\int_{-1}^{+1}\psi_j n_e \psi_i\, d\xi]$.

The vector $\vec{b}$ is given by $\vec{b}=(b_1,\ b_2,\ \ldots)$

with $b_j=K(1,2)\int_{-1}^{+1} n_e\psi_j(\xi)\,d\xi$;

the vector $\vec{a}=(\alpha_1,\ \alpha_2,\ \ldots)$ is to be determined. First, it should be noted that when the problem has reflection symmetry, as is often the case, all odd eigenfunctions must vanish from the solution, i.e., the corresponding expansion coefficients must be zero. The matrix elements of B for the even eigenfunctions can be calculated from their expansion in Tschebyscheff polynomials of the second kind, using Eq. (6). The off-diagonal elements of the matrix B couple the different modes to one another. Their role here is quite similar to the part they play in quantum mechanics, where they describe the transitions between different states. At the end of this section, we shall introduce for these modes the concept quasilevels, so that the off-diagonal elements describe transitions between different quasilevels. After calculation of the matrix elements of B, the inverse of the matrix $A+B$ is calculated numerically and the vector $\vec{a}$ is found from the relation

$$\vec{a}=(A+B)^{-1}\vec{b}\,.$$

The matrices are infinite. It is therefore necessary to truncate them in the numerical calculation at certain orders n and $n+1$. If no appreciable changes occur for the coefficients of the vector $\vec{a}$, the order is considered high enough and the solution has been determined. One can estimate that in many cases $n=5$ will be sufficient.

Let us now simplify by assuming that n_e is constant (or that n_e varies only very smoothly compared with variations in the density of excited atoms). The matrix B becomes a diagonal matrix because of the orthogonality of the eigenfunctions. The solution of Eq. (10) is immediately obtained and we have (for $k_0L\gg 1$)

$$\frac{n(2)}{n(1)}=\sum_{j=1}^{\infty}\alpha_j\psi_j=n_eK(1,2)\,\frac{\sum_{j=1}^{\infty}\psi_j(\xi)\int_{-1}^{+1}\psi_j(\xi')\,d\xi'}{\tilde{A}_j(2,1)+n_eK(2,1)}\ . \tag{11}$$

This case has been treated numerically for a Doppler profile by Hearn[10] and for a Voigt profile by Hummer.[11] The integrals in Eq. (11) are readily evaluated by using the representation of the eigenfunctions in the Tschebyscheff polynomials and Eq. (6). After that, the summation is carried out numerically. For a Doppler profile with or without hfs, the integrals become particularly simple for

$$\int_{-1}^{+1}\psi_j(\xi')\,d\xi' = \tfrac{1}{2}\pi\, a_{0,j}\,, \tag{12}$$

$a_{0,j}$ being the first expansion coefficient of the eigenfunctions ψ_j in Eq. (4). For a Lorentz profile (and for a Voigt profile with $a\neq 0$), we obtain from Eq. (5)

$$\int_{-1}^{+1}\psi_j(\xi')\,d\xi'=\tfrac{1}{4}\sqrt{\pi}\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{1}{4})}{\Gamma(m+\frac{7}{4})}\,b_{2m,j}\,, \tag{12'}$$

where the $b_{2m,j}$ are the expansion coefficients of the even eigenfunctions. The series is quickly convergent. Note that, apart from the summation, Eq. (11) is formally the same as the expression that would have been obtained if we had neglected reabsorption of resonance radiation in Eq. (7):

$$\frac{n(2)}{n(1)}=\frac{n_eK(1,2)}{A(2,1)+n_eK(2,1)}\ . \tag{13}$$

C. Limiting Cases and Discussion

When the radiative loss is small compared to the loss by collisional deexcitation, thermodynamic equilibrium must be attained. In order to show that this is correctly yielded by the theory, we assume that $\tilde{A}_j(2,1)\ll n_eK(2,1)$ in Eq. (11) so that

$$\frac{n(2)}{n(1)}=\frac{K(1,2)}{K(2,1)}\sum_{j=1}^{\infty}\psi_j(\xi)\int_{-1}^{+1}\psi_j(\xi')\,d\xi'\ . \tag{14}$$

We recognize the sum in Eq. (14) as the expansion of the function that is equal to unity for $|\xi|<1$ in the complete set ψ_j. Therefore, we have[12]

$$\frac{n(2)}{n(1)}=\frac{K(1,2)}{K(2,1)}=\frac{g_2}{g_1}\exp\left(-\frac{E(2)-E(1)}{kT_e}\right),$$

so that the correct thermodynamic limit is indeed attained. Another interesting limiting case is $\tilde{A}_j(2,1)\gg n_eK(1,2)$. It can be considered as the opposite of thermodynamic equilibrium [Eq. (14)]. Equation (11) takes the form for a simple Doppler line

$$\frac{n(2)}{n(1)} = \frac{n_e K(1,2)}{A(2,1)} \frac{k_0 L}{\sqrt{\pi}} \left(\ln \frac{k_0 L}{2\sqrt{\pi}}\right)^{1/2} \sum_{j=1}^{\infty} \tfrac{1}{2}\pi \frac{a_{0,j}\psi_j(\xi)}{\mu_j} . \tag{15}$$

The summation is to be carried out numerically. The series is quickly convergent so that 3 or 4 terms are sufficient to give an accuracy of a few percent.

However, the value of this sum can be determined in another way. Let us reconsider Eq. (7). By neglecting the collisions of the second kind $n(2) n_e K(2, 1)$, it can be written symbolically

$$(I - K)\, n(2) = \frac{n_e n(1) K(1,2)}{A(2,1)} , \tag{16}$$

I being the identity operator, K the operator corresponding to the Biberman-Holstein integral equation, and the other symbols having their common meaning. The problem is to determine the inverse operator $(I-K)^{-1}$. Now the essence of the work by Widom,[13] already cited in I, was the determination of the first order asymptotics of the operator $(I-K)^{-1}$. Only the result, valid here, is stated. For a simple Doppler profile, we have for $k_0L \to \infty$ (see also Appendix B of I)

$$(I-K)^{-1} \sim \frac{k_0 L}{\pi^2} (\ln k_0 L/2\sqrt{\pi})^{1/2} \times \ln\left(\frac{1-\xi\xi' + [(1-\xi^2)(1-\xi'^2)]^{1/2}}{1-\xi\xi' - [(1-\xi^2)(1-\xi'^2)]^{1/2}}\right) . \tag{17}$$

If n_e is independent of place, the solution of Eq. (16) becomes

$$\frac{n(2)}{n(1)} = \frac{n_e K(1,2)}{A(2,1)} \frac{k_0 L}{\pi^2} (\ln k_0 L/2\sqrt{\pi})^{1/2} \int_{-1}^{+1} d\xi' \times \ln\left(\frac{1-\xi\xi' + [(1-\xi^2)(1-\xi'^2)]^{1/2}}{1-\xi\xi' - [(1-\xi^2)(1-\xi'^2)]^{1/2}}\right) .$$

By partial integration the result is readily shown to be[14]

$$\frac{n(2)}{n(1)} = \frac{2 n_e K(1,2)}{\pi A(2,1)} K_0 L \,(\ln k_0 L/2\sqrt{\pi})^{1/2} (1-\xi^2)^{1/2} . \tag{18}$$

It can be ascertained numerically that the sum in Eq. (15) indeed yields $2(1-\xi^2)^{1/2}/\sqrt{\pi}$).[15] For a Doppler line with hfs, the term $(\ln k_0 L/2\sqrt{\pi})^{1/2}$ is to be replaced by [see Eq. (22) of I]

$$2[(\ln R_1\, k_0 L/2\sqrt{\pi})^{-1/2} + (\ln R_n\, k_0 L/2\sqrt{\pi})^{-1/2}]^{-1} .$$

The behavior of $n(2)/n(1)$ as a function of k_0L, Eq. (11), all things [$n_eK(1, 2)$ and $n_eK(2, 1)$] being equal,[16] is therefore as follows: Eq. (18) holds good if k_0L, the electron density, and the electron temperature are such that $\tilde{A}_j(2,1) \gg n_e K(1,2)$ (but with k_0L large enough for the asymptotic expansion to apply). The relative density $n(2)/n(1)$ increases essentially linearly with k_0L. Within this range for k_0L the space dependence of $n(2)/n(1)$ remains the same. $\tilde{A}_j(2,1)$ decreases with increasing k_0L and, at a certain value of k_0L, $\tilde{A}_j(2,1)$ and $n_eK(1, 2)$ become of comparable magnitude. For this case, the general equation (11) applies. As a function of place, $n(2)/n(1)$ lies somewhere between the function $(1-\xi^2)^{1/2}$ and the constant function. This limit (thermodynamic equilibrium) is attained for $\tilde{A}_j(2,1) \ll n_e K(2,1)$, see Eq. (14). The rate of approach to equilibrium as a function of k_0L is different for the various types of radiative transfer. Radiation is much more easily transferred by the mechanism of repeated absorption and emission with frequency changes than without such changes (diffusion equation). The radiative leak is therefore larger in the former case than in the latter. Hence, thermodynamic equilibrium should be less quickly attained as a function of k_0L in the former case than in the latter. This is also seen by comparing the various expressions for $\tilde{A}_j(2,1)$: no frequency changes, $\tilde{A}_j(2,1) \propto (k_0L)^{-2}$; frequency changes, Doppler $\tilde{A}_j(2,1) \propto (k_0 L(\ln k_0 L/2\sqrt{\pi})^{1/2})^{-1}$; frequency changes, Lorentz, Voigt $\tilde{A}_j(2,1) \propto (k_0L)^{-1/2}$. The condition for equilibrium $\tilde{A}_j(2,1) \ll n_e K(2,1)$ will be fulfilled for the various types of transfer at values of k_0L increasing in this order. The difference between Doppler and Lorentz is, of course, due to the fact that a Doppler profile has a far less extended wing, so that the radiative leak is smaller. The analogs of Eqs. (15) and (18) for a Lorentz profile (or Voigt profile with $a \neq 0$) can be obtained in a similar way. The analog of Eq. (15) becomes (independent of hfs)

$$\frac{n(2)}{n(1)} = \frac{n_e K(1,2)}{A(2,1)} (k_0 L)^{1/2} \sum_{j=1}^{\infty} \frac{\psi_j(\xi)}{\mu_j'} \times \sum_{m=0}^{\infty} \tfrac{1}{4}\sqrt{\pi} \frac{\Gamma(m+\frac{1}{4})}{\Gamma(m+\frac{7}{4})} b_{2m,j} . \tag{19}$$

As in the Doppler case, we can also determine $n(2)/n(1)$ by using the asymptotic expression for $(I-K)^{-1}$ for a Lorentz profile. It can be shown that[17]

$$\frac{n(2)}{n(1)} = \frac{n_e K(1,2)}{A(2,1)} (k_0 L)^{1/2} \frac{3}{\pi} (1-\xi^2)^{1/4} . \tag{20}$$

Again it can be verified that Eq. (19) is indeed equal to Eq. (20). Just as in the Doppler case,

$n(2)/n(1)$ given by Eq. (11) is intermediate as a function of position between $(1-\xi^2)^{1/4}$ and the constant function. In the above, we have given a general scheme for the calculation of the expansion coefficients α_i in Eq. (8). From now on, these coefficients α_i will be considered as known. In Sec. IV, the representation of Eq. (8) will be used to calculate the line shape of the spectral line emitted by the optically thick slab.

The treatment of a two-level atom is, of course, only a first step to the calculation of number densities in the levels of real atoms. Bates, Kingston, and McWhirter[1,4] have given a scheme for such a calculation and applied it to a hydrogen-ion plasma. Their treatment of self-absorption was fairly rough since all $\tilde{A}_j(2,1)$ were put equal to zero. It is easy to see that a treatment of the self-absorbed levels, according to the lines given here, can be fitted directly into their scheme. The rate equation for a level with self-absorption is, according to Bates *et al.*, Eq. (7) with terms added describing transitions from and to other levels $p(p=3,4,\ldots)$ and from and to the continuum. Again, as in Eqs. (9) and (10), the equation is split up into rate equations for the α_j. It is useful to consider these as equations for number densities in quasilevels. The rate equation for every quasilevel, labeled by j, is formally the same as the rate equation for the level without self-absorption (hence, without integral equation), but $A(2,1)$ is replaced by $\tilde{A}_j(2,1)$. Therefore, the problem has been formally reduced to one for an optically *thin* plasma. Compare in this respect Eq. (11), where the number density in level 2 has been written as a superposition of the densities in the quasilevels with Eq. (13), the solution for a level without self-absorption. The terms describing collision-induced transitions between the quasilevels among themselves (the off-diagonal elements of B) and transitions between the quasilevels and the other levels (the generalization of the coefficients of the vector $\vec{b}$), etc., can easily be identified in the new Eqs. (9) and (10). There is no radiative coupling between the quasilevels among themselves, however. The further calculations may follow the lines given by Bates, Kingston, and McWhirter.[4] Most of the calculations given here also apply to other geometries, the only point being that in these cases the eigenvalues and eigenfunctions of Eq. (3) are not yet known.

III. STATIONARY PROBLEM IN RADIATIVE TRANSFER: ABSORPTION OF EXTERNAL RADIATION

A. Solution for a Cold Plasma

Another interesting stationary problem is the one encountered when atoms are excited by absorption of external radiation. It is of importance for the analysis of decay experiments and for some problems in astrophysics.

First, we shall consider a special case. It is assumed that parallel rays are incident at an angle α to the normal on a slab at $x=-\frac{1}{2}L$. The optically dense slab absorbs and reemits the light. Excitation or deexcitation by collisions does not occur [cold plasma, all thermal energies small compared to $E(2)-E(1)$]. However, we shall indicate below how this requirement can be dispensed with.

The incident external energy per cm^2 and per sec in a frequency interval $d\nu$ within an element of the solid angle $d\Omega$ is denoted by $I_0\,d\nu$. Further, it is assumed that I_0 does not vary appreciably over the spectral line. The integral equation for this problem becomes, for $-\frac{1}{2}L\leq x\leq\frac{1}{2}L$,

$$A(2,1)n(2)=\frac{I_0}{h\nu_0}\int_0^\infty d\nu k(\nu)\exp\left(\frac{-k(\nu)(\frac{1}{2}L+x)}{\cos\alpha}\right)$$

$$+A(2,1)\int d\vec{r}\,'K(|\vec{r}-\vec{r}\,'|)n(2). \qquad (21)$$

The first term on the right-hand side describes the rate of excitation due to absorption of the external radiation, the second one the rate of excitation due to absorption of the scattered radiation. It is understood that $n(2)$ depends on place though this is not explicitly indicated. In the first term I_0 could be placed outside the integral sign since it has been assumed that it does not vary over the spectral line. This corresponds to most experimental situations because the excitation mainly takes place due to a broad line. We proceed now to the solution of Eq. (21).

The expansion of $n(2)$ [Eq. (8)] is substituted in Eq. (21). However, the expansion now contains both the even and odd eigenfunctions since the problem has no reflection symmetry. Following entirely the same procedure that has led us from Eq. (7) to Eq. (9), we obtain for the expansion coefficients α_j

$$\tilde{A}_j(2,1)\alpha_j=\frac{I_0}{n(1)h\nu_0}\int_{-1}^{+1}d\xi\psi_j(\xi)\int_0^\infty d\nu\,k(\nu)$$

$$\times\exp\left(-\frac{k(\nu)L(1+\xi)}{2\cos\alpha}\right). \qquad (22)$$

It is assumed that the line shape is a simple Doppler profile. By substituting the expression for the eigenfunctions [Eq. (4′)] and by using Eq. (6), Eq. (22) takes the form with $u=2(\nu-\nu_0)(\ln 2)^{1/2}/\Delta\nu_D$ and $k_0\Delta\nu_D/2(\ln 2)^{1/2}=h\nu_0B(1,2)n(1)/4\pi$. [$B(1,2)$ is the well-known Einstein coefficient.[18]]

$$\tilde{A}_j(2,1)\alpha_j = \frac{B(1,2)I_0}{2\pi k_0}\sum_{m=0}^{\infty} a_{m,j}\int_0^{\pi} d\vartheta \sin\vartheta$$

$$\times \sin(m+1)\vartheta\int_0^{\infty} du k(u)\exp\left(-\frac{k(u)L(1+\cos\vartheta)}{2\cos\alpha}\right)$$

$$= \frac{B(1,2)I_0\cos\alpha}{k_0 L}\sum_{m=0}^{\infty}(-1)^m(m+1)a_{m,j}\int_0^{\infty} du$$

$$\times \exp\left(-\frac{k(u)L}{2\cos\alpha}\right)I_{m+1}\left(\frac{k(u)L}{2\cos\alpha}\right) .$$

Here I_{m+1} is the modified Bessel function of order $m+1$.[19] The new variable $y = k_0 L\, e^{-u^2}/2\pi^{1/2}\cos\alpha$ is introduced. The resulting expression can readily be expanded asymptotically for $k_0 L \to \infty$. We then have

$$\tilde{A}_j(2,1)\alpha_j \sim \frac{B(1,2)I_0}{2k_0 L}\,\frac{\cos\alpha}{[\ln(k_0 L/2\pi^{1/2}\cos\alpha)]^{1/2}} \times \sum_{m=0}^{\infty}(-1)^m a_{m,j} . \tag{23}$$

The expression for $\tilde{A}_j(2,1)(=\beta_j)$ [Eq. (4)] is substituted. In order to be able to make a further analytic reduction of Eq. (23), we do not use the numerical expressions for μ_j but $\mu_j = \sqrt{\pi}/2\lambda_j$, λ_j being the jth eigenvalue of the asymptotic integral equation given by Widom[13] for $\alpha = 1$. [See I, Eqs. (17), (18), (20), and Appendix B.] The summation in Eq. (23) can now be carried out for the even eigenfunctions using Eq. (A5) of Appendix A. For the odd ones, the sum can be transformed into a more rapidly converging one by Eq. (A6). For the even eigenfunctions, we therefore have

$$\alpha_j \sim \frac{B(1,2)I_0}{\pi A(2,1)}\cos\alpha\left(\frac{\ln(k_0 L/2\pi^{1/2})}{\ln(k_0 L/2\pi^{1/2}\cos\alpha)}\right)^{1/2}\tfrac{1}{4}\pi a_{0,j} , \tag{24}$$

and a similar expression for the odd ones. These formulas are useful when we want to generalize the problem by including collisions.

A closed-form solution of the problem can be obtained when Eq. (23) is resubstituted in Eq. (8). By applying the orthogonality relation to be derived in Appendix A [see Eq. (A10)] and Eq. (6), the resulting expression is identified as the expansion of $\cos^{-1}\xi$ in the Tschebyscheff polynomials.[20] The solution of Eq. (21) therefore becomes (for $k_0 L \gg 1$)

$$\frac{n(2)}{n(1)} = \frac{B(1,2)I_0}{\pi A(2,1)}\cos\alpha\left(\frac{\ln(k_0 L/2\pi^{1/2})}{\ln(k_0 L/2\pi^{1/2}\cos\alpha)}\right)^{1/2}\frac{\cos^{-1}\xi}{\pi} . \tag{25}$$

B. Discussion and Extension

Now suppose that radiation from a black body at a temperature T_r from all directions $(0 \le \alpha < \frac{1}{2}\pi)$ is incident at $x = -\frac{1}{2}L$. It is known [but will be shown again in Eq. (30) of Sec. IV] that at large absorption local thermodynamic equilibrium exists between the radiation field and the two-level atoms. At the left boundary of the slab, therefore, we should find that the relative density $n(2)/n(1)$ is given by the Boltzmann factor corresponding to the radiation temperature T_r. We want to check that this is indeed the case in Eq. (25). Carrying out the integration over a half sphere in the right-hand side of Eq. (25), and making use of the fact that black body radiation is isotropic, we obtain (for $k_0 L \gg 1$)

$$\frac{n(2)}{n(1)} = \frac{B(1,2)I_0}{A(2,1)}\,\frac{\cos^{-1}\xi}{\pi} = \frac{g_2}{g_1}\exp\left(-\frac{h\nu}{kT_r}\right)\frac{\cos^{-1}\xi}{\pi} . \tag{26}$$

For $\xi = -1$, we therefore find the Boltzmann factor, as expected.

In deriving Eq. (26) the term due to stimulated emission in Planck's law for I_0 was neglected. In fact, we used Wien's law, valid if $h\nu \equiv E(2) - E(1) \gg kT_r$. This is consistent with the requirement – always imposed here – that the density in level 2 should be small compared to that in level 1. From Eq. (26), it is readily shown that, if also black-body radiation of temperature T_r is incident on $x = +\frac{1}{2}L$, the relative density is given everywhere by the Boltzmann factor as it should. This can also be obtained directly from the expansion into the eigenfunctions by combining Eqs. (24), (12), and (14). Moreover it can be checked with the aid of the formulas to be derived in Sec. IV and the asymptotic expansion[21] of the expression for the energy absorbed per cm^2 and per sec[18] that all absorbed radiation is emitted. This must be the case since it has been assumed that no deexciting collisions take place.

In the above it has been assumed that the plasma is cold. When excitation and deexcitation by, for instance, electrons occurs, in addition to absorption of external radiation, Eq. (7) can be generalized with the aid of Eq. (21). Again the relative density $n(2)/n(1)$ is expanded according to Eq. (8). By following the reduction procedure of Eq. (7) to Eqs. (9) and (10), it can be verified that the expansion coefficients α_j are calculated from Eq. (10) when the right-hand side of Eqs. (22) or (23) is added to every element b_j of the vector $\vec{b}$. A closed-form solution is possible when n_e is constant and $\tilde{A}_j(2,1) \gg n_e K(2,1)$. Simply by adding Eqs. (18) and (25), we find

$$\frac{n(2)}{n(1)} = \frac{[\ln(k_0L/2\pi^{1/2})]^{1/2}}{\pi A(2,1)} \left(2n_e K(1,2)\,k_0 L(1-\xi^2)^{1/2} + \frac{B(1,2)\,I_0 \cos\alpha \cos^{-1}\xi}{\pi[\ln(k_0L/2\pi^{1/2}\cos\alpha)]^{1/2}}\right). \qquad (27)$$

In the above, Eq. (21) has been solved for a simple Doppler profile. It is also possible to obtain the analog of Eq. (23) for a Lorentz or Voigt profile ($a \neq 0$). This will not be given since with a slightly different representation of the eigenfunctions, the calculations become much easier.

This problem will be discussed further in Appendix B. The difficulties in determining the analog of the closed-form solution [Eq. (25)] have not as yet been overcome. The impression exists that it is not a known function.

IV. SPECTRAL LINE SHAPE

A. Basic Formulas

The two foregoing sections have shown how in various cases the density of excited atoms can be found as a superposition of the eigenfunctions of the Biberman-Holstein integral equation. Once this is known, the radiation is also determined. In this section, it is our purpose to derive explicit expression for this field. It is assumed in the following that the relative density can be written according to Eq. (8), with coefficients α_i as calculated, for instance, in Secs. II or III. These coefficients vary from case to case. The most appropriate method is, therefore, to calculate for every eigenfunction the corresponding radiation field. The final field is found as a superposition of the fields corresponding to every eigenfunction. An advantage of this method is that hfs offers no special difficulties. The eigenfunctions are independent of it, see Eqs. (4′) and (5′). Therefore, we include it in the discussion. The radiation emitted by a slab and its line shape is, of course, a very interesting quantity. In astrophysics, for example, it represents the only possibility of obtaining information concerning the radiating body.

The intensity of the radiation (energy/cm^2 sec) at the point in space $\vec{r}$ transported in the direction denoted by the unit vector $\vec{s}$ within an element of solid angle $d\Omega$ at the frequency ν is designated by $I_\nu(\vec{r},\vec{s})\,d\nu$. A well-known differential equation for $I_\nu(\vec{r},\vec{s})$ in terms of $n(1)$ and $n(2)$ gives

$$\vec{s}\cdot\nabla I_\nu(\vec{r},\vec{s}) = \frac{h\nu_0}{4\pi}\,\mathfrak{L}(\nu)[A(2,1)n(2) - (B(1,2)n(1) - B(2,1)n(2))I_\nu(\vec{r},\vec{s})]. \quad (28)$$

$\vec{s}\cdot\nabla$ is the directional derivative. All the other symbols have their common meaning. As is customary we neglect stimulated emission in Eq. (28), and introduce the dimensionless frequency u defined by Eqs. (4) and (5). For a Doppler profile we write $k_0\Delta\nu_D/2(\ln 2)^{1/2} = h\nu_0 B(1,2)n(1)/4\pi$, and for a Lorentz profile $k_0\Delta\nu_L/2 = h\nu_0 B(1,2)n(1)/4\pi$, in accordance with the definitions in Eqs. (4) and (5),[18] and $I_u du = I_\nu d\nu$. Equation (28) is a linear inhomogeneous differential equation of order 1 along the line $(l\vec{s})$, l being a real number. The method of integrating this equation is well known. Coordinates (α,β) on the unit sphere are introduced such that the normals on the right-hand and left-hand sides of the slab correspond to $\alpha = 0$ and π, respectively. The azimuth is β, $0 \leq \beta \leq 2\pi$. The problem has symmetry for rotations about the normal. Since $\vec{s}\cdot\nabla = \cos\alpha\, d/dx$, the solution of Eq. (28) becomes

$$\begin{aligned} I_u(x,\alpha) = {} & \frac{h\nu_0}{4\pi} A(2,1)\,\mathfrak{L}(u) \int_0^{(\pm L/2 + x/\cos\alpha)} dl \\ & \times\ n(2;\ \mp\tfrac{1}{2}L + l\cos\alpha) \\ & \times\ \exp\left(-\frac{k(u)}{\cos\alpha}\,(\pm\tfrac{1}{2}L + x - l\cos\alpha)\right) \\ & + I_u'(\mp\tfrac{1}{2}L,\alpha)\exp\left(-\frac{k(u)}{\cos\alpha}\,(\pm\tfrac{1}{2}L + x)\right), \end{aligned} \quad (29)$$

in which the + sign is used for $0 \leq \alpha < \frac{1}{2}\pi$, the − sign for $\frac{1}{2}\pi < \alpha \leq \pi$.

From a mathematical point of view, the second term in the right-hand side of Eq. (29) is due to an integration constant: $I_u(\mp\frac{1}{2}L,\alpha) = I_u'(\mp\frac{1}{2}L,\alpha)$. Physically, it represents radiation incident on the left (−) or right (+) boundary of the slab, of the type described in Sec. III. Since it is of no further interest for us, we shall put it equal to zero.

It is instructive to develop Eq. (29) asymptotically for large values of $k(u)L$ by the Laplace method,[22] which applies when x is more than a few absorption lengths remote from the boundary:

$$I_\nu(x,\alpha)\,d\nu \sim \frac{A(2,1)}{B(1,2)}\frac{n(2)}{n(1)}\,d\nu - \frac{A(2,1)}{k(u)} \times [\cos\alpha(d/dx)n(2)/B(1,2)n(1)]\,d\nu\cdots. \quad (30)$$

Now, $dn(2)/dx$ is of the order $n(2)L^{-1}$. Consequently, the second term in the right-hand side of Eq. (30) is an order of $\cos\alpha/k(u)L$ smaller than the first one. Since only the frequency range is considered in which this factor is small, it can be neglected. From the first term, we see that the intensity is isotropic, independent of frequency, and corresponds to thermodynamic equilibrium with a Bolzmann factor at an excitation temperature defined in the usual way by $n(2)/n(1)$. Local thermodynamic equilibrium therefore exists between the radiation field and the two-level atoms in this frequency range. Equation (30) has been derived under fairly restrictive conditions. The enclosure has been taken to be a slab; stimulated emission has been neglected and the density in the ground state has been assumed to be independent of place. All these conditions are not essential for the method and can be removed.

B. Line Shape: Doppler Profile

The new variables $\mp\frac{1}{2}L + 1\cos\alpha = \pm\frac{1}{2}L\xi$ and $x = \pm\frac{1}{2}L$ are introduced in Eq. (29). For the intensity emerging at the right $(0 \leq \alpha < \frac{1}{2}\pi)$, and at the left $(\frac{1}{2}\pi < \alpha \leq \pi)$, we obtain

$$I_u(\pm\tfrac{1}{2}L,\alpha) = \frac{h\nu_0}{4\pi}\,\frac{A(2,1)\mathfrak{L}(u)L}{2|\cos\alpha|}\,\exp\left(-\frac{k(u)L}{2|\cos\alpha|}\right)$$
$$\times\int_{-1}^{+1} d\xi\, n(2;\pm\tfrac{1}{2}L\xi)\exp\left(\frac{k(u)L\xi}{2|\cos\alpha|}\right), \qquad (31)$$

and zero for $x = \frac{1}{2}L$, $\frac{1}{2}\pi < \alpha \leq \pi$ and $x = -\frac{1}{2}L, 0 \leq \alpha < \frac{1}{2}\pi$ (no light entering the vessel from the outside). Since we want to calculate this expression for every eigenfunction of the Biberman-Holstein integral equation as explained at the head of this section, we insert $n(2) = n(1)\psi_j(\xi)$ in Eq. (31), with ψ_j given in Eq. (4′). Hence, when $\mathfrak{L}(u)$ is a Doppler profile with or without hfs, we obtain for the emerging intensity due to the jth eigenfunction

$$I_u(\pm\tfrac{1}{2}L,\alpha) = \pi\,\frac{A(2,1)}{B(1,2)}\,\frac{\Delta\nu_D}{2(\ln 2)^{1/2}}\left(\frac{\cos\alpha}{|\cos\alpha|}\right)^{j-1}$$
$$\times\exp\left(-\frac{k(u)L}{2|\cos\alpha|}\right)\sum_{m=0}^{\infty}(m+1)a_{m,j}I_{m+1}\left(\frac{k(u)L}{2|\cos\alpha|}\right), \qquad (32)$$

in which I_{m+1} is the modified Bessel function of order $m+1$.[19] The factor $(\cos\alpha/|\cos\alpha|)^{j-1}$ requires further discussion. For the even eigenfunction, $j = 1, 3, \ldots,$ it is equal to one. Therefore the intensity emerging at $+\frac{1}{2}L$ at an angle α is equal to that which emerges at $-\frac{1}{2}L$ at an angle $\pi - \alpha$. This should be the case since the problem has reflection symmetry. For the odd eigenfunctions, $j = 2, 4, \ldots,$ it is equal to one for $x = \frac{1}{2}L$, $0 \leq \alpha < \frac{1}{2}\pi$ and equal to minus one for $x = -\frac{1}{2}L$, $\frac{1}{2}\pi < \alpha \leq \pi$. At first sight, it might appear impossible for negative radiation to be emitted. It should, however, be noted that an odd eigenfunction can never exist by itself. It must always be accompanied by one or more even eigenfunctions in order to ensure that the density $n(2)/n(1)$ as a function of place is everywhere positive.

The net radiation emitted by an odd eigenfunction (i.e., the radiation emitted at both $x = \pm\frac{1}{2}L$), is always equal to zero. This is clear since the mean value of an odd eigenfunction is zero. We shall now examine Eq. (32) as a function of the frequency, the line shape of the spectral line (see Fig. 1). The discussion will be restricted to the case of even eigenfunctions since similar arguments apply to the odd ones.

For small values of $k_0L\mathfrak{L}(u)/2\cos\alpha$ (i.e., in the far wings of the line), Eq. (32) takes the form

$$I_u(\pm\tfrac{1}{2}L,\alpha) = \pi\,\frac{A(2,1)}{B(1,2)}\,\frac{\Delta\nu_D}{2(\ln 2)^{1/2}}\,a_{0,j}\,\frac{k_0L\mathfrak{L}(u)}{4|\cos\alpha|}\,. \qquad (33)$$

Therefore, in the far wings, the intensity falls off according to a common Doppler line.

For large values of $k_0L\mathfrak{L}(u)/2\cos\alpha$ (i.e., large $k_0L/2\cos\alpha$ and in the core of the line), the asymptotic behavior of the modified Bessel functions yields

$$I_u(\pm\tfrac{1}{2}L,\alpha) = (\tfrac{1}{2}\pi)^{1/2}\frac{A(2,1)}{B(1,2)}\,\frac{\Delta\nu_D}{2(\ln 2)^{1/2}}$$
$$\times\left(\frac{2|\cos\alpha|}{k(u)L}\right)^{1/2}\sum_{m=0}^{\infty}(2m+1)a_{2m,j}\,. \qquad (34)$$

In the center of the line, the intensity shows a dip. This is the self-reversal of the line. See, for the

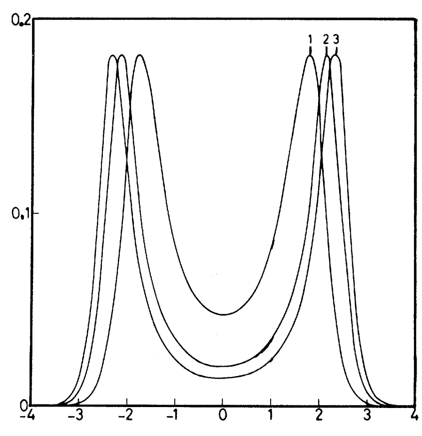

FIG. 1. Intensity emitted by an homogeneous layer of gas in a direction perpendicular to it in units of $\frac{1}{2}\pi A(2,1)\Delta\nu_D\, B(1,2)\,(\ln 2)^{1/2}$ [i.e., $2I_u(\frac{1}{2}L,0)B(1,2)\,(\ln 2)^{1/2}/\pi A(2,1)\Delta\nu_D$] corresponding to the first even eigenfunction ($j=1$) for a simple Doppler profile as a function of the dimensionless frequency $u = 2(\nu-\nu_0)(\ln 2)^{1/2}/\Delta\nu_D$. (a) $k_0L=100$, (b) $k_0L=500$, (c) $k_0L=1000$. See Eqs. (38)–(40).

features described by Eqs. (33) and (34), also Fig. 1. Note further that the intensity at the center decreases with the square root of k_0L.

As has been said, the radiation emitted by a layer of gas is found by multiplying Eq. (32) with the pertinent expansion coefficients α_j and summing. We want to pay somewhat more attention to the problem of which the solution was obtained in Sec. II.

When the electron density is constant, the general situation is described by Eq. (11). This case is intermediate between the two limits described by Eqs. (14) and (15) or (18). We want to know the corresponding radiation. When Eq. (14) is inserted in Eq. (31), and the well-known expression for $A(2, 1)/B(1, 2)$ is substituted, the result takes the form

$$I_u(\pm\tfrac{1}{2}L, \alpha) = \frac{2h\nu^3}{c^2}\, e^{-h\nu/kT_e}\, \frac{\Delta\nu_D}{2(\ln 2)^{1/2}} \times\left[1-\exp\left(-\frac{k(u)L}{|\cos\alpha|}\right)\right] \quad . \qquad (35)$$

Hence, the gas layer radiates as a black body at the electron temperature T_e in the core of the line, just as it should do. Instead of Planck's law, Wien's law is found, since it has been assumed in this paper that stimulated emission is absent or $n(2) \ll n(1)$. It should be noted that Eq. (35) shows no self-reversal. This point will be discussed below.

The emitted radiation corresponding to the limiting case [Eq. (15)] is found by inserting Eq. (18) in Eq. (31).[23] We have, for a simple Doppler profile,

$$I_u(\pm\tfrac{1}{2}L, \alpha) = \frac{n_e K(1,2)}{B(1,2)}\, \frac{\Delta\nu_D}{(\ln 2)^{1/2}}\, k_0 L\left(\ln\frac{k_0L}{2\sqrt{\pi}}\right)^{1/2} \times \exp\left(-\frac{k(u)L}{2|\cos\alpha|}\right) I_1\left(\frac{k(u)L}{2|\cos\alpha|}\right) . \qquad (36)$$

For a Doppler profile with hfs, the term $[\ln(k_0L/2\sqrt{\pi})]^{1/2}$ is to be replaced by [see Eq. (22) of I]

$$2[(\ln R_1 k_0 L/2\sqrt{\pi})^{-1/2} + (\ln R_n k_0 L/2\sqrt{\pi})^{-1/2}]^{-1} .$$

The discussion of the line shape described by Eq. (36) is entirely similar to the one given for the general case [Eq. (32)]. It has been argued in Sec. II that, if $n(2)/n(1)$ is given by Eq. (11), $n(2)/n(1)$ as a function of position lies somewhere between $(1-\xi)^{1/2}$ and a constant function. Therefore, the corresponding line shape will lie between Eqs. (35) and (36). In particular, the self-reversal is less than that exhibited by Eq. (36).

The part that describes the line shape in Eq. (36), namely,

$$\mathfrak{L}'(u, \alpha) \equiv \exp\left(-\frac{k(u)L}{2|\cos\alpha|}\right) I_1\left(\frac{k(u)L}{2|\cos\alpha|}\right) \qquad (37)$$

has already been derived previously by Ivanov[24] as an approximating formula, for $\tilde{A}_j(2,1) \gg n_e K(1,2)$. It appears to express the numerical results obtained by Hearn[10] for this case well as it should do. As noticed by Ivanov, in the limit of $k_0L/\cos\alpha \to 0$, Eq. (37) becomes proportional to $\mathfrak{L}(u)$. It is therefore a reasonable approximation to the line shape when $\tilde{A}_j(2,1) \gg n_e K(1,2)$ for all values of k_0L since it gives an exact result for $k_0L \to 0$ and $k_0L \to \infty$.

The physical interpretation of the broadening of the spectral line and its self-reversal (see Fig. 1) is well known. In the wings of the line the medium is optically thin, and the photons escape immediately once they are created. In the core, however, photons are strongly absorbed. After absorption, there is a chance that they may be re-emitted to the wings, where they can escape. This enhances the intensity in the wings and decreases it in the core.[25]

In the theory of this type of radiative transfer, the basic assumption is that the emission profile is proportional to the absorption profile (complete redistribution). The exact redistribution function is known[7] but the problem seems to be intractable with this function. Since the approximation of complete redistribution is fulfilled quite well, it is expected that this approximation yields fairly good eigenfunctions and eigenvalues. However, this assumption naturally has a direct effect on the line shape of the emergent intensity. Hummer,[26] using a simplified redistribution function, has shown numerically that the differences in the far wings can become as large as 80% in the worst case, and in the core can amount to 20%. This might be considered discouragingly high. However, in the far wings also, natural broadening shows itself, in most cases masking the error there entirely.[27] Furthermore, the total integrated intensity, being directly related to the eigenfunctions and eigenvalues (see below), is far less sensitive to the assumption. In addition, it is possible that the assumption of complete redistribution has a larger range of validity than expected. The exact redistribution function has been derived on the assumption that no changes in the velocity of the particle occur during the absorption and reemission process. This will often happen but the problem is complicated since collisions will also contribute a certain amount of pressure broadening. It therefore remains an open question.

The total radiative loss of a slab per cm^2 and per sec (i.e., the total radiation emitted at both $x=\pm\frac{1}{2}L$) can be calculated by integrating over all frequencies and the total solid angle. The procedure is similar to that used in deriving Eq. (23). In first-order asymptotics, we have for the radiation corresponding to an even eigenfunction $[n(2)/n(1)=\psi_j(\xi);\ j=1,3,\ldots]$ and for a simple Doppler profile

$$\int_0^{2\pi} d\beta \int_0^{\pi} d\alpha \sin\alpha\,|\cos\alpha| \int_{-\infty}^{+\infty} du\, I_u(\alpha) = 2\pi^2 \frac{A(2,1)}{B(1,2)} \frac{\Delta\nu_D}{2(\ln 2)^{1/2}} \frac{1}{(\ln k_0 L/2\sqrt{\pi})^{1/2}} \sum_{m=0}^{\infty} a_{2m,j}\ . \tag{38}$$

It is interesting to note that it decreases as a function of k_0L. We are now able to check the internal consistency of the theory. Suppose that a density $n(2)=n(1)\psi_1(\xi)$ could be created in the slab. This function is positive everywhere as a function of place, so that this is possible in principle. By the definition of the eigenfunction and the decay constant $\tilde{A}_1(2,1)$, the radiative loss per cm^2 and per sec is [see Eq. (12)]

$$h\nu_0 n(1)\tilde{A}_1(2,1)\int_{-L/2}^{+L/2} dx\,\psi_1(x)\int_0^{\infty} d\nu\, \mathfrak{L}(\nu) = h\nu_0 n(1)\tfrac{1}{2}L\tilde{A}_1(2,1)\tfrac{1}{2}\pi\, a_{0,1}\ . \tag{39}$$

Equation (38), for $j=1$, and Eq. (39) should be equal. If the proper expression for $\tilde{A}_1(2,1)$ is inserted [Eq. (4)], it is found that

$$\tfrac{1}{2}\pi\lambda_1 \sum_{m=0}^{\infty} a_{2m,1} = \tfrac{1}{8}\pi^2 a_{0,1}\ ,$$

a relation between the eigenvalue and the expansion coefficients of the first even eigenfunction of the integral equation treated in Appendix B of I. It can be proven by induction that the formula is valid for all even eigenfunctions. It should be noted that the proof fails for the odd ones, since both Eqs. (38) and (39) give then zero. Actually, another relation is valid which will be proven together with the above mentioned one in Appendix A.

Finally, we want to discuss an important point. The reader will have noted that Eq. (35) shows no self-reversal while the general formula Eq. (32) does. Moreover, the net radiative loss in Eq. (35) is $\propto(\ln k_0L/2\sqrt{\pi}\,)^{1/2}$ [21] while the general formula [Eq. (38)] yields $\propto(\ln k_0L/2\sqrt{\pi})^{-1/2}$. First we note that this is not at all surprising. The convergence of the series in Eq. (14) is certainly not uniform. Hence, when this series is substituted in Eq. (31), the interchange of summation and integration is not allowed. Therefore, in this special case, the resulting emerging intensity *cannot* be written as a superposition of the intensities corresponding to the eigenfunctions, as for a nonequilibrium case.

When, however, the density of excited atoms *continuously* decreases to zero at the boundary, a certain departure from thermodynamic equilibrium is always present. This is connected with self-reversal however slight it may be. In order to be able to represent arbitrarily small self-reversals, the emerging intensities corresponding to the eigenfunctions with higher j should show less self-reversal for higher values of j. This is indeed the case.

Therefore, we may expect that in every nonequilibrium situation the above mentioned difficulty does not occur. The theory is essentially a nonequilibrium one. Thermodynamic equilibrium is a somewhat singular limiting case (see also Ref. 12).

A more quantitative discussion is impossible at this time since nothing is known about the asymptotic properties of the eigenfunctions ψ_j for $j\to\infty$.

C. Line Shape: Lorentz (Voigt) Profile

The line shape for a Lorentz or Voigt profile $(a\neq 0)$ with or without hfs is found by inserting $n(2)=n(1)\psi_j(\xi)$ in Eq. (31), $\psi_j(\xi)$ being given by Eq. (5′). However, with a slightly different representation of the eigenfunctions, viz.,

$$\psi_j(\xi)\sim(1-\xi^2)^{1/4}\sum_{m=0}^{\infty} b'_{m,j}\, C_m^{3/4}(\xi)\ , \tag{40}$$

the resulting formula becomes much easier in form. Here, $C_m^{3/4}(\xi)$ is the Gegenbauer polynomial of order $\frac{3}{4}$ and degree m.[28] The coefficients $b'_{m,j}$ are related to the original $b_{m,j}$ [Eq. (5′)] by a linear transformation. This point will be discussed further in Appendix B.

When $n(2)=n(1)\psi_j(\xi)$, with $\psi_j(\xi)$ given by Eq. (40), is inserted in Eq. (31), we obtain for the emerging intensity due to the jth eigenfunction

$$I_u(\pm\tfrac{1}{2}L,\alpha)=2^{3/4}\Gamma(\tfrac{5}{4})\frac{A(2,1)}{B(1,2)}\,\Delta\nu_L \times\left(\frac{\cos\alpha}{|\cos\alpha|}\right)^{j-1}\left(\frac{k(u)L}{2|\cos\alpha|}\right)^{1/4}\exp\left(-\frac{k(u)L}{2|\cos\alpha|}\right) \times\sum_{m=0}^{\infty} b'_{m,j}\frac{\Gamma(m+\frac{3}{2})}{m!}\, I_{m+3/4}\left(\frac{k(u)L}{2|\cos\alpha|}\right). \tag{41}$$

$I_{m+3/4}$ is the modified Bessel function of order[29] $m+\frac{3}{4}$ and $\mathfrak{L}(u)$ is a Lorentz or Voigt profile $(a\neq 0)$ with or without hfs. Compare this with Eq.

(32). The discussion of Eq. (41) is identical to that given for Eq. (32). Here it is only intended to give the analogs of Eq. (33) through Eq. (36). In discussing the line shape described by Eq. (41), we shall confine ourselves to the even eigenfunctions $j = 1, 3, \ldots$.

For small values of $k_0 L \mathfrak{L}(u)/2\cos\alpha$ (i.e., in the far wings of the line), Eq. (41) takes the form

$$I_u(\pm\tfrac{1}{2}L,\,\alpha) = \frac{\Gamma(\frac{5}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{7}{4})}\,\frac{A(2,1)}{B(1,2)}\,\Delta\nu_L\; b'_{0,j}\,\frac{k_0 L \mathfrak{L}(u)}{2|\cos\alpha|}\,. \tag{42}$$

Therefore, in the far wings, the intensity has the common Lorentz or Voigt shape.

For large values of $k_0 L \mathfrak{L}(u)/2\cos\alpha$ (i.e., large $k_0 L/2\cos\alpha$ and in the core of the line), the asymptotic behavior of the modified Bessel functions yields

$$I_u(\pm\tfrac{1}{2}L,\,\alpha) = \frac{2^{3/4}\Gamma(\frac{5}{4})}{(2\pi)^{1/2}}\,\frac{A(2,1)}{B(1,2)}\,\Delta\nu_L$$

$$\times\left(\frac{2|\cos\alpha|}{k_0 L \mathfrak{L}(u)}\right)^{1/4}\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{3}{2})}{m!}\,b'_{m,j}\,. \tag{43}$$

This is the self-reversal of the line. Compare the analytical results of Eqs. (42) and (43) with Fig. 2. The self-reversal (increasing with the $\frac{1}{4}$ power of $k_0 L$) is less than for a Doppler profile. This is brought about by the fact that the eigenfunctions are flatter than for a Doppler profile (see Ref. 25 and the discussion of the eigenfunctions in I).

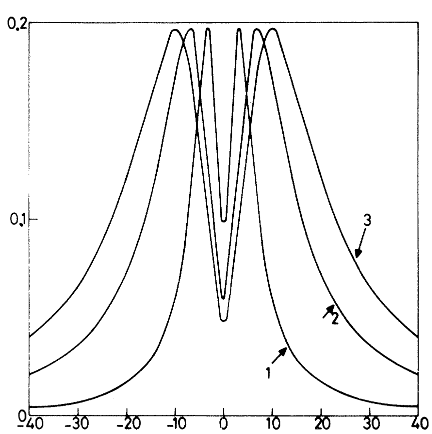

FIG. 2. Intensity emitted by an homogeneous layer of gas in a direction perpendicular to it in units of $2^{3/4}\Gamma(\frac{5}{4})A(2,1)\Delta\nu_L/B(1,2)$ [i.e., $I_u(\frac{1}{2}L,0)B(1,2)/2^{3/4}\Gamma \times (\frac{5}{4})A(2,1)\Delta\nu_L$] corresponding to the first even eigenfunction ($j=1$) for a simple Lorentz profile as a function of the dimensionless frequency $u = 2(\nu-\nu_0)/\Delta\nu_L$. (a) $k_0L=100$, (b) $k_0L=500$, (c) $k_0L=1000$. See Eqs. (47)–(49). It should be noted that $\Delta\nu_L$ is also a function of k_0L if resonance broadening is important.

In order to obtain a general idea of the emission of a layer of gas, we shall study the two limiting cases Eq. (14) (thermodynamic equilibrium) and Eq. (20) (opposite of thermodynamic equilibrium). The emission is obtained by inserting these equations in Eq. (31), just as in the Doppler case. At thermodynamic equilibrium Eq. (35) again applies, but now $\mathfrak{L}(u)$ is a Lorentz or Voigt profile ($a \neq 0$) instead of a Doppler profile.

For the opposite of thermodynamic equilibrium [Eqs. (19) or (20)] we have for a line with or without hfs:

$$I_u(\pm\tfrac{1}{2}L,\,\alpha) = \frac{3\Gamma(\frac{5}{4})2^{-1/4}}{\pi^{1/2}}\,\frac{n_e K(1,2)}{B(1,2)}$$

$$\times\,\Delta\nu_L (k_0 L)^{1/2}\left(\frac{k(u)L}{2|\cos\alpha|}\right)^{1/4}$$

$$\times\exp\left(-\frac{k(u)L}{2|\cos\alpha|}\right) I_{3/4}\left(\frac{k(u)L}{2|\cos\alpha|}\right)\,. \tag{44}$$

Equations (42)–(44) should be compared with Eqs. (33)–(36), given for a Doppler profile, and the complete analogy should be noted. In particular, the physical interpretation of the broadening and the self-reversal are the same. Moreover, the arguments given for the use of Eq. (37) as an approximating formula for all values of k_0L apply as well to that part of Eq. (44) that describes the line shape.

Finally, the total radiative loss of a slab per cm^2 and per sec. is calculated by integrating Eq. (41) over all frequencies and the total solid angle. In first-order asymptotics for $k_0 L \to \infty$, we obtain, in exactly the same way as for a Doppler profile,

$$\int_0^{2\pi} d\beta \int_0^{\pi} d\alpha\,\sin\alpha\,|\cos\alpha| \int_{-\infty}^{+\infty} du\, I_u(\alpha)$$

$$\sim \tfrac{4}{3}\pi\,\frac{A(2,1)}{B(1,2)}\,\Delta\nu_L (k_0 L)^{1/2}$$

$$\times\sum_{m=0}^{\infty} b'_{m,j}\,\frac{\Gamma(m+\frac{3}{2})\Gamma(m+\frac{1}{2})}{m!\,(m+1)!}\,. \tag{45}$$

V. CONCLUSIONS AND REMARKS

It has been shown how the eigenvalues and eigenfunctions of the Biberman-Holstein integral equation can be used for the solution of various stationary problems at high optical depth. Sometimes even closed-form expressions could be found. Though numerical calculations cannot entirely dispensed with, those required here have a major advantage in addition to being simple. The point is that we do not deal with the identity operator I and the operator corresponding to the Biberman-Holstein integral equation K separately, but subtract them analytically and use the resulting expression for I-K. At high optical depth, where I-K is very small, this enhances the accuracy considerably. Although the assumption of high optical depth is quite frequently fulfilled, it is an interesting question to consider whether an extension of the theory to lower optical depth is feasible. Usually the levels lying immediately above the resonance level(s) are neither optically thick nor thin. This problem must be solved to enable us to calculate the number densities in all levels. A solution of the Biberman-Holstein integral equation for all values of k_0L is considered as being infeasible for the moment, as it very probably will continue to be in the near or perhaps even distant future as well. Therefore, it appears that only two possibilities remain.

First, we may perform these calculations fully numerically. Secondly, we may try to devise a reasonable approximation procedure. Since we have used the analytical method up to now, we shall adopt the second procedure. The problem is related to the question posed already in Sec. III of I, in which a guess was made about the second term in the asymptotic expansion of $\tilde{A}_j(2, 1)$ for $k_0L \to \infty$. Now two questions can be put. (i) Can a function be proposed for $\tilde{A}_j(2, 1)$ such that all $\tilde{A}_j(2, 1)$ reduce to $A(2, 1)$ for $k_0L \to 0$ and that they have the correct asymptotic behavior for $k_0L \to \infty$? (ii) How do the eigenfunctions behave as a function of k_0L? As to the first question: There are many possibilities. For example,

$$\tilde{A}_j(2, 1) = \left(\frac{\mu_j A(2, 1)}{\mu_j + (k_0L/\sqrt{\pi})(\ln k_0L/2\sqrt{\pi})^{1/2}}\right)$$

fulfills the above requirements for a simple Doppler profile. But if the second term in the asymptotic expansion of this expression is considered, it becomes clear that it is in fact

$$\propto \mu_j \left(k_0L \ln \frac{k_0L}{2\sqrt{\pi}}\right)^{-1} .$$

This is contrary to the conjecture made in I, about which we are fairly confident, that the higher-order terms are logarithms. Therefore, the possibility proposed above is ruled out. A better approximation for $\tilde{A}_j(2, 1)$ is likely to be the Fourier transform of the Biberman-Holstein integral kernel, $f(\sigma)$ under Eq. (6) of I, if we substitute a value for $L\sigma_x$ such that the correct asymptotic formula for k_0L is found.[3] The higher-order terms in the asymptotic expansion are now, correctly, logarithms. Although this seems to be a better proposal, a fundamental difficulty remains. In these papers, it has always been assumed that the emission coefficient is proportional to the absorption coefficient. It is believed that this does not significantly affect the results at high optical depth. At low and moderate optical depth, however, it is an entirely different matter. Therefore, a first step towards solving the problem posed would be to determine the solutions of a theory of radiative transfer from which this assumption has been removed.

The importance of this problem has already been pointed out in Sec. IV in connection with the calculation of the radiation emitted by a layer of gas. When these solutions are known it is very probable that we can make a justified guess about $\tilde{A}_j(2, 1)$ and the eigenfunctions for all values of k_0L. The guess can be tested experimentally by the determination of $\tilde{A}_j(2, 1)$ from decay experiments.

ACKNOWLEDGMENT

The author is greatly indebted to Professor H. L. Hagedoorn for a thorough reading of the manuscript, many suggestions for its improvement, and discussions.

APPENDIX A

In this Appendix, a number of relations are derived between the eigenvalues and the expansion coefficients of the eigenfunctions of the integral equation

$$\lambda f(\xi) = \frac{1}{2\pi} \int_{-1}^{+1} d\xi' \times \ln \left(\frac{1 - \xi\xi' + [(1 - \xi^2)(1 - \xi'^2)]^{1/2}}{1 - \xi\xi' - [(1 - \xi^2)(1 - \xi'^2)]^{1/2}}\right) f(\xi') . \tag{A1}$$

As has been shown in I, these quantities have a bearing on the eigenfunctions and eigenvalues of the Biberman-Holstein integral equation when the line shape is a Doppler profile with or without hfs. In Appendix B of I, the following expansion was introduced for the normalized eigenfunctions:

$$f_j(\xi) = (1 - \xi^2)^{1/2} \sum_{m=0}^{\infty} a_{m,j} U_m(\xi) , \tag{A2}$$

$U_m(\xi)$ being the Tschebyscheff polynomials of the second kind. It was also shown there that the expansion coefficients $a_{m,j}$, for the even eigenfunctions, are the solutions of the matrix problem

$$\tfrac{1}{2}\pi\lambda_j a_{2k,j}=\sum_{m=0}^{\infty}\alpha_{2k,2m}a_{2m,j}\,,\qquad j=1,3,\ \ldots \tag{A3}$$

and, for the odd ones:

$$\tfrac{1}{2}\pi\lambda_j a_{2k+1,j}=\sum_{m=0}^{\infty}\alpha_{2k+1,2m+1}a_{2m,j}\,,\qquad j=2,4,\ \ldots\,. \tag{A4}$$

The matrix elements are given by

$$\alpha_{k,m}=[(k+1)^2-m^2]^{-1}-[(k+1)^2-(m+2)^2]^{-1}\,.$$

We now derive a few interesting identities. Both the left- and right-hand sides of Eq. (A3) are summed over k. The sum over the matrix elements can be calculated using the expansion in partial fractions of $tg\frac{1}{2}\pi x$, viz.,[30]

$$tg\tfrac{1}{2}\pi x=\frac{4x}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2-x^2}\,.$$

Hence, $\frac{1}{2}\pi\lambda_j\sum_{k=0}^{\infty}a_{2k,j}$

$$=\sum_{m=0}^{\infty}\left(\frac{\pi tg\pi m}{8m}-\frac{\pi tg\pi(m+1)}{4(2m+1)}\right)a_{2m,j}=\tfrac{1}{8}\pi^2 a_{0,j}\,. \tag{A5}$$

This relation has been proved in Sec. IV, Eqs. (38) and (39), using physical arguments. With the aid of the expansion

$$\left(\frac{1}{\pi x}-ctg\pi x\right)\frac{\pi}{8x}=\sum_{k=0}^{\infty}\frac{1}{(2k+2)^2-4x^2}\,,$$

it can similarly be proved for the odd eigenfunctions that

$$\tfrac{1}{2}\pi\lambda_j\sum_{k=0}^{\infty}a_{2k+1,j}=\sum_{m=0}^{\infty}\frac{2(2m+2)}{[(2m+2)^2-1]^2}a_{2m+1,j}\,. \tag{A6}$$

Since the integral kernel in Eq. (A1) is symmetric, the eigenfunctions fulfill an orthogonality relation.[31] For the even eigenfunctions, we have

$$\int_{-1}^{+1}d\xi f_i(\xi)f_j(\xi)=\sum_{m,n=0}^{\infty}a_{2m,j}a_{2n,j}$$
$$\times\int_0^{\pi}d\vartheta\sin(2m+1)\vartheta\sin(2n+1)\vartheta\sin\vartheta$$
$$\equiv\sum_{m,n=0}^{\infty}a_{2m,j}u_{2m,2n}a_{2n,i}=\delta_{ij}\,.$$

A straightforward calculation shows that

$$u_{2m,2n}=(2n+1)\,\alpha_{2n,2m}\,.$$

Hence, it is immediately inferred by using Eq. (A3) that

$$\tfrac{1}{2}\pi\lambda_j\sum_{n=0}^{\infty}(2n+1)\,a_{2n,j}a_{2n,i}=\delta_{i,j}\,. \tag{A7}$$

Similarly, for the odd functions, using Eq. (A4), we obtain

$$\tfrac{1}{2}\pi\lambda_j\sum_{n=0}^{\infty}(2n+2)a_{2n+1,j}a_{2n+1,i}=\delta_{i,j}\,. \tag{A8}$$

The relations (A7) and (A8) are orthogonality relations for the columns of the matrix consisting of the eigenvectors of the systems (A3) and (A4). Also, an orthogonality relation exists for the rows. The spectral representation[32] of the operator $K^{(1)}(\xi,\xi')$ in Eq. (A1) is

$$K^{(1)}(\xi,\xi')=\sum_{j=1}^{\infty}\lambda_j f_j(\xi)f_j(\xi')\,. \tag{A9}$$

We introduce the new variables $\xi=\cos\vartheta$, $\xi'=\cos\vartheta'$, multiply Eq. (A9) with $\sin(m+1)\vartheta$ and $\sin(n+1)\vartheta'$ and integrate over ϑ and ϑ' from 0 to π. With the aid of Eq. (A2) the right-hand side of Eq. (A9) takes the form

$$=(\tfrac{1}{2}\pi)^2\sum_{j=1}^{\infty}\lambda_j a_{m,j}a_{n,j}\,.$$

On the other hand, the left-hand side of Eq. (A9) can be evaluated as well, using the Fourier expansion of $K^{(1)}(\cos\vartheta,\cos\vartheta')$ [see Eq. (A4) of I], viz.,

$$K^{(1)}(\cos\vartheta,\cos\vartheta')=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\vartheta\sin n\vartheta'}{n}\,.$$

A short calculation shows that

$$\sum_j\tfrac{1}{2}\pi\lambda_j a_{m,j}a_{n,j}=\delta_{m,n}/(m+1)\,. \tag{A10}$$

APPENDIX B

In Sec. IV C, an expression was derived for the emerging intensity due to the jth eigenfunction when $\mathfrak{L}(u)$ was a Lorentz or Voigt profile ($a\neq 0$) with or without hfs.

For this, a representation for the eigenfunctions was used different from that given in Eq. (5′), namely,

$$\psi_j(\xi)=(1-\xi^2)^{1/4}\sum_{m=0}^{\infty}b'_{m,j}\,C_m^{3/4}(\xi)\,. \tag{B1}$$

$C_m^{3/4}(\xi)$ is the Gegenbauer polynomial of order $\frac{3}{4}$ and degree m. Here the transformation matrix will be calculated which makes it possible to express the coefficients $b'_{m,j}$ in the $b_{m,j}$ of the usual representation [Eq. (5′)].

By applying the orthogonality relationship for the Gegenbauer polynomials, we have[33]

$$\frac{\pi 2^{-1/2}\Gamma(\frac{3}{2}+n)}{n!(n+\frac{3}{4})[\Gamma(\frac{3}{4})]^2}\, b'_{n,j} = \sum_{m=0}^{\infty} b_{m,j}\int_{-1}^{+1} d\xi$$

$$\times (1-\xi^2)^{1/4} C_n^{3/4}(\xi) U_m(\xi) = \sum_{m=0}^{\infty} b_{m,j}$$

$$\times \int_0^{\pi} d\varphi\, (\sin\varphi)^{1/2} C_n^{3/4}(\cos\varphi)\sin(m+1)\varphi\ . \quad \text{(B2)}$$

The latter integral can be calculated by using a formula due to Szegö,[34] expressing $(\sin\varphi)^{1/2}$ $C_n^{3/4}(\cos\varphi)$ in a sine series. By the orthogonality of the sine functions on $[0, \pi]$, it is immediately inferred that, in matrix form,

$$\vec{b}' = B\vec{b}\ ,$$

the matrix B having the matrix elements for $n, m = 0, 1, 2, \ldots$:

$$\beta_{n,m} = 0,\quad n > m,\quad \beta_{2n,2m+1} = \beta_{2n+1,2m} = 0\ ,$$

$$\beta_{2n,2m} = (2n+\tfrac{3}{4})\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\,\frac{\Gamma(m-n+\frac{1}{4})}{\Gamma(m-n+1)}\,\frac{\Gamma(n+m+1)}{\Gamma(n+m+\frac{7}{4})}\ , \quad \text{(B3)}$$

$$\beta_{2n+1,2m+1} = (2n+\tfrac{7}{4})\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\,\frac{\Gamma(m-n+\frac{1}{4})}{\Gamma(m-n+1)}\,\frac{\Gamma(n+m+2)}{\Gamma(n+m+\frac{11}{4})}\ .$$

It has been shown in Sec. IV C that with the representation [Eq. (B1)] a close analogy exists between the formulas for the emerging intensity for a Doppler profile and for a Lorentz or Voigt profile. The analogy extends also to other formulas. For example, Eq. (12′) becomes

$$\int_{-1}^{+1} d\xi\,\psi_j(\xi) = \frac{\pi(2\pi)^{1/2}}{3[\Gamma(\frac{3}{4})]^2}\, b'_{0,j}\ . \quad \text{(B4)}$$

This should be compared with Eq. (12). Also the expansion coefficients α_j for a Lorentz or Voigt profile $(a \neq 0)$ with or without hfs can be given for the problem treated in Sec. III A. We start with Eq. (22).

Using $u = 2(\nu-\nu_0)/\Delta\nu_L$, $k_0\Delta\nu_L/2 = h\nu_0 B(1,2)n(1)/4\pi$, and the representation Eq. (B1), we have

$$\tilde{A}_j(2,1)\alpha_j = \frac{I_0B(1,2)}{2\pi k_0}\sum_{m=0}^{\infty} b'_{m,j}\int_0^{\pi} d\vartheta(\sin\vartheta)^{3/2}C_m^{3/4}(\cos\vartheta)\int_0^{\infty} du\,k(u)\exp\left(-\frac{k(u)L(1+\cos\vartheta)}{2\cos\alpha}\right)$$

$$= \frac{2^{7/4}\Gamma(\frac{5}{4})}{\pi}\,\frac{I_0B(1,2)}{k_0L}\cos\alpha\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(m+\frac{3}{2})}{m!}\,b'_{m,j}\int_0^{\infty} du\left(\frac{k(u)L}{2\cos\alpha}\right)^{1/4}\exp\left[-\frac{k(u)L}{2\cos\alpha}\right] I_{m+3/4}\left(\frac{k(u)L}{2\cos\alpha}\right).$$

Here, $I_{3/4+m}$ is the modified Bessel function of order $\frac{3}{4}+m$.[29] The new variable $y = k_0L/2\pi(x^2+1)$ $\times\cos\alpha$ is introduced. Upon asymptotic expansion for $k_0L\to\infty$, we obtain

$$\tilde{A}_j(2,1)\alpha_j \sim \frac{I_0B(1,2)}{2\pi}\left(\frac{\cos\alpha}{k_0L}\right)^{1/2}$$

$$\times\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(m+\frac{3}{2})\Gamma(m+\frac{1}{2})}{m!(m+1)!}\,b'_{m,j}\ . \quad \text{(B5)}$$

This should be compared with the same formula for a simple Doppler profile [Eq. (23)]. Upon substitution of the expression for $\tilde{A}_j(2,1)$[Eq. (5)], Eq. (B5) takes the form

$$\alpha_j = \frac{I_0B(1,2)}{2\pi A(2,1)}\left(\frac{\cos\alpha}{\mu'_j}\right)^{1/2}$$

$$\times\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(m+\frac{3}{2})\Gamma(m+\frac{1}{2})}{m!(m+1)!}\,b'_{m,j}\ . \quad \text{(B6)}$$

Equation (B6) has been derived for a Lorentz profile but is as well valid for a Voigt profile $(a\neq 0)$.

It has been shown in Sec. III that in the equation analogous to Eq. (B5), Eq. (23), the summation over m could be carried out by Eq. (A5). The reader might ask if something like it is possible for Eq. (B5). This requires the solution of the integral equation given in Appendix C of I in terms of the representation [Eq. (B1)]. This problem is under investigation. However, the answer to the question is probably in the affirmative. The relations between the coefficients $a_{m,j}$ of the representation for a Doppler profile, Eq. (4′), derived in Appendix A, could also be found by physical arguments. An example of this has been given in Sec. IV as regards Eq. (A5). These physical arguments are the same for a Lorentz profile.

Since, moreover, a close analogy exists between the formulas for a Doppler profile and for a Lorentz profile, it is expected that relations analogous to these exist as well for the coefficients $b'_{m,j}$ of Eq. (B1). This point is under investigation.

Note added in proof. In the mean time the above mentioned problems have been solved. The results will be reported elsewhere.[35]

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[12]R. H. Fowler, Statistical Mechanics (Cambridge University Press, Cambridge, 1955). It should be noted that the convergence in Eq. (14) is not uniform since $\psi_j(\pm 1)=0$, so that $n(2)/n(1)$ is not a continuous function of ξ. The proof that for $\tilde{A}_j(2,1) \ll n_e K(1,2)$ [Eq. (11)] reduces to Eq. (14) is not entirely correct. Actually, for a Doppler profile, for instance, $\tilde{A}_j(2,1)/\tilde{A}_1(2,1) \sim j$, $j \to \infty$ (see Ref. 25 of I). Therefore, the requirement $\tilde{A}_j(2,1) \ll n_e K(1,2)$ cannot be fulfilled for all values of j. Consequently, $\tilde{A}_j(2,1)$ cannot be neglected in all terms of Eq. (11). These terms describe a nonequilibrium behavior which is sometimes genuine and corresponds to physical reality. However, sometimes it is attributed to a breakdown of the validity of the asymptotic expansion. To see this more clearly, suppose that the ψ_j and $\tilde{A}_j(2,1)$ represent the true (and not the asymptotic) solutions of the Biberman-Holstein integral equation. Hence, $\tilde{A}_j(2,1) < A(2,1)$ and $\tilde{A}_j(2,1) \to A(2,1)$, $j \to \infty$. If $A(2,1) \ll n_e K(1,2)$, all $\tilde{A}_j(2,1)$ can be neglected in the denumerator of Eq. (11), and Eq. (14) is established rigorously. If we now replace in Eq. (11) the ψ_j and $\tilde{A}_j(2,1)$ by their asymptotic values, we see that the nonequilibrium terms with $\tilde{A}_j(2,1) \approx$ or $> n_e K(1,2)$ also have the property $\tilde{A}_j(2,1) \gg A(2,1)$. For these, the asymptotic expansion is not valid. Therefore, they describe no real effect. Suppose now that $\tilde{A}_1(2,1) \ll n_e K(1,2) \ll A(2,1)$. Hence, there are eigenvalues $\tilde{A}_j(2,1)$ with $n_e K(1,2) \ll \tilde{A}_j(2,1) \ll A(2,1)$. These eigenvalues and the corresponding eigenfunctions are asymptotically correct. Hence, they correspond to physical reality. In the center of a slab, thermodynamic equilibrium is established but, at the boundary where the higher eigenfunctions contribute relatively more, certain slight departures remain. In order to suppress these fully, it is necessary that $A(2,1) \ll n_e K(1,2)$. In conclusion, the requirement for Eq. (14) is not fully correct. It describes the experimental fact that, in an optically thick plasma, thermodynamic equilibrium is earlier established (except very close to the boundary) than in an optically thin one.

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[16]Because of stepwise ionization, a change in the density $n(2)$ may influence the electron temperature.

[17]C. van Trigt, thesis, University, Utrecht, The Netherlands (unpublished). The fact that Eqs. (15) and (18) and Eqs. (19) and (20) are identical is useful for the numerical calculation of Eq. (11). By subtracting Eq. (15) or (19) from Eq. (11), the latter is converted to a more quickly convergent series, since $\tilde{A}_j(2,1)/\tilde{A}_1(2,1)$ is an increasing function of j.

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[20]L. B. W. Jolley, Summation of Series, (Chapman and Hall Ltd., Dover, 1961).

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[23]However, it can also be found with the aid of the pertinent expansion coefficients Eq. (15) and the orthogonality relation Eq. (A9).

[24]V. V. Ivanov, Theory of Stellar Spectra (National Aeronautics and Space Administration, Washington, D.C., 1967), p. 112; D. I. Nagirner, Astrophysika 3, 293 (1967) [English transl.: Astrophysics 3, 133 (1967)]. Dr. Ivanov has informed me that a new book by him on radiative transfer is in preparation, in which in a sense the Fourier transform [Eq. (17) of I] plays the central role as well.

[25]Another interpretation with the aid of Eq. (31) is also known. In the core of the line, the optical depth is such that the contribution by the layers near the boundary to the emitted radiation is dominant. The local excitation temperature defined there in the usual way by $n(2)/n(1)$ is low. Therefore, the radiation temperature at this frequency is low as well. Proceeding along the frequency scale to the wings, the optical depth decreases and more layers with a higher excitation temperature contribute. Hence, the radiation temperature at these frequencies increases. For frequencies defined by $k(u)L/\cos\alpha \leq 1$, the slab is optically thin. All layers are now equivalent. The decrease in the intensity of the radiation follows the common Doppler profile, as already proved in Eq. (33).

[26]D. G. Hummer, Monthly Notices Roy. Astron. Soc. (to be published).

[27]When the radiative transfer is determined by the Doppler line, the emergent intensity is found by substituting for $\mathfrak{L}(u)$ in Eq. (32) a Voigt profile. As can be inferred from Eqs. (19) and (A2) of I, the radiative transfer is determined by the Doppler part of the line if $\sqrt{\pi}/2k_0L \gg a\ln(k_0L/2\sqrt{\pi})$. This requires a small value of a and not excessively large k_0L.

[28]Reference 8, p. 174. A further advantage of this representation is that a close analogy is brought about between all the formulas given in this paper for a Doppler profile on the one hand and for a Lorentz or Voigt profile ($a \neq 0$) on the other.

[29]Reference 8, pp. 5 and 57.

[30]E. T. Whittaker and G. N. Watson, A Course of Modern Analysis (Cambridge University Press, Cambridge, 1962).

[31]R. Courant and D. Hilbert, Methods of Mathematical Physics (Wiley-Interscience Publishers Inc., New York, 1953), Vol. I, p. 129.

[32]Reference 31, Vol. I, p. 134.

[33]Reference 8, p. 174.

[34]G. Szegö, Orthogonal Polynomials (American Mathematical Society Colloquium Publications, New York, 1959), Vol. XXII, p. 96.

[35]C. van Trigt, thesis, University, Utrecht, The Netherlands (unpublished).

PHYSICAL REVIEW A VOLUME 1, NUMBER 5 MAY 1970

Appraisal of an Iterative Method for Bound States*

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An iterative method for determining bound-state eigenvalues and properties of the radial Schrödinger equation is appraised. The method stems from iterating the integral equation $\psi = \mu(T + \frac{1}{2}\gamma^2)^{-1}(-V\psi)$, where T and V are the kinetic- and potential-energy operators. The basic theory is briefly reviewed, and calculations are performed for the Coulomb and screened-Coulomb potentials. The lowest three μ eigenvalues, together with the expected values of $(\gamma r)^{-1}$, γr, and $(\gamma r)^2$, are obtained from a single iterated eigenfunction sequence. Convergence is rapid for eigenvalues but slow for expected values. There is some sensitivity to the choice of the numerical integration formula. Regarded as a numerical method, this approach may be most competitive for the determination of zero-energy potential-strength eigenvalues. Its disadvantages are listed. Analytical improvements to eigenfunctions can be easier to obtain by iteration than by perturbation, and some success has been achieved. A simple example suggests that the rate of convergence of an iterated eigenfunction sequence is less than that of a related perturbation sequence unless the choice of starting function is bad.