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[13]See Ref. (4) p. 58 for a discussion of this point.

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Wave-Radiation Model for the Onset of Dissipation at the Roton Critical Velocity in Superfluid Helium†

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A linearized form of the Gross-Pitayevski (GP) equation is used to calculate the rate at which quasiparticles are created by a small sphere moving through a superfluid at a velocity greater than $(\omega_k/k)_{\min}$, the Landau roton critical velocity. Because the excitations described by the linearized GP equation can be given a hydrodynamic interpretation, the qualitative features of the model porposed here for the roton critical velocity are analogous to the many critical-velocity phenomena of classical physics. The quantitative results of the model indicate that energy dissipation sets in so rapidly that, once the creation of quasiparticles is allowed kinematically, it should be impossible experimentally to force a negative-ion complex to move observably faster than the roton critical velocity. An investigation of the meaning of the linearized GP equation is also given.

INTRODUCTION

The roton critical velocity predicted by Landau in 1941[1] was first observed experimentally by Meyer and Reif in 1961.[2] By measuring the energy dissipated by a negative-ion complex moving through superfluid helium at low temperature and high pressure,[3] a threshold was observed for the creation of excitations in the superfluid. Their measured critical velocities agreed with those predicted by Landau, namely, $(\omega_k/k)_{\min} \approx 52$–$57$ m/sec depending on the pressure; and the distinction between quasiparticle[4] and vortex creation has been particularly clarified in the later experiments of Rayfield[5] and of Neeper and Meyer.[6]

The original theory predicting the roton critical velocity was a kinematic argument showing that it becomes possible for a foreign body to create quasiparticles in the superfluid only when the foreign body moves faster than $v_c = (\omega_k/k)_{\min}$, ω_k being the quasiparticle excitation spectrum. Since that time, only one theoretical discussion has attempted to investigate the magnitude and velocity dependence of the rate of energy dissipation due to this allowed creation of the quasiparticles when the foreign body moves faster than v_c.[7] As suggested by Reif and Meyer and elaborated in Appendix A, one would at first expect to determine this energy dissipation by carrying out a Fermi "golden rule" calculation of the probability rate for the quasiparticle creation processes. However, the interaction between an ion complex and the roton quasiparticles of the superfluid is unknown.[8] One needs to know this interaction and needs a model analogous, for instance, to the deformation potential commonly used for the electron-phonon interaction in metals.[9]

With the excitations of the superfluid described by a linearized form of the Gross-Pitayevaki (GP) equation,[10] Secs. IV-VI of the present paper use a hydrodynamic model both for the excitations of the superfluid and for the interaction between the moving ion complex and the excitations of the superfluid. The linearized GP equation and concomitant description of the excitations of the system are extensions of Gross's hydrodynamic description of the superfluid condensate for a zero-temperature system of weakly interacting bosons.[11] With $\psi(\vec{x}, t)$ expanded as $f + \varphi(\vec{x}, t)$, where f^2 is the density of the undisturbed condensate, the fluctu-

ation part of this wave function, $\varphi(\vec{x}, t)$, is associated with the quasiparticle excitations. Substituting this expansion for $\psi(\vec{x}, t)$ into the GP equation, and assuming $|\varphi(\vec{x}, t)| \ll f$, leads to a linearized equation of motion for $\varphi(\vec{x}, t)$. Similarly, expressions for particle and momentum densities and for the energy-flow density can be expressed in terms of the fluctuation $\varphi(\vec{x}, t)$. Section III sets forth the relationship between the real and imaginary parts of $\varphi(\vec{x}, t)$ and a local density fluctuation and velocity potential $\tilde{\rho}(\vec{x}, t)$ and $S(\vec{x}, t)$, and hence suggests a hydrodynamically meaningful model of superfluid excitation.

What is proposed in the present paper is the use of the property $\mathrm{Im}\varphi(\vec{x}, t) \propto S(\vec{x}, t)$ to set up a hydrodynamic type of boundary-value problem for the velocity potential of the excitations created in the superfluid by the ion complex. The moving complex is first replaced by a solid sphere and then by a hydrodynamic dipole source that is used with the Green's function for the equation of motion for $S(\vec{x}, t)$ in order to obtain a solution describing the roton radiation when $v > (\omega_k/k)_{\min}$. However, it should be pointed out that, although the mathematics of this approach is classical in appearance and the terms "phase velocity," "group velocity," "constructive interference," and "boundary-value problem" will be used, the problem treated is basically quantum mechanical in nature because Planck's constant appears throughout the equations in the ratio $(\hbar/m)$. While any wave function can be given a hydrodynamic interpretation,[12] it is only for the condensed Bose liquid that the phase of a one-body wave function can be associated with the velocity potential for the motions of an excited system.

At this point, one may question the sense in which the quanta emitted in the individual roton creation processes may be assumed coherent when described in the fluctuation $\varphi(\vec{x}, t)$. According to the "golden rule" estimates of Appendix A, the quasiparticles created by an ion complex moving faster than v_c are emitted in such rapid succession that they may not be regarded as independent and nonoverlapping. In this regard, the situation is quite analogous to that of Čerenkov radiation where classical electromagnetic theory rather than quantum electrodynamics is used to describe the photon emission problem.[13] There, the classical theory follows from the quantum theory by the replacement of photon operators with c numbers in the sense of large-number occupation of the photon states. It is in this same sense that the one-body wave function $\varphi(\vec{x}, t)$ can be used in the superfluid problem to describe the aggregate behavior of a large number of quasiparticles superimposed on a uniform condensate. The development of this point of view is given in Sec. I and then applied in Secs. II and III to derive various physical properties in terms of the $\varphi(\vec{x}, t)$'s.

In another sense, the Čerenkov and roton radiation problems are quite different. When there is no dispersion in a medium, the wave disturbance created by a moving foreign body is confined to large-amplitude shock fronts. Such is the case for Mach waves and Čerenkov radiation. On the other hand, when the group velocity can take on a range of values in a dispersive medium, the wave-radiation disturbance created by a moving foreign body can be extended over a considerable region of space, and hence be of small amplitude. Such is the case for several water-wave phenomena[14] – in particular, the surface waves created by a small object moving slightly faster than 23 cm/sec.[15] For the superfluid problem the linearization condition $|\varphi(\vec{x}, t)| \ll f$ is a reasonable assumption based on the fact that the quasiparticle spectrum has pronounced dispersion in the roton region. The assumption, however, turns out to be valid only at large distances from an object moving at the roton critical velocity. Because of this difficulty, the linearized theory can be used to calculate only a lower bound for the roton radiation.

Since the material to be presented covers several disciplines, a general outline is given here in the Introduction. Section I addresses basic questions concerning the meaning of the linearized GP equation when applied to the roton radiation problem, especially the analogy with the classical, large-occupation limit used implicitly in the standard theory of Čerenkov radiation. Using the formulations of Sec. I, Secs. II and III express various physical properties of the aggregate quasiparticle excitations in terms of $\varphi(\vec{x}, t)$: The discussion in Sec. II is devoted to the particle, momentum, energy, and energy-flow densities; Sec. III is devoted to equations of motion, the velocity potential, and approximate boundary conditions.

Although the results at this point are all self-consistent within the approximations of the generalized Bogoliubov theory,[16] the formulas have to be simplified for the purpose of developing the roton wave-radiation model of Sec. IV–VI. For instance, while the equation of motion first derived for $\varphi(\vec{x}, t)$, namely, Eq. (3.7), is a generalization of the linearized GP equation, it is also too formidable to use in a simple model. Dropping the intractable terms is equivalent to making the approximations of standard Bogoliubov theory[17] and reduces the equation of motion to Eq. (3.8), the linearized GP equation. It is Eq. (3.15), a variant of this simplified equation for $\varphi(\vec{x}, t)$, that is used to describe the velocity potential $S(\vec{x}, t)$ in the "boundary-value problem" solved in Secs. IV–VI. Since a derivation of the excitation spectrum is not within the objectives of the present paper, the interaction potential in Eq. (3.15) is simply replaced by an effective potential, adjusted phenom-

enologically so as to make the Bogoliubov spectrum agree with the empirically known ω_k in the roton region.

In Sec. IV the Green's function for $S(\vec{x}, t)$ is used with a suitable source distribution to derive a formula for $S(\vec{R})$, the approximate velocity potential for the disturbance created in the model superfluid system by a moving sphere of 10–20 Å radius. When the velocity of the moving sphere passes from less than to greater than $(\omega_k/k)_{\min}$, the disturbance that it creates passes abruptly from retarded dipole flow to a roton wave-radiation field. Following the discussion in Sec. V for the general form of this wave-radiation field, a derivation is given in Sec. VI for the rate of energy lost into the roton radiation. The results of Sec. V are analogous to the classical water-wave problem already mentioned.[15]

Estimates based on a "golden rule" calculation are given in Appendix A. Appendix B suggests that, since time-retardation effects are essential in the present model for understanding the onset of roton radiation, they may also play a key role in the problem of how primordial vorticity can form in a superfluid.

I. WAVE-FUNCTION APPROXIMATION

The relationship between quantum electrodynamics and electromagnetic theory is well known – when the number of photons is large, the potential field operators of quantum electrodynamics can be replaced by the classical potential field variables of electromagnetic theory. The corresponding relationship between the second-quantized formulation for bosons and Schrödinger wave functions is perhaps not as common – when the number of bosons is large, the field operator $\underline{\psi}(\vec{x}, t)$ can be replaced by a one-particle wave function $\psi(\vec{x}, t)$.[18,19] For instance, if the operator $\underline{\psi}(\vec{x}, t)$ is expanded on a complete set of states according to

$$\underline{\psi}(\vec{x}, t) = \sum u_{\vec{l}}(x, t)\underline{a}_{\vec{l}}, \tag{1.1}$$

then the wave function for the single-particle state labeled by $\vec{k}$ can be expressed

$$u_{\vec{k}}(\vec{x}, t) \propto \langle \vec{k} | \underline{\psi}(\vec{x}, t) | \vec{k} \rangle, \tag{1.2}$$

where the number of particles in the level $\vec{k}$ of the state $|\vec{k}\rangle$ is large compared to unity, in order that the expectation value of $\underline{a}_{\vec{k}}$ may be treated (in a sense to be carefully defined) as nonzero. If the state used in taking the expectation value represents macroscopic occupation over a spread of $\vec{k}$ values, then the operator can be replaced by a general one-body wave function.

Now, for the condensed system of interacting bosons the expansion of the operator $\psi(\vec{x}, t)$ is not as simple as Eq. (1.1). The approximate one-body wave function for this system with a given quasiparticle level highly occupied follows from the replacement of a Bogoliubov operator $\underline{\alpha}_{\vec{k}}(t)$ by a complex function of time in the equations[20,21]

$$\underline{\psi}(\vec{x}, t) = e^{-i(\mu/\hbar)t}[f + \underline{\varphi}(\vec{x}, t)], \tag{1.3}$$

$$\underline{\varphi}(\vec{x}, t) = (1/\sqrt{\Omega}) \sum{}' e^{i\vec{l}\cdot\vec{x}} \underline{a}_{\vec{l}}(t), \tag{1.4}$$

$$\underline{a}_{\vec{l}}(t) = A_+(\vec{l})\underline{\alpha}_{\vec{l}}(t) + A_-(\vec{l})\underline{\alpha}_{-\vec{l}}^{\dagger}(t). \tag{1.5}$$

Here Ω is the volume, $\vec{l}$ the discrete momentum variable (sometimes also denoted by $\vec{k}$), and f^2 is the condensate density N_0/Ω (N_0 being the number of particles in the condensate). The prime indicates the sufficient restriction that $\vec{l} = 0$ is excluded from the summation, while the coefficients $A_+(\vec{l})$ and $A_-(\vec{l})$ are restricted by the condition

$$A_+^2(\vec{l}) - A_-^2(\vec{l}) = 1 \tag{1.6}$$

in order that the quasiparticle operators obey simple commutation relations. Because of the time dependence of $\underline{\alpha}_{\vec{l}}(t)$, a convenient way of writing Eqs. (1.4) and (1.5) is

$$\underline{\varphi}(\vec{x}, t) = (1/\sqrt{\Omega}) \sum{}' [A_+(\vec{l})\underline{\alpha}_{\vec{l}}(0) e^{i(\vec{l}\cdot\vec{x} - \omega_{\vec{l}} t)} + A_-(\vec{l})\underline{\alpha}_{\vec{l}}^{\dagger}(0) e^{-i(\vec{l}\cdot\vec{x} - \omega_{\vec{l}}(t)}]. \tag{1.7}$$

This equation is of precisely the same form as the operator expansions for the vector and scalar potentials in quantum electrodynamics. The replacement of an $\underline{\alpha}_{\vec{l}}(0)$ by a c number in Eq. (1.7) gives the wave-function limit for the condensed Bose system, namely,

$$\psi_{\vec{k}}(\vec{x}, t) = e^{-i(\mu/\hbar)t}\{f + \varphi_{\vec{k}}(\vec{x}, t)\}$$

and

$$\varphi_{\vec{k}}(\vec{x}, t) = (n/\Omega)^{1/2}[A_+(\vec{k}) e^{i(\vec{k}\cdot\vec{x} - \omega_{\vec{k}} t)} + A_-(\vec{k}) e^{-i(\vec{k}\cdot\vec{x} - \omega_{\vec{k}} t)}], \tag{1.8}$$

where $n \gg 1$ is the number of quasiparticles all approximately in the same momentum level $\vec{l} = \vec{k}$, or more generally

$$\psi(\vec{x}, t) = e^{-i(\mu/\hbar)t}[f + \varphi(\vec{x}, t)]$$

and $$\varphi(\vec{x},t)=\sum{}'\{\mathfrak{N}(\vec{k})/\Omega\}^{1/2}[A_+(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega_{\vec{k}}t)} + A_-(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega_{\vec{k}}t)}], \tag{1.9}$$

where $\mathfrak{N}(\vec{k})$ is a general number distribution. Although the wave-function expressions will at first appear to be unusual, the time averages of the wave-function expressions always turn out to be equal to a rigorous expectation value of the operator expressions.

Among the operator expressions of interest are $\underline{\psi}^\dagger(\vec{x},t)\underline{\psi}(\vec{x},t)$ and $\mathrm{Re}\{\underline{\psi}^\dagger(\vec{x},t)[-i\hbar\vec{\nabla}]\underline{\psi}(\vec{x},t)\}$. The corresponding wave-function expressions give approximate particle and momentum flow densities. One can expect the equation of motion for $\varphi(\vec{x},t)$, the fluctuation part of the wave function $\psi(\vec{x},t)$, to give the Bogoliubov spectrum when $|\underline{\varphi}(\vec{x},t)|$ is assumed small compared to f.

However, for expressions involving products of $\underline{\psi}(\vec{x},t)$ operators, it is not clear *a priori* whether or not the products of Bogoliubov operators should be written in normal-order form before they are replaced by c numbers. It is argued here that all operator-product expressions should first be normal ordered. The various complications of the theory that result are shown in Sec. II to work out self-consistently. For instance, with proper normal ordering the equation of motion for $\varphi(\vec{x},t)$ gives the generalized Bogoliubov excitation energy ω_k, namely, Eq. (2.26) in Paper II, rather than just the standard Bogoliubov spectrum; and the average energy density for n quasiparticles of momentum $\vec{k}$ turns out to be $n\hbar\omega_k/\Omega$, where ω_k is again given by the generalized Bogoliubov theory.

To understand the need for normal ordering before making the c-number replacements for the $\underline{\alpha}_{\vec{k}}^\dagger$'s and $\underline{\alpha}_{\vec{k}}$'s, one may compare the expectation values of an operator product in the ground state and in the state with $n\gg 1$ quasiparticles all in approximately the same momentum level $\vec{k}$. The ground-state expectation value is not zero, but rather is equal to the c-number terms that arise from writing the product expression in normal-order form with only $\alpha_{\vec{k}}^\dagger$'s operating on the vacuum-state vector. Now, when the expectation value is taken in the excited state, one might at first expect these normal-order correction terms to be negligible compared to the large wave-function contributions. However, even with the large total number of quasiparticle present in a wave packet state of momentum $\vec{k}$, there must be enough uncertainty in the quasiparticle number distribution such that any given excitation level has an average occupation number much less than unity in the limit of an infinite-volume system. Then the normal-order correction terms are all as significant as for the ground state, and the excited-state properties show contributions in addition to, and are not complete alterations of, the ground-state properties.

For instance, the energy of the wave packet of momentum $\vec{k}$ should be E_0 plus an excitation energy $\hbar\omega_k$, where E_0 and $\hbar\omega_k$ are given for the generalized Bogoliubov theory by Eqs. (2.25) and (2.26) of Paper II. In the electromagnetic case where there is no photon-photon interaction, $\hbar\omega_k$ reduces to $\epsilon_k-\mu$, and E_0 corresponds to just the zero-point vacuum fluctuation energy $\sum'\epsilon_k A_-{}^2(\vec{k})$ which is usually dismissed as unobservable. The zero-point effects in the condensed system of interacting bosons, on the other hand, influence the excitation spectrum as well as the ground-state energy and cannot be dismissed as insignificant.

The various properties of the Bose system are defined in terms of products of the field operators $\underline{\psi}(\vec{x},t)$. After the operator displacement of Eq. (1.3) for the condensed-boson system, the various properties are expressed in terms of the operators $\underline{\varphi}^\dagger(\vec{x},t)$ or $\underline{\varphi}(\vec{x},t)$ and their spatial derivatives. Then an operator product like $\underline{\varphi}_1\underline{\varphi}_2$ can be written

$$\underline{\varphi}_1\,\underline{\varphi}_2=\langle\underline{\varphi}_1\underline{\varphi}_2\rangle_0+:\underline{\varphi}_1\underline{\varphi}_2: \tag{1.10}$$

where only the normal-order correction term $\langle\underline{\varphi}_1\,\underline{\varphi}_2\rangle_0$, a c number, has to be elaborated for the various forms of products. Then the normal-ordered products such as $:\underline{\varphi}_1\underline{\varphi}_2:$ are simply replaced by the corresponding products of wave functions given by Eq. (1.8) or, in the more general case, by Eq. (1.9).

For later purposes, the most useful normal-order correction terms are

$$\begin{aligned}
&\langle\underline{\varphi}^\dagger(\vec{x},t)\varphi^{(\dagger)}(\vec{x}+\vec{r},t)\rangle_0 \\
&\quad=(1/\Omega)\sum{}'A_{(\mp)}(\vec{l})A_-(\vec{l})e^{-i\vec{l}\cdot\vec{r}}, \\
&\langle\underline{\varphi}^{(\dagger)}(\vec{x}+\vec{r},t)\vec{\nabla}\underline{\varphi}(\vec{x},t)\rangle_0 \\
&\quad=(-i/\Omega)\sum{}'\vec{l}A_{(\mp)}(\vec{l})A_-(\vec{l})e^{i\vec{l}\cdot\vec{r}}, \\
&\langle\underline{\varphi}^{(\dagger)}(\vec{x}+\vec{r},t)\nabla^2\underline{\varphi}(\vec{x},t)\rangle_0 \\
&\quad=(1/\Omega)\sum{}'l^2A_{(\mp)}(\vec{l})A_-(\vec{l})e^{i\vec{l}\cdot\vec{r}}, \\
&\langle-(\hbar^2/2m)\nabla^2\underline{\varphi}^\dagger(\vec{x},t)\vec{\nabla}\underline{\varphi}(\vec{x},t)\rangle_0 \\
&\quad=(-i/\Omega)\sum{}'\vec{l}\epsilon_l A_-{}^2(\vec{l}).
\end{aligned} \tag{1.11}$$

The question of quasiparticle coherence in the large occupation limit is related to the argument just given for keeping the normal-order correction terms. The assertion is that for a large-volume

system the fine details of the quasiparticle-number eigenvalues $n_{\vec{k}}$ are indeterminate and irrelevant in the large-occupation limit in which operators are replaced by c numbers. Rather than by $n_{\vec{k}}$, this large-occupation limit is fixed by only a coarse-grained number distribution $\mathfrak{N}(\vec{k})$ defined in the infinite volume limit by

$$\mathfrak{N}(\vec{k}) = \lim_{\Omega\to\infty}\left[\sum_{D(\vec{l}\,')} n_{\vec{l}}\,' \Big/ \sum_{D(\vec{l}\,')} 1\right], \tag{1.12}$$

where the sums over the momentum levels $\vec{l}\,'$ are limited to a domain $D(\vec{l}\,')$ about the value $\vec{l}$, $D(\vec{l}\,')$ being very small but larger than the diminishing spacing $\delta\vec{l}\,'$ between the levels of the system as $\Omega\to\infty$. Although both $n_{\vec{l}}\,'$ and $\delta\vec{l}$ become small, and in fact $n_{\vec{l}}\,'$ may become quite erratic in this limit, $\mathfrak{N}(\vec{k})$ is well defined, retaining only the smooth-averaged behavior of $n_{\vec{l}}\,'$. When the total number of quasiparticles is large, $\mathfrak{N}(\vec{k})$ can be a nonzero function of $\vec{k}$, for instance, a sharply peaked function about $\vec{l}=\vec{k}$ for a wave packet. The large-occupation limit, then, is not related uniquely to a state of the system, both because of the erratic behavior of $n_{\vec{l}}$ for increasing Ω and because a given $n_{\vec{l}}\,'$ does not define a state of fixed phase.

It is not hard to show that the space- or time-average expressions involving $\varphi_{\vec{k}}(\vec{x},t)$ with the associated normal-ordering corrections included are identical to the result obtained by taking a rigorous expectation value of the corresponding operator expressions. The following formulas are used in Sec. II:

$$\begin{aligned}
&\langle n,\vec{k}|:\text{(odd number of } \underline{\varphi}\text{'s)}:|n,\vec{k}\rangle = 0,\\
&\langle n,\vec{k}|:\underline{\varphi}^{\dagger}(x,t)\underline{\varphi}(\vec{x}+\vec{r},t):|n,\vec{k}\rangle = \{[A_+^{\,2}(\vec{k})+A_-^{\,2}(\vec{k})]\cos(\vec{k}\cdot\vec{r})+i\sin(\vec{k}\cdot\vec{r})\}n/\Omega\ ,\\
&\langle n,\vec{k}|:\underline{\varphi}^{\dagger}(\vec{x},t)\underline{\varphi}^{\dagger}(\vec{x}+\vec{r},t):|n,\vec{k}\rangle = 2A_+(\vec{k})\cos(\vec{k}\cdot\vec{r})n/\Omega,\\
&\langle n,\vec{k}|:\underline{\varphi}^{\dagger}(\vec{x}+\vec{r},t)\vec{\nabla}\underline{\varphi}(\vec{x},t):|n,\vec{k}\rangle = \vec{k}\{[A_+^{\,2}(\vec{k})+A_-^{\,2}(\vec{k})]\sin(\vec{k}\cdot\vec{r})+i\cos(\vec{k}\cdot\vec{r})\}n/\Omega\ ,\\
&\langle n,\vec{k}|:\underline{\varphi}(\vec{x}+\vec{r},t)\vec{\nabla}\underline{\varphi}(\vec{x},t):|n,\vec{k}\rangle = \vec{k}A_+(\vec{k})A_-(\vec{k})\sin(\vec{k}\cdot\vec{r})n/\Omega\ ,\\
&\langle n,\vec{k}|:\underline{\varphi}^{\dagger}(\vec{x}+\vec{r},t)\nabla^2\underline{\varphi}(\vec{x},t):|n,\vec{k}\rangle = -k^2\{[A_+^{\,2}(\vec{k})+A_-^{\,2}(\vec{k})\}\cos(\vec{k}\cdot\vec{r})-i\sin(\vec{k}\cdot\vec{r})\}n/\Omega\ ,\\
&\langle n,k|:\underline{\varphi}(\vec{x}+\vec{r},t)\underline{\varphi}(\vec{x},t):|n,\vec{k}\rangle = -2k^2A_+(\vec{k})A_-(\vec{k})\cos(\vec{k}\cdot\vec{r})\,n/\Omega\ ,
\end{aligned} \tag{1.13}$$

and $\langle n,\vec{k}|:(-\hbar^2/2m)\nabla^2\underline{\varphi}(\vec{x},t)\vec{\nabla}\underline{\varphi}(\vec{x},t):|n,\vec{k}\rangle = i\vec{k}\,\epsilon_k n/\Omega$.

Here, $|n,\vec{k}\rangle$ is the ket vector with n quasiparticles of momentum $\vec{k}$. The expectation values are rigorous involving no c-number replacements, and hence in this case n may or may not be large.

II. EXCITATIONS OF WELL-DEFINED WAVE NUMBER

In this section the particle and momentum densities and the average particle, momentum, energy-flow densities of the wave-function excitations with wave number $\vec{k}$ are derived within the approximation of the generalized Bogoliubov theory and the assumption that $n/\Omega \ll f^2$. Before expressions for these quantities are derived, it is convenient to define notation by pointing out that Eq. (1.8) for $\varphi_{\vec{k}}(\vec{x},t)$ may be written in the form

$$\varphi_{\vec{k}}(\vec{x},t) = B_+(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + B_-(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\ , \tag{2.1}$$

or as

$$\varphi_{\vec{k}}(\vec{x},t) = \varphi^1_{\vec{k}}(\vec{x},t) + i\ \varphi^2_{\vec{k}}(\vec{x},t)\ , \tag{2.2}$$

where $B_\pm(\vec{k}) = (n/\Omega)^{1/2}A_\pm(\vec{k})$, (2.3)

$$\varphi^1_{\vec{k}}(\vec{x},t) = \zeta_1(\vec{k})\cos(\vec{k}\cdot\vec{x}-\omega_k t)\ , \tag{2.4}$$

$$\varphi^2_{\vec{k}}(\vec{x},t) = \zeta_2(\vec{k})\sin(\vec{k}\cdot\vec{x}-\omega_k t)\ , \tag{2.5}$$

$$\zeta_1(\vec{k}) = B_+(\vec{k}) + B_-(\vec{k})\ , \tag{2.6}$$

and $$\zeta_2(\vec{k}) = B_+(\vec{k}) - B_-(\vec{k})\ . \tag{2.7}$$

The B's and ζ's, of course, are restricted through Eq. (1.6) according to

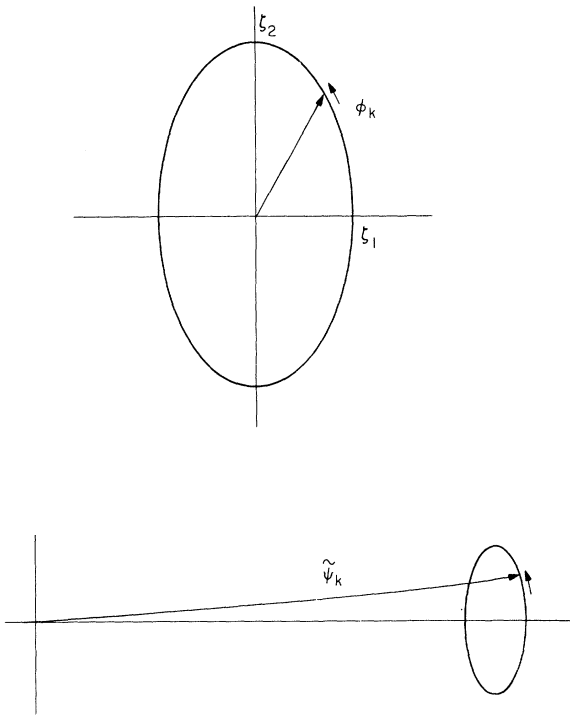

FIG. 1. The trajectories of $e^{i(\mu/\hbar)t}\psi_{\vec{k}}(\vec{x},t)$ and $\psi_{\vec{k}}(\vec{x},t)$. The arrows represent the complex values of these wave functions, the trajectories being traversed in the clockwise sense for increasing time.

$$B_+^2(\vec{k}) - B_-^2(\vec{k}) = n/\Omega = \zeta_1(\vec{k})\zeta_2(\vec{k}) \ . \tag{2.8}$$

Thus, as shown in Fig. 1, $\varphi_{\vec{k}}(\vec{x},t)$ traverses an elliptic trajectory around the origin with intercepts $\zeta_1(\vec{k})$ and $\zeta_2(\vec{k})$ on the real and imaginary axes and area n/Ω. In the approximation of standard Bogoliubov theory for which all normal-order corrections are neglected, one can show that the following limiting values for the A's and ζ's hold:

$$\begin{aligned} A_+(\vec{k}) &\to \infty && 1 \ , \\ A_-(\vec{k}) &\to -\infty && 0 \ , \\ \hbar\omega_k &\to \hbar(f^2U_k/4m)^{1/2}k && \epsilon_k = \hbar^2k^2/2m, \quad \text{and} \\ \zeta_1(\vec{k}) &\to 0 && n/\Omega \ , \\ \zeta_2(\vec{k}) &\to \infty && n/\Omega \ , \\ k &\to 0 && k \to \infty \ . \end{aligned} \tag{2.9}$$

Hence one can expect the trajectory of $\varphi_{\vec{k}}(\vec{x},t)$ to be a circle in the high-k limit (or when $\hbar\omega_k$ and the kinetic energy cross) and to become a very elongated ellipse in the low-k limit.

A. Particle and Momentum Densities

The particle- and momentum-density operators are

$$\rho(\vec{x},t) = \underline{\psi}^\dagger(\vec{x},t)\underline{\psi}(\vec{x},t) = f^2 + (1/\Omega)\sum{}' A_-^2(\vec{k}) + f[\underline{\varphi}^\dagger(\vec{x},t) + \underline{\varphi}(\vec{x},t)] + :\underline{\varphi}^\dagger(\vec{x},t)\underline{\varphi}(\vec{x},t): \tag{2.10}$$

and $$\vec{\underline{P}}(\vec{x},t) = \mathrm{Re}\{\underline{\psi}^\dagger(\vec{x},t)[-i\hbar\vec{\nabla}]\underline{\psi}(\vec{x},t)\} = \mathrm{Im}\hbar[(-i/\Omega)\sum{}'\vec{1} + f\vec{\nabla}\underline{\varphi}(\vec{x},t) + :\varphi^\dagger(\vec{x},t)\vec{\nabla}\underline{\varphi}(\vec{x},t):], \tag{2.11}$$

where Eqs. (1.11) have been used. The momentum contribution from the zero-point oscillations is zero by symmetry as usual. Then the particle- and momentum-density wave function expressions for excitations in the momentum level $\vec{k}$ are

$$\rho_{\vec{k}}(\vec{x},t) = f^2 + (1/\Omega)\sum{}' A_-^2(\vec{k}) + f[\varphi_{\vec{k}}^*(\vec{x},t) + \varphi_{\vec{k}}(\vec{x},t)] + |\varphi_{\vec{k}}(\vec{x},t)|^2 \tag{2.12}$$

and $$\vec{P}_{\vec{k}}(\vec{x},t) = \mathrm{Im}\hbar[f\vec{\nabla}\varphi_{\vec{k}}(\vec{x},t) + \varphi_{\vec{k}}^*(\vec{x},t)\vec{\nabla}\varphi_{\vec{k}}(\vec{x},t)] \ . \tag{2.13}$$

By using Eqs. (1.13), one can see that the average values of $\rho_{\vec{k}}(\vec{x},t)$ and $\vec{P}_{\vec{k}}(\vec{x},t)$ (distinguished notationally by the omission of position and time arguments) are

$$\rho_{\vec{k}} = f^2 + (1/\Omega)\sum{}' A_-^2(\vec{k}) + [A_+^2(\vec{k}) + A_-^2(\vec{k})]n/\Omega \tag{2.14}$$

and $$\vec{P}_{\vec{k}} = \hbar\vec{k}/\Omega \ . \tag{2.15}$$

Thus, on the average, each of the n quasiparticles in the excited state has a momentum of $\hbar\vec{k}$, as one would expect, but on the other hand involves the effective number of particles $A_+^2(\vec{k}) + A_-^2(\vec{k})$.[22] Since the total density of the system must be the same with or without the presence of excitations, the condensate density f^2, or more correctly the density $f^2 + (1/\Omega)\sum{}' A_-^2(\vec{k})$, must be depleted by the amount $[A_+^2(\vec{k}) + A_-^2(\vec{k})]n/\Omega$. Thus the n excita-

tions of momentum $\hbar\vec{k}$ can be interpreted as a collective motion of n times $[A_+^2(\vec{k})+A_-^2(\vec{k})]$ condensate or superfluid particles. On the basis of Eqs. (2.9), one expects that this effective number of particles removed from the condensate per excitation is one, only in the high-k limit or when $\hbar\omega_k$ crosses the free-particle energy ϵ_k, and that for low k a very large number of particles becomes involved in the excitation.[23]

While net momentum flow and the renormalization of f^2 are due to terms quadratic in $\varphi(\vec{x},t)$, the linear terms in Eqs. (2.12) and (2.13) dominate the spatial and time dependence of the particle and momentum densities – provided that coefficients of these terms on the order of $f(n/\Omega)^{1/2}$ are much larger than coefficients of the quadratic term on the order of n/Ω. Although large compared to unity, n is assumed small compared to the number of condensate particles Ωf^2. Then the density and momentum fluctuations are approximately given by

$$\begin{aligned}\tilde{\rho}_{\vec{k}}(\vec{x},t) &\equiv \rho_{\vec{k}}(\vec{x},t) - [f^2+\textstyle\sum'A_-^2(\vec{k})]\\ &= 2f^1\varphi_{\vec{k}}(\vec{x},t)\\ &= 2f\zeta_1\cos(\vec{k}\cdot\vec{x}-\omega_k t)\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}\vec{P}_{\vec{k}}(\vec{x},t) &\approx \hbar f\vec{\nabla}\varphi^2_{\vec{k}}(\vec{x},t)\\ &= \hbar\vec{k}f\zeta_2\sin(\vec{k}\cdot\vec{x}-\omega_k t),\end{aligned} \tag{2.17}$$

where Eq. (2.2) has been used.

These are sinusoidal fluctuations satisfying the continuity equation. In the wave-function approximation, then, the excitation formed from the aggregate of many quasiparticles, all approximately with the same $\vec{k}$ value, takes on a wave nature with sinusoidal density fluctuations traveling through the medium as a longitudinal compression wave. These density fluctuations occur for all momenta, including the $k\to\infty$ free-particle limit, and can be pictured as the fluctuations of the projection of $e^{i(\mu/\hbar)t}\psi_{\vec{k}}(\vec{x},t)$ on the real axis as it traverses the real trajectory shown in Fig. 1.

B. Average Energy Density

The Hamiltonian for the system is expressed in normal-order form for the Bogoliubov operators in Paper II. It follows that the average energy when the system contains n excitations, all in approximately the same momentum level $\vec{k}$, is

$$E_{\vec{k}} = E_0 + \mu[A_+^2(\vec{k})+A_-^2(\vec{k})]n + \hbar\omega_k n\,, \tag{2.18}$$

where quartic terms have been neglected and E_0, μ, and ω_k are the ground-state energy, chemical potential, and generalized Bogoliubov spectrum given by Eqs. (2.25), (2.20), and (2.26) of Paper II. However, just as in Eq. (2.14), the f that appears in the formulas quoted in Paper II is now that for a depleted condensate with N_0 reduced from the ground-state value by $\delta N_0 = [A_+^2(\vec{k})+A_-^2(\vec{k})]n$. Then, too, for the zero-temperature theory, μ can be expressed as $(1/\Omega)\partial E_0/\partial f$; hence the fact that $[A_+^2(\vec{k})+A_-^2(\vec{k})]$ particles are effectively removed from the condensate for each of the n quasiparticle excitations works out self-consistently. $E_0+\mu[A_+^2(\vec{k})+A_-^2(\vec{k})]n$ is actually the true ground-state energy expressed in terms of the N_0 that obtains for the unperturbed system. In addition to this ground-state energy, Eq. (2.18) shows that each of the n excitations has energy $\hbar\omega_k$ given exactly, as it should be, by the generalized Bogoliubov theory.

C. Average Energy-Flow Vector

The excitations must also give rise to a transport of energy through the medium. This energy transport will be described by an energy-flow vector defined by the local continuity equation

$$\dot{\underline{\rho}}_E(\vec{x},t) = \vec{\nabla}\cdot\vec{\underline{J}}_E(\vec{x},t) = 0\,, \tag{2.19}$$

where $\underline{\rho}_E(\vec{x},t)$ is an energy-density operator. However, a difficulty arises because the interaction energy is not local, and an energy-density operator cannot be defined uniquely. The integral over space of the energy-density operator must give the total Hamiltonian, but the double-integral term in Eq. (1.1) of Paper II may be expressed in various ways. For later algebraic convenience, the symmetric form is chosen:

$$\underline{\rho}_E(\vec{x},t) = (\hbar^2/2m)\vec{\nabla}\underline{\psi}^\dagger(\vec{x},t)\cdot\vec{\nabla}\underline{\psi}(\vec{x},t) + \tfrac{1}{2}\int d^3r\, V(\vec{r})\underline{\psi}^\dagger(\vec{x}-\tfrac{1}{2}\vec{r},t)\underline{\psi}^\dagger(\vec{x}+\tfrac{1}{2}\vec{r},t)\underline{\psi}(\vec{x}+\tfrac{1}{2}\vec{r},t)\underline{\psi}(\vec{x}-\tfrac{1}{2}\vec{r},t)\,. \tag{2.20}$$

Using the equation of motion for $\underline{\psi}(\vec{x},t)$, Eq. (1.3) of Paper II, one finds the following time derivative of Eq. (2.20) for $\underline{\rho}_E(\vec{x},t)$:

$$\dot{\underline{\rho}}_E(\vec{x},t) = -\vec{\nabla}\cdot\mathrm{Im}(\hbar/m)\{[(-\hbar^2/2m)\nabla^2\underline{\psi}^\dagger(\vec{x},t)]\vec{\nabla}\underline{\psi}(\vec{x},t) + \int d^3r\, V(\vec{r})\vec{\underline{F}}^{(1)}(\vec{x},\vec{r})\}$$

$$-\mathrm{Im}(\hbar/m)\int d^3r\, V(r)[\underline{F}^{(2)}(\vec{x},\vec{r})-\underline{F}^{(2)}(\vec{x}-\tfrac{1}{2}\vec{r},\vec{r})] \quad , \tag{2.21}$$

where $$\underline{F}^{(i)}(\vec{x},\vec{r})\equiv\underline{\psi}^{\dagger}(\vec{x},t)\underline{\psi}^{\dagger}(\vec{x}+\vec{r},t)\underline{\psi}(\vec{x}+\vec{r},t)\,\vec{\nabla}^{(i)}\underline{\psi}(\vec{x},t) \quad , \tag{2.22}$$

$\vec{\nabla}^{(1)}$ and $\nabla^{(2)}$ being the vector and scalar operators $\vec{\nabla}$ and ∇^2, and where the commutability of $\underline{\psi}(\vec{x},t)$ and $\nabla^2\underline{\psi}(\vec{x},t)$ has been used.

The first part of this expression has been written as the divergence of a vector. To write the rest of this expression in the same form, one must use the fact that

$$\vec{r}\cdot\vec{\nabla}_{\vec{x}}\int_{-1/2}^{0}d\lambda\,\underline{F}^{(2)}(\vec{x}+\lambda\vec{r},\vec{r})=\underline{F}^{(2)}(\vec{x},\vec{r})-\underline{F}^{(2)}(\vec{x}-\tfrac{1}{2}\vec{r},\vec{r}) \quad . \tag{2.23}$$

Then, from Eqs. (2.19), (2.21), and (2.23) the operator for the energy-flow vector can be defined as

$$\underline{\vec{J}}_E(\vec{x},t)=\mathrm{Im}(\hbar/m)\{[(-\hbar^2/2m)\nabla^2\underline{\psi}^{\dagger}(\vec{x},t)]\vec{\nabla}\underline{\psi}(\vec{x},t)\}+\mathrm{Im}(\hbar/m)[\int d^3r\, V(\vec{r})\,\underline{\vec{F}}^{(1)}(\vec{x},\vec{r})]$$

$$+\mathrm{Im}(\hbar/m)[\int d^3r\, V(\vec{r})\vec{r}\int_{-1/2}^{0}d\lambda\,\underline{F}^{(2)}(\vec{x},\lambda\vec{r},\vec{r})] \quad . \tag{2.24}$$

Now, the quantity of interest is the average energy flow of the excited state with momentum $\vec{k}$. As discussed at the end of Sec. I, this average value is just a straightforward expectation value of $\underline{\vec{J}}_E(\vec{x},t)$. The actual calculation is lengthy, however. The $\underline{F}^{(i)}(\vec{x},\vec{r})$ must be expanded in terms of normal-ordered $\underline{\varphi}$ products by using Eq. (1.3) and then Eq. (1.10) and similar expressions for products of three and four $\underline{\varphi}$ operators. Eqs. (1.11) and (1.13) are used as well as the fact that $A_+(\vec{k})$, $A_-(\vec{k})$, and $V_{\vec{k}}$ are even functions of $\vec{k}$. The $\int d\lambda$ integration turns out to give only a factor of $\frac{1}{2}$, and one eventually finds[24]

$$\vec{J}_{E,k}=(\hbar\vec{k}/m)\{\epsilon_{\vec{k}}+V_0[f^2+(1/\Omega\,\Sigma' A_-^{\,2}(\vec{l})]\}n/\Omega+(\hbar/m)(\vec{k}+\tfrac{1}{2}k^2\vec{\nabla}_{\vec{k}})[f^2V_{\vec{k}}+(1/\Omega)$$

$$\times\sum{}' A_-^{\,2}(\vec{l})\,V_{\vec{k}+\vec{l}}]n/\Omega+(1/\Omega)\sum{}''[A_+(\vec{l})A_-(\vec{l})(\hbar/m)(\vec{l}+\tfrac{1}{2}l^2\vec{\nabla}_{\vec{l}})\,V_{\vec{k}-\vec{l}}]n/\Omega \quad . \tag{2.25}$$

This equation is consistent within the approximations of the generalized Bogoliubov theory, including $n/\Omega\ll f^2$. In the approximation of standard Bogoliubov theory, where normal-order correction terms are neglected, Eq. (2.25) reduces to

$$\vec{J}_{E,\vec{k}}=(\hbar\vec{k}/m)f^2U_0n/\Omega+\vec{U}(\vec{k})\hbar\omega_{\vec{k}}n/\Omega \quad , \tag{2.26}$$

where $\omega_{\vec{k}}$ is given by the standard Bogoliubov formula and $\vec{U}(\vec{k})$ is the group velocity $\vec{\nabla}_{\vec{k}}\omega_{\vec{k}}$. Thus only part of the energy flow is propagated by the wave excitations at the group velocity.

The first term in this equation is a contribution associated with mass flow, $\hbar\vec{k}/m$ being the mass translation velocity of the n excitations and f^2U_0 being the approximate chemical potential per particle. Yet, the Feynman-Cohen backflow argument suggests that this term is unphysical for stationary states of the system.[25] It is sufficient here to state that, since only a lower bound will be calculated for the energy dissipation by a moving ion complex, only the propagation part of $\vec{J}_{E,\vec{k}}$ need be used.

Although all the symbols needed have not yet been defined, the actual formula to be used for $\vec{J}_{E,\vec{k}}$ will be given here. As discussed following Eq. (3.16), the Bogoliubov approximation with a phenomenological interaction potential is used to calculate a quantity $S_{\mathrm{amp}}(\vec{k})$ defined and related to $\zeta_1(\vec{k})$ in Eq. (3.3). Since n/Ω in Eq. (2.26) is just $\zeta_1(\vec{k})\zeta_2(\vec{k})$ by Eq. (2.8), and $\zeta_1(\vec{k})$ and $\zeta_2(\vec{k})$ are related according to the imaginary part of Eq. (3.8), the energy propagation part of Eq. (2.26) can be written as

$$\vec{J}_{E,\vec{k}}<(\vec{k}/k)g(k)\,S_{\mathrm{amp}}^{\,2}(k) \quad , \tag{2.27}$$

where $$g(k)=\epsilon_k(mf^2/\hbar^2)U(k) \quad . \tag{2.28}$$

Eq. (2.27) will be used in Sec. VI to estimate the energy lost into the roton radiation field created by a small sphere moving slightly faster than the roton critical velocity.

III. GENERAL EXCITATIONS AND TIME-DEPENDENT BOUNDARY CONDITIONS

The condensate wave function $\psi(\vec{x},t)$ is usually determined by using the GP equation and suitable boundary conditions. Alternatively, with a homo-

geneous condensate it has been shown that quasiparticle excitations may be described approximately by the fluctuation part $\varphi(\vec{x}, t)$ of $\psi(\vec{x}, t)$, this $\varphi(\vec{x}, t)$ representing excitations that exist in addition to rather than as part of the condensate. In particular, the imaginary part of $\varphi(\vec{x}, t)$ in Eq. (1.9) is suggested here as the convenient unknown for certain types of eigenfunction problems because it is proportional to the velocity potential for the flow field of the general small-amplitude excitation and can be associated with simplified fluid-flow boundary conditions. The equation of motion and the time-dependent Green's function for this velocity potential are derived in the present section.

A. Velocity Potential and Boundary Conditions

A velocity potential can be defined for any one-particle wave function[12] and in particular for the expression for $\psi(\vec{x}, t)$ in Eq. (1.9). While Gross's discussion of the condensate hydrodynamics in Paper II is for general solutions of the form $\psi(\vec{x}, t) = R(\vec{x}, t) e^{iS(\vec{x}, t)}$, the same approach will be applied here for the case of small-amplitude fluctuations. The formula for $\vec{P}(\vec{x}, t)$ given by Eq. (2.13) implies the velocity field

$$\vec{v}(\vec{x}, t) = \mathrm{Im}(\hbar/m)[f\vec{\nabla}\varphi(\vec{x}, t) + \varphi^*(\vec{x}, t)\vec{\nabla}\varphi(\vec{x}, t)] \ . \tag{3.1}$$

Again, linearization holds if the amplitude of $\varphi(\vec{x}, t)$ is less than f, in which case

$$\vec{v}(\vec{x}, t) = \vec{\nabla} S(\vec{x}, t) \ , \tag{3.2}$$

where $$S(\vec{x}, t) = (\hbar/mf)\varphi_2(\vec{x}, t) \ , \tag{3.3}$$

and $\varphi_2(\vec{x}, t)$ is the imaginary part of $\varphi(\vec{x}, t)$ as denoted by the separation

$$\varphi(\vec{x}, t) = \varphi_1(\vec{x}, t) + i\varphi_2(\vec{x}, t) \ . \tag{3.4}$$

Equation (3.3) identifies the fluid-flow velocity potential of the general excitation. Since

$$\int \vec{v}(\vec{x}, t) \cdot d\vec{l} = (\hbar/mf) \int d\{\varphi_2(\vec{x}, t)\}$$

along any flow line, a necessary condition for the linearization procedure is

$$\int \vec{v}(\vec{x}, t) \cdot d\vec{l} \ll \hbar/m \ . \tag{3.5}$$

Unfortunately, this condition is not satisfied for the problem of interest here, namely, an ion complex moving at the roton critical velocity. For a sphere of radius a and velocity v, Eq. (3.5) requires

$$v \ll \hbar/\pi am \approx 30\,\mathrm{m/sec} \ , \tag{3.6}$$

but the roton critical velocity is on the order of 50 m/sec. However, the linearized theory can still be applied for calculating lower bounds in the roton emission problem. The linearized theory will first be applied for low v in order to determine an effective source strength for the influence of the moving sphere on the liquid. A lower-bound estimate of this source strength is then made for the velocities of interest. Alternatively, one may think of this manipulation as reducing the problem to that of a fictitiously small sphere with its radius reduced by a factor of 10 or so, but with hydrodynamic boundary conditions still applied.

Now, the true boundary condition that should be applied to $S(\vec{x}, t)$ is nonlinear because on any infinite-potential boundary surface, $\psi(\vec{x}, t)$ must be zero. However, on the basis of the fact that the condensate density $|\psi(\vec{x}, t)|^2$ reaches the value it has in the bulk liquid at a distance on the order of the healing length $(\hbar^2/2m\mu)^{1/2} \approx 1$ Å from the boundary surface,[11] while the radius of an ion complex is at least 10–15 Å, the approximation of simply conserving mass at a boundary surface is made. The quantum healing-length region around this sphere is neglected, and the solution for $S(\vec{x}, t)$ is assumed to be limited simply by the condition that the fluid velocity $\vec{\nabla} S(\vec{x}, t)$ have no perpendicular component into or out of the sphere. Then, as for classical hydrodynamics, this simple boundary condition is equivalent to a dipole source at the center of the sphere provided the sphere is moving slowly enough for time-retardation effects to be negligible. The difference between a sphere moving through a classical fluid and the present model, then, is the fact that Planck's constant will appear in the ratio $\hbar/m$ in the equation of motion for $S(\vec{x}, t)$.

B. Velocity-Potential Equation of Motion and Green's Function

The equation of motion for $\varphi(\vec{x}, t)$ follows from that for $\psi(\vec{x}, t)$, namely Eq. (1.3) in Paper II, by translating according to Eq. (1.3), isolating normal-order correction terms according to Eq. (1.10) and similar expansions for the products of three and four operators, and then replacing $\underline{\varphi}(\vec{x}, t)$ by $\varphi(\vec{x}, t)$ according to the general wave-function prescription for normal-ordered operator products. This procedure, very similar to the discussion already given for the energy-flow vector, leads to the following linearized equation of motion:

$$i\hbar\varphi(\vec{x}, t) = -(\hbar^2/2m)\nabla^2\varphi(\vec{x}, t) + f^2 \int d^3r\, V(\vec{r})[\varphi^*(\vec{x} + \vec{r}, t) + \varphi(\vec{x} + \vec{r}, t)]$$

$$+f\{V_0 f^2+(1/\Omega)\sum{}'[(V_0+V_{\vec{l}})A_-^2(\vec{l})+V_{\vec{l}}A_+(\vec{l})A_-(\vec{l})]-\mu\}+(1/\Omega)\sum{}'\int d^3r\,V(\vec{r})e^{i\vec{l}\cdot\vec{r}}$$

$$\times\{A_+(\vec{l})A_-(\vec{l})\,[\varphi^*(\vec{x}+\vec{r},t)-\varphi(\vec{x},t)]+A_-^2(\vec{l})[\varphi(\vec{x}+\vec{r},t)-\varphi(\vec{x},t)]\}\ . \tag{3.7}$$

Not surprisingly, μ, given by Eq. (2.20) in Paper II, is equivalent to the condition that this equation has no constant term in order that $\varphi(\vec{x},t)$ has no linear increase with time and represents only oscillations about a time-constant mean value. The normal-order correction terms arise from the nonlinearity of the original operator equation of motion. In the approximation of standard Bogoliubov theory these terms do not appear, the equation of motion being

$$i\hbar\dot{\varphi}(\vec{x},t)=-(\hbar^2 2m)\nabla^2\varphi(\vec{x},t)+f^2\int d^3r\,U(\vec{r})$$

$$\times\{\varphi^*(\vec{x}+\vec{r},t)+\varphi(\vec{x}+\vec{r},t)\}\ . \tag{3.8}$$

With μ, $A_+(\vec{k})$, and $A_-(\vec{k})$ given by the generalized Bogoliubov theory, Eq. (3.7) is such a complicated wave equation that solutions for $\varphi(\vec{x},t)$ cannot be expressed in closed form if at all. The special solution for momentum $\vec{k}$ can, however, be derived as follows. The Fourier transform of Eq. (3.7) is

$$\left(i\hbar\frac{\partial}{\partial t}-\gamma_{\vec{l}}\right)\varphi(\vec{l},t)=2\delta_{\vec{l}}\varphi^*(-\vec{l},t)\ , \tag{3.9}$$

where $$\varphi(\vec{l},t)=\int d^3r\,e^{i\vec{l}\cdot\vec{x}}\varphi(\vec{x},t) \tag{3.10}$$

and $\gamma_{\vec{l}}$ and $\delta_{\vec{l}}$ are given by Eq. (2.23) in Paper II. If the interaction potential was zero or at least small compared to the kinetic energy, $\delta_{\vec{l}}$ would be zero or negligible and there would be no coupling of the Fourier components. Except for large l this coupling is always present, however. Thus rather than being plane waves, the solutions of Eq. (3.9) for momentum $\vec{k}$ have the form

$$\varphi_{\vec{k}}(\vec{l},t)=[C_+(\vec{k})e^{-i\omega_{\vec{k}}}]\delta_{\vec{l},\vec{k}}$$

$$+[C_-(\vec{k})e^{-i\omega_{\vec{k}}}]^*\delta_{\vec{l},-\vec{k}}\ , \tag{3.11}$$

where $\omega_{\vec{k}}$ and $C_+(\vec{k})/C_-(\vec{k})$ are to be determined. Substituting this form for the solution into Eq. (3.9) for $\vec{l}=\vec{k}$ and for $\vec{l}=-\vec{k}$ gives

$$\gamma_{\vec{k}}C_+(\vec{k})C_-(\vec{k})+\delta_{\vec{k}}\{C_+^2(\vec{k})+C_-^2(\vec{k})\}=0 \tag{3.12}$$

and $$\hbar^2\omega_{\vec{k}}^2=\gamma_{\vec{k}}^2-4\delta_{\vec{k}}^2\ , \tag{3.13}$$

which are equivalent to Eqs. (2.22) and (2.26) of Paper II. As it should be, the Fourier inversion of Eq. (3.11) is proportional to Eq. (1.8) for $\varphi_{\vec{k}}(\vec{x},t)$.

In Paper I the GP equation

$$i\hbar\dot{\psi}(\vec{x},t)=-(\hbar^2/2m)\nabla^2\psi(\vec{x},t)$$

$$+\psi(\vec{x},t)\int d^3r\,V(\vec{r})\,|\psi(\vec{x}+\vec{r},t)|^2 \tag{3.14}$$

is given the meaning of a self-consistent Hartree equation for the condensate wave function. In the context of the present paper the GP equation may be regarded, alternatively, as the equation of motion for a one-body wave function $\psi(\vec{x},t)$ that describes both a uniform condensate and possible quasiparticle excitations superimposed on this condensate. Equation (3.8) for $\varphi(\vec{x},t)$ is simply a linearization of Eq. (3.14), as implied in Paper I. Corrections to the GP equation could be made by including the nonlinear and normal-order correction terms dropped in Eq. (3.8), but this generalization is not within the objectives of the present paper. The equation of motion for $\varphi_2(\vec{x},t)$ is to be derived.

Perhaps it is worth noting here that while Eq. (3.8) has been derived as an approximation to the rigorous operator theory, one may, instead, regard it as an initial consideration in deciding how to go about solving the rigorous theory. The condensate can be thought of as affecting the optimum choice of expansion wave functions for the operator $\underline{\psi}(\vec{x},t)$. While the plane-wave functions usually used for the expansion are the free-field noninteracting particle states, one may regard the coherent interaction from the macroscopic number of particles in the uniform condensate as an external potential to be included in the otherwise free-field equation of motion for the boson excited states. Such an equation of motion must be manufactured by a Bogoliubov displacement and a concomitant linearization of Eq. (3.14) taken as an *ad hoc* "classical" equation of motion. From this point of view, the external condensate potential causes not only a constant energy shift of $V_0 f^2$, the leading term in the chemical potential, but also a coupling of the real and imaginary parts of $\varphi(\vec{x},t)$ as in Eqs. (3.8) and (3.11) with the results that the particular solutions have both positive- and negative-frequency parts and that the sign of the excitation energy is indeterminable from the mathematics.

Since the sign of the excitation spectrum is indeterminate, one expects, as for the Klein-Gordon

and Dirac fields, that the φ-operator expansion should have both positive- and negative-frequency parts – an assertion equivalent to the Bogoliubov transformation in Eq. (1.7). Only for high-k values does the kinetic energy term in Eq. (3.8) dominate the condensate-potential term, in which case the equation is the usual phase-invariant free-particle equation of motion, and the operator expansion of Eq. (1.7) reduces to only a positive-frequency part with particle number a good quantum number.

Now concerning the main discussion, the equation of motion for the imaginary part of $\varphi(\vec{x},t)$ is to be derived. For the Bogoliubov approximation Eq. (3.8) gives

$$\hbar^2\left(\frac{\partial}{\partial t}\right)^2 S(\vec{x},t)+(\hbar^2/2m)^2\nabla^4 S(\vec{x},t) +2f\int d^3y\, U(\vec{x}-\vec{y})(-\hbar^2/2m)\nabla_{\vec{y}}^2 S(\vec{y},t)=0\ , \quad (3.15)$$

both for $\varphi_2(\vec{x},t)$ and, neglecting surface terms, for $\varphi_1(\vec{x},t)$. Eq. (3.15) is an approximate equation of motion for the velocity potential that follows the one-body wave-function description of the quasiparticle excitations of the liquid. As they should be, the momentum eigenfunctions and eigenfrequencies of Eq. (3.15) are just Eq. (1.8) for $\varphi(\vec{x},t)$ and the standard Bogoliubov formula

$$\hbar^2\omega_k^2=\epsilon_k^2+2f^2U_k\epsilon_k\ . \quad (3.16)$$

Unlike many of the theories concerning superfluid helium, that of the present paper does not attempt to derive the experimentally derived spectrum. Instead, Eq. (3.15) is used for the equation of motion with the assumption that U_k is an effective interaction potential that can be adjusted to make the Bogoliubov formula for ω_k agree with the known empirical spectrum. Although one might regard this choice for U_k as a T-matrix approximation,[26] the theory is simply a model for the excitations of superfluid helium. However, it turns out that, because of the Fourier transformation and integral approximation techniques to be used, all that is needed for an equation of motion for $S(\vec{x},t)$ is some linear equation that gives the correct spectrum only in the immediate vicinity of k_0, the k value for which $\omega_{\vec{k}}/k$ is a minimum. Then even if the equation of motion for $S(\vec{x},t)$ were improved, the essential properties used in the later mathematics of this paper would be the same as those of Eq. (3.15), with U_k an empirical interaction potential.

Although the equation of motion involves the ratio $\hbar/m$, the solution of Eq. (3.15) with $v_\perp=0$ on boundaries, or, equivalently, the Green's function

$$G(\vec{x}-\vec{x}',t-t')=0, \quad t>t'$$

$$G(x-x',t-t')\propto|\vec{x}-\vec{x}'|^{-1}\int_0^\infty dk\,(k/\omega_k) \times[\cos(k|\vec{x}-\vec{x}'|-\omega_k t) -\cos(k|\vec{x}-\vec{x}'|-\omega_k t)], \quad t<t', \quad (3.17)$$

with a suitable source distribution replacing the boundary conditions, is tantamount to solving a classical hydrodynamic boundary-value problem. Such a boundary-value problem is actually an approximate but unique criterion for determining the wave-function description of the excitations of the condensed system of interacting bosons. Time-retardation effects and time-varying boundaries can be included in this way.

IV. ROTON RADIATION

In this and Secs. V and VI the linearized GP equation [Eq. (3.8)] for the fluctuation part of $\psi(\vec{x},t)$, will be applied to construct a model for the roton emission from an ion complex moving at a velocity greater than the roton critical velocity. Just as classical electrodynamics suffices to describe the behavior of the large number of photons being emitted during Čerenkov radiation, it should be clear from Secs. I–III that the fluctuation $\varphi(\vec{x},t)$ can be used to describe the aggregate, coherent behavior of the large number of rotons being emitted by the moving-ion complex. That the roton emission is indeed rapid is inferred both from Appendix A and from the self-consistent results of the model to be developed. Because of the significance of dispersion in the water-wave problem,[15] one can expect that the roton-radiation model will not show the large-amplitude shock waves characteristic of Čerenkov radiation and Mach waves, and hence that the linearized theory for $\varphi(\vec{x},t)$ should be adequate. However, it was shown in Eq. (3.6) that, for a different reason, the linearized theory is still insufficient for the problem at hand, and that one can calculate only a lower bound for the roton radiation.

According to the linearized theory for $\varphi(\vec{x},t)$ the velocity potential for the disturbance around the moving-ion complex should be expressible in the form

$$S(\vec{R},t)=\int d^3x'\int dt' A(\vec{x}',t')G(\vec{R}-\vec{x}',t-t')\ , \quad (4.1)$$

where $G(\vec{R}-\vec{x}',t-t')$ is the Green's function given in Eq. (3.17) and $A(\vec{x}',t')$ is a suitable source distribution. Because $A(\vec{x}',t')$ and $S(\vec{R},t)$ must have the spatial dependencies $\vec{x}'-\vec{v}t'$ and $\vec{R}-\vec{v}t$, this equation is equivalent at time $t=0$ to

$$S(\vec{R})=\int d^3x''\int_0^\infty d\tau A(\vec{x}'')G(\vec{R}+\vec{v}\tau-\vec{x}'',\tau)\ . \quad (4.2)$$

Now, for low v, for which time-retardation effects are unimportant, the velocity potential around a sphere of radius a moving in the x direction should be

$$S(\vec{R}) = -\tfrac{1}{2}va^3x/R^3 \ . \tag{4.3}$$

After integration by parts Eq. (4.2) does reduce to this value provided the source distribution is given by

$$A(\vec{x}) = (1/2\pi)vc^2a^3\delta'(x)\delta'(y)\delta'(z) \ , \tag{4.4}$$

where the δ' are derivatives of the δ function and c is the phonon velocity appearing in the low-k part of the spectrum where $w_{\vec{k}} \approx ck$. In other words, with

$$\vec{r} = \vec{R} + \vec{v}t \ ,$$

$$r^2 = \xi^2 + y^2 \ , \tag{4.5}$$

and $\xi = x + vt$,

the expression

$$S(\vec{R}) = \frac{-1}{2\pi} vc^2a^3 \frac{\partial}{\partial x}\int_0^\infty dt \int_0^\infty dk \frac{k}{\omega_k}\frac{1}{r} \times[\cos(kr + \omega_k t) - \cos(kr - \omega_k t)] \tag{4.6}$$

reduces to Eq. (4.3) when $v \ll V_{\min}$.

For the velocities of interest near 50 m/sec, Eq. (4.6) is imprecise for at least three reasons. As pointed out by Eq. (3.6), such high velocities are an order of magnitude too large for the validity of the small-amplitude theory. Then, too, when the speed of propagation of waves through the medium is not instantaneous but relatively slow, it is not reasonable for a dipole source localized at the center of a sphere to replace the effects of the boundary of the sphere. The delay time for a disturbance of the medium to travel from this dipole to the edge of the sphere should be important, and one would expect to have to consider a source distribution that covers the whole interior of the sphere. Finally, as discussed in Appendix B, another difficulty is that the process by which vortex formation occurs may be competing with roton emission for limiting the speed of the ion complex.

Yet, the far-field effects of the moving source distribution should be qualitatively independent of its precise form, and quantitatively dependent on only its effective amplitude and strength. Since it appears quite difficult to analyze a general source distribution, the problem will be greatly simplified by considering just the dipole source term. One would at least expect Eq. (4.6) when evaluated at the v of maximum validity to provide a lower bound for the true source strength at higher velocities. In other words, for the high velocities of interest, the use of Eq. (4.4) should not be an overestimate of the true source strength by much more than, say, a couple of orders of magnitude. With this in mind, Eq. (4.6) will be used to determine the velocity potential even at high velocities. The results will indicate that the source strength given in Eq. (4.4) needs to be reduced in fact by a factor of 10^{-3} or 10^{-4} in order to ensure the validity of the linearized theory.

The velocity potential $S(\vec{R})$ will need to be determined only at large distances, ultimately in a surface integral over a cylinder of arbitrarily large radius about the sphere. For large R the integrations in Eq. (4.6) for $S(\vec{R})$ may be calculated either by the method of stationary phase or by integration by parts.

For $v < V_{\min}$ there are no points of stationary phase, and integration by parts may be performed – the major contribution to the integrations coming from the lower limits $k = 0$ and $t = 0$. In this case, the result varies roughly as R^{-1} for each integration and $S \propto R^{-2}$, which is certainly not a R^{-1} radiation field.

The points of stationary phase that exist for $v > V_{\min}$ will be denoted by k_S and t_S. Since the method of stationary phase gives a contribution in proportion to $R^{-1/2}$ for each of the integrations, the result is a R^{-1} radiation field. The fact that only narrow regions about k_S and t_S contribute significantly to the integrations in this case gives two particularly useful simplifications. The phenomenological spectrum ω_k need not be fixed for all k. In fact, it will turn out that only a very small range of ω_k in the roton region near the minimum phase velocity ω_k/k will be needed for numerical work.

Another simplification is that the x derivative may be commuted with the k integration with the understanding that although the resulting k integration does not formally converge, it is to be evaluated by the method of stationary phase. With local field terms and the trigonometric functions with positive definite arguments omitted, Eq. (4.6) thus reduces to

$$S(\vec{R}) = \frac{1}{2\pi} vc^2a^3 \int_0^\infty dk \int_0^\infty dt \frac{\xi k^2}{r\omega_k} \sin(kr - \omega_k t). \tag{4.7}$$

The stationary-phase conditions are

$$\cos\alpha_S = V(k_S)/v \tag{4.8}$$

and $r_S = U(k_S)t_S$, (4.9)

where the angle α_S has been defined by

$$\cos\alpha = \xi/r \tag{4.10}$$

and $V(k)$ and $U(k)$ are the phase and group velocities

$$V(k) = \omega_k/k\ , \tag{4.11}$$

$$U(k) = d\omega_k/dk\ . \tag{4.12}$$

The stationary-phase conditions determining k_S and t_S may be interpreted physically by using the meaning of the phase and group velocities. As illustrated in Fig. 2, the first condition states that the component of $\vec{v}$ in the α_S direction must match a phase velocity of the medium. The equation is a simple mathematical statement of the physics contained in many phenomena – in order to maintain a radiation wave pattern by continuously putting energy into the medium, the moving foreign body must be able to move and exert a force that is always in phase with waves of the medium. In particular, the onset of roton emission occurs at the critical velocity $V_{\min} = (\omega_k/k)_{\min}$.

The second stationary-phase condition, Eq. (4.9), stipulates that the waves of wave number k_S traveling in the α_S direction for the time t_S interfere constructively at the distance $U_S t_S$ from the source point where they were emitted. This is also illustrated in Fig. 2.

By using these physical interpretations for the stationary-phase conditions, one can see that dispersion in ω_k causes the wave pattern to be spread over a region of space. For $v\cos\alpha_S > V_{\min}$, Eq. (4.8) can be satisfied for two values of k, one on either side of k_0, where $V(k_0) = V_{\min}$. Then Eq. (4.9) gives two constructive interference positions for the k_S values emitted in this α_S direction. For the larger of the two k_S values, $U(k) > V(k)$ and the constructive interference tends to occur in front of the moving sphere; while for the smaller of the two k_S values just the opposite is true.

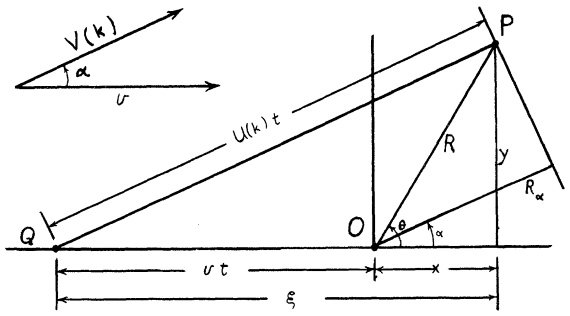

FIG. 2. The relationship between various quantities defined in the text. "O" is the instantaneous $t = 0$ position of the sphere as it moves to the right along the x axis, "P" a point at position $\vec{R}$ in the wave pattern and "Q" the previous position of the sphere at time $-t$ when it created the disturbance that has propagated to point P.

The method of stationary phase can be applied to Eq. (4.7) for large R, giving

$$S(\vec{R}) = c^2 a^3 (k/r)\{\sin(\vec{k}\cdot\vec{r})/[B(k)^2 - A(k)C(k)]\}\ , \tag{4.13}$$

where $A(k) = -t\dfrac{\partial U(k)}{\partial k}$,

$$B(k) = V(k) - U(k)\ , \tag{4.14}$$

$$C(k) = ykv^2/r^3\ .$$

$\vec{k}$ is in the α direction from the x axis, and the subscripts "s" have been omitted from the stationary-phase values of $\vec{k}$, t, and α. Since $\vec{k}$ will be shown to be a slowly varying function of $\vec{R}$, Eq. (4.13) shows that the far-field disturbance created in the liquid by the moving sphere is locally of the form of the momentum wave functions described in Sec. II with amplitude falling off roughly as R^{-1}. This roton wave-radiation field describes the aggregate behavior of the large number of rotons being created by the moving sphere.

V. ASYMPOTIC FORM OF THE ROTON WAVE-RADIATION FIELD

When the stationary-phase conditions and the formula for $S(\vec{R})$ are evaluated, it is convenient to use the variable α instead of t. Since the power lost through the roton-emission processes turns out to be large compared to that available to moving-ion complexes in the actual experiments, v will be assumed to be only slightly greater than the critical velocity $V_{\min}$. All calculations will be made by taking the lowest-order terms from the expansions of quantities in terms of the small fractional increase of v above $V_{\min}$. With Δ defined by

$$v = V_{\min}(1 + \Delta)\ , \tag{5.1}$$

the phase and group velocities in the roton region can be written

$$V(k) = V_{\min}(1 + \Delta l^2) \tag{5.2}$$

and $$U(k) = V_{\min}\{1 + 2(W\Delta)^{1/2} l\}, \tag{5.3}$$

where l bounded by $|l| \leq 1$ is a parameter defined by

$$k = k_0[1 + (\Delta/W)^{1/2} l]\ , \tag{5.4}$$

k_0 being slightly smaller than the usually quoted roton momentum for which ω_k rather than ω_k/k is a minimum. Although the available experimental data are not sufficient to fix k_0, $V_{\min}$, and W at high pressures and low temperatures,[27] the estimates given in Table I can be made.[28]

Using these parametrizations, one finds that Eq. (4.8), the first stationary-phase condition, becomes

$$\beta^2 \approx 1 - l^2 \ , \tag{5.5}$$

where $$\beta \equiv \alpha/\alpha_{\max} \ , \tag{5.6}$$

with $$\alpha_{\max} = (2\Delta)^{1/2} \ , \tag{5.7}$$

while Eqs. (4.9) and (4.10) become

$$l \approx \pm (1+2W\tan\theta)^{-1/2} \ , \tag{5.8}$$

where θ is the direction of $\vec{R}$ from the forward direction. Positive l corresponds to $k > k_0$, $U(k) > V(k)$, and hence $\theta > \frac{1}{2}\pi$, namely, constructive interference occurring in front of the moving sphere, while just the opposite is true for negative l. In order to express β as a function of θ, Eq. (5.5) is written as

$$\beta = (2W)^{1/2}\tan\theta/(1+2W\tan^2\theta) \ . \tag{5.9}$$

Thus β is approximately unity over a wide range of θ about $\frac{1}{2}\pi$, dropping to the value $(2)^{-1/2}$ at θ_c and $\pi - \theta_c$, where

$$\theta_c = (2W)^{-1/2} \ . \tag{5.10}$$

Equations (5.8) and (5.9) for $l(\theta)$ and $\beta(\theta)$ are plotted in Fig. 3 based on a value of 25 for W.

The locus of a wave front in the radiation wave pattern can be determined as the envelope of tangents to the wave front, provided only that the dependence of $\vec{k}$ on α is known. As shown in Fig. 2 and Eq. (4.13), the wave vector at point P is in the $\vec{k}$ direction and is perpendicular to the locus of constant phase passing through this point. The tangent to the wave crest passing through the point P is also tangent to an auxiliary line drawn in the α direction from the origin, the perpendicular distance of the tangent from the origin being R_α. Thus, the tangent to the wave crest at the point P can be established by constructing a line perpendicular to the auxiliary line and passing through it at the distance R_α from the origin. The actual wave crest passing through P is the envelope of all such tangents for various α.

For a fixed phase of the argument of the sine function in Eq. (4.13), R_α is a function of $\vec{k}$ and hence a function of α. In particular,

TABLE I. The empirical pressure-dependent values of the parameters for the phase velocity near its minimum value.

k_0 (Å^{-1})	$V_{\min}$ (m/sec)	W	Pressure (atm)
1.97	58	16.7	0
2.04	51	(25)	(13)
2.11	44	(33)	25.3

$$R_d = D/k(\alpha), \tag{5.11}$$

where D is a constant and $k(\alpha)$ is determined from Eqs. (5.4) and (5.5). Thus the locus of points of a given phase in the wave pattern is constructed graphically by drawing lines perpendicular to the α direction and passing at a distance $D/k(\alpha)$ from the origin. For a given constant D the envelope of many such lines is the locus of constant phase points. Changing D by 2π and repeating the procedure gives the locus of the next period in the wave pattern. The result of such a graphic construction procedure is displayed in Fig. 4, the formulas for $\alpha_{\max}$, θ_c, and $|k-k_0|_{\max}$ being given in Eqs. (5.7), and (5.4), respectively. For $\Delta = 0.001$ one has

$$\begin{aligned} \alpha_{\max} &= 2.6 \text{ deg} \ , \\ \theta_c &= 6.6 \text{ deg} \ , \\ |k-k_0|_{\max} &= 0.0065 k_0 \ . \end{aligned} \tag{5.12}$$

In a discussion of the amplitude of the roton-radiation field, it is perhaps more meaningful to evaluate the magnitude of the density fluctuations than the velocity potential itself. Since $S(\vec{R})$ has

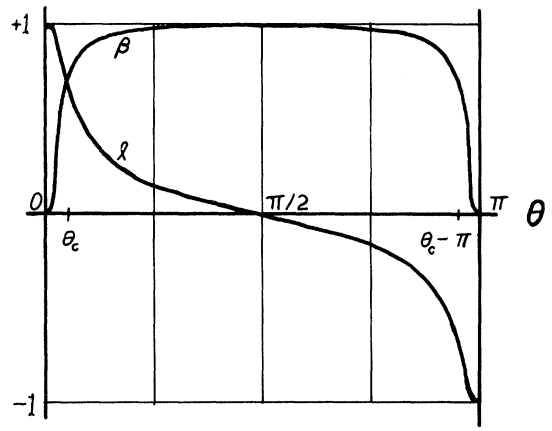

FIG. 3. The variables $l(\theta)$ and $\beta(\theta)$ as functions of Δ, $\theta_c \approx 7°$ being independent of Δ.

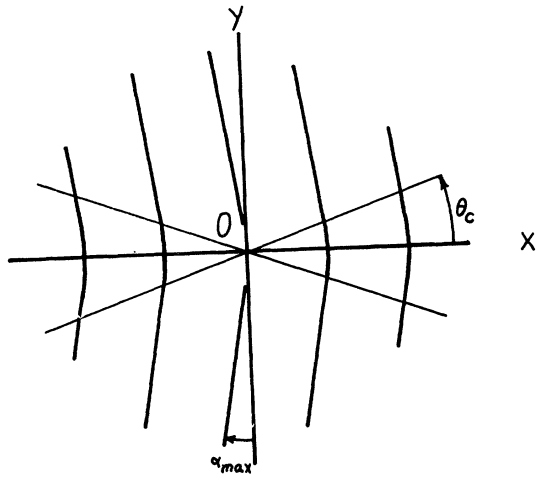

FIG. 4. The wave pattern around a small moving sphere. Outside fore and aft cones of angle θ_c, the asymptotic direction of the wave crests is at an angle α_{max} from the y axis.

been shown to be locally of the momentum type discussed in Sec. II, S_{amp} and $\tilde{\rho}_{amp}$ are simply related by means of Eqs. (2.16) and (2.17), and the real part of Eq. (3.8), namely, by

$$\tilde{\rho}_{amp}/f^2 = [k/V(\vec{k})]S_{amp} \ . \qquad (5.13)$$

After carrying out the algebra for the expansions of Eqs. (4.13) and (4.14) in terms of Δ, one finds

$$\tilde{\rho}/f^2 = (2/W)^{1/2}(c^2a^2k_0/V_{min})(a/y)\beta(\theta) + \theta(\Delta^{1/2})$$

$$\approx 2\times 10^3 (a/y)\beta(\theta) \ , \qquad (5.14)$$

where the data compiled in Table II have been used, and $\beta(\theta)$ is given in Fig. 3.[29]

As discussed following Eq. (3.6), the linear theory of helium wave excitations was not expected to hold near the foreign body when it moves at high velocities. Although only valid at positions far away from the foreign body, Eq. (5.14) indicates an unreasonable large-amplitude disturbance at positions nearer than 10^3–10^4 times the ion radius, depending on whether or not one chooses to distinguish between the superfluid and condensate densities. Of course, this result is inconsistent with the linear theory assumed in Sec. IV to evaluate the dipole source strength. However, the geometric far-field properties of the wave pattern must depend only on the form of the Green's function for excitations of the medium, not on the strength of the source which gives rise to the wave pattern. Since the Green's function, Eq. (3.17), has been used as the basis for all the formulas and discussion, the geometric form given in Fig. 4 must hold for the asymptotic wave pattern; and the spatial dependence of Eq. (5.14) must apply at large distances around any foreign body moving through the medium at a velocity slightly greater than V_{min}. For quantitative results, lower bounds follow by reducing the source strength of Eq. (4.4) by even more than 10^{-2}, say, by 10^{-3} or 10^{-4}, in order to ensure the validity of the linearized theory.

Referring to $\beta(\theta)$ in Fig. 3, one sees that Eq. (5.14) gives an unusual spatial dependence for the amplitude of the wave pattern. Outside a cone of 7° or so around the forward axis, $\tilde{\rho}$ is proportional to $1/y$ rather than $1/R$. On the other hand, for θ near zero or π, Eq. (5.9) indicates that $\beta \approx \tan\theta$; and Eq. (5.14) becomes $\tilde{\rho} \propto 1/x$ along the x axis, the spatial dependence passing smoothly between these two extremes according to $\beta(\theta)$ in Fig. 3.

The most surprising aspect of Eq. (5.14) is its independence of Δ. Physically, this means that when the radiation wave pattern sets in at $v = V_{min}$, it does so quite abruptly with an amplitude independent of v, so long as v is not too much greater than V_{min}. One might expect the amplitude of the wave pattern to grow in proportion to some positive power of Δ. However, since the time integration in Eq. (4.2) extends to infinity, Eq. (5.14) applies only after the foreign body has been moving at the same velocity for an infinitely long time. For this equilibrium situation the amplitude of the wave pattern is independent of Δ for small Δ.

VI. ENERGY LOSS BY THE ROTON RADIATION

Sections I–V have estimated the far-field disturbance created by a foreign body moving through the superfluid component of helium at an equilibrium velocity v slightly greater than the roton critical velocity of about 50 m/sec. The amplitude of the wave pattern in Fig. 4 is independent of Δ, while the asymptotic direction of the wave crests is proportional to $\Delta^{1/2}$. Since the far-field disturbance created in the medium is locally sinusoidal, energy flow through the wave pattern is given approximately by Eq. (2.27) – which is proportional to the square of the local amplitude and is in the direction of the local wave vector of this dis-

TABLE II. The speed of sound, the fluid density, and the negative-ion bubble radius in helium II.

c (m/sec)	mf^2 (g/cm^3)	a (approximate) (Å)	Pressure (atm)
239	0.145	20	0
313	0.160	15	12
365	0.170	14	25

turbance, as one would expect. Thus, at least at positions far from the foreign body, the magnitude of the energy flow through the medium is independent of Δ, while its direction is perpendicular to the wave crests shown in Fig. 4 or, in other words, outward from the path of the foreign body at an angle proportional to $\Delta^{1/2}$. The total energy being lost by the moving foreign body in the form of energy radiation through the roton wave pattern will be calculated in this section by integrating the energy flow through a closed surface around the foreign body, in particular, through a cylinder of arbitrarily large radius y with axis along the path of the moving sphere. Since the main contribution to this integral is over a length on the order of $y\cot\theta_c$, which is independent of Δ, the energy loss should be proportional to $\Delta^{1/2}$.

Where y is the radius of the cylindrical surface, the power loss P is

$$P = \int_{\text{cylinder}} d\vec{s}\cdot\vec{J}_E$$

$$= 2\pi y \int dk\,(dx/dk)\quad (k) S^2_{\text{amp}}(k)\sin\alpha(k), \quad (6.1)$$

where $\mathcal{J}(k)$ is given by Eq. (2.28) and the limits on the k integration are determined by the condition that $V(k) \leq v$, which is equivalent to $|l| \leq |$. In the determination of the lower-bound estimate of P any multiplicative factor for the source strength and hence S_{amp} should be squared for adjusting the resulting values of P. In spite of the lower-bound adjustment factor on the order of 10^{-6} to 10^{-8}, the numerical result for P turns out to be unrealizable experimentally.

The algebra involved in the evaluation of Eq. (6.1) to lowest order in Δ is lengthy, although somewhat simplified by a fortuitous cancellation of some rather involved terms in the product of dx/dk and $S_{\text{amp}}{}^2$. The result is independent of y as it should be.

Of more practical interest than the power P is the electric field that must be applied to an ion complex in order to maintain its velocity as it undergoes the loss of energy due to roton emission. The required electric field is P/ve, which can be evaluated to give

$$E(\Delta) = \frac{6}{\sqrt{W}}\,\frac{\mu c^2 k_0}{e}\left(\frac{c}{V_{\min}}\right)^2$$

$$\times\,(k_0 a)^3\,\Delta^{1/2} + \theta(\Delta^{3/2}) \quad , \qquad (6.2)$$

where M is defined as $\frac{4}{3}\pi a^3 m f^2 \approx 1.7\times10^{-21}$ g.

Since $Mc^2/e \approx 0.40$ V, this result becomes

$$E(\Delta) = \approx 2.6\times10^{14}\,\Delta^{1/2} \text{ V/cm} \quad , \qquad (6.3)$$

while the lower-bound estimate is

$$E(\Delta)_{\substack{\text{lower}\\\text{bound}}} \approx 10^6\,\Delta^{1/2} \text{ V/cm} \quad , \qquad (6.4)$$

which are to be compared to experimentally used voltages on the order of hundreds of V/cm [30] and an electric breakdown voltage on the order of 10^6 V/cm. [31]

CONCLUSION

Both the hydrodynamic model and the "golden rule" estimate of Appendix A indicate that it should be impossible experimentally to force a negative-ion complex to move through superfluid helium at a velocity observably greater than the Landau roton critical velocity. Neeper and Meyer have made an initial test of this prediction and find no contradiction. [6] However, the time-of-flight method used to date in the ion experiments is not sensitive enough to measure small Δ. For Δ equal to 1%, the lower-bound E value of Eq. (6.4) is on the order of 10^4 V/ cm. Yet, in the experiments an electric field is switched in a square-wave signal applied across certain grids and is apparently limited to E values on the order of a few hundred V/cm. [30] Although electric breakdown occurs in liquid helium at about 10^6 V/cm, [31] it is unlikely that the present experimental techniques will ever be able to detect an increase of ion velocity above the roton critical velocity.

From the point of view of the "golden rule" calculation, the prediction of large power dissipation at v_c is due to an avalanche of quasiparticle emissions from the ion complex once it moves fast enough for these creation processes to be allowed kinematically. From the hydrodynamic point of view, this sudden onset of energy dissipation is due to the creation of a wave radiation field by the ion complex once it moves at least as fast as the minimum phase velocity ω_k/k of the possible excitations in the model superfluid system. Either way, the roton energy dissipation is relatively macroscopic, compared to the single electronic charge of the ion complex, and hence its ability to be forced through the fluid by an electric field.

It is, perhaps, somewhat unfortunate that the increase of v above the roton critical velocity is not observable. If future experiments could determine v accurately enough to measure its increase above v_c for large electric fields, one would be able to determine the strength of the interaction between the negative-ion complex and the superfluid quasiparticles. The pressure dependence of this interaction strength would be particularly interesting since Eq. (4.4) indicates that this interaction

strength is proportional to the volume of the ion complex rather than to its cross-sectional area, which is what one might expect from a scattering cross-section point of view. Although the pressure dependence of Eq. (6.2) is at best, qualitative, it suggests that for a given Δ the required E at 25 atm is about 15% larger than that required at 12 atm.

One of the objectives of the present paper was to give a theoretical model for the interaction between an ion complex and a superfluid quasiparticle. Because of the success of the deformation-potential theory of the electron-phonon interaction in metals, a hydrodynamic model for the ion-quasiparticle interaction was investigated. Although this model has direct intuitive meaning, further theoretical efforts are needed to determine the interaction quantitatively.

Perhaps the most significant aspect of the hydrodynamic model given here for the roton critical velocity is the inclusion of time-retardation effects due to the time that it takes for a disturbance created at one point in the superfluid to propagate to a second point in the superfluid, effects which have not been discussed before in the literature. From the hydrodynamic point of view the roton radiation field is, in fact, nothing but a constructive interference effect due to this time retardation.

The dipole backflow of Feynman and Cohen[25] has been shown by Miller, Pines, and Nozieres[32] to be due, on the microscopic scale, to the self-energy cloud of the long-wavelength phonons coupled to the impurity atom moving much slower than the speed of sound. Such a description does not allow for time-retardation effects in the response of the liquid around the moving impurity atom. The present wave-radiation model suggests that one should try to incorporate these time-retardation effects in any classical picture of the local roton self-energy cloud.

Also, as discussed in Appendix B, time-retardation effects may be important for understanding the process by which primordial vorticity can form in a normally curl-free superfluid.

The physical interpretation of the hydrodynamic model given here for the roton critical velocity is the same as that for many classical phenomena.[14,15] To build up and sustain a wave-radiation field, a foreign body must supply energy in a continuous manner, by, say, pushing at a position of fixed phase in the wave pattern it is trying to maintain. Thus the motion of a foreign body can stimulate the radiation of energy only when it moves at least as fast as the minimum phase velocity of the wave excitations of the medium in which it moves.

ACKNOWLEDGMENTS

The author would like to thank Dr. R. L. Mills for many helpful discussions. The slant of Sec. I, for instance, grew out of his questioning as to what the linearized GP equation should mean from the matter-field approach to second quantization, and he suggested Eq. (2.23) at a time when the derivation of Eq. (2.25) seemed impossible. The author is also indebted to the National Aeronautics and Space Administration and to the National Science Foundation for financial assistance during the course of this research.

APPENDIX A

An ion complex moving through superfluid helium with momentum $\vec{p}=M\vec{v}$ can undergo an inelastic creation process wherein a quasiparticle of momentum $\hbar\vec{k}$ and energy $\hbar\omega_k$ is excited in the liquid. Although the interaction between the ion complex and the quasiparticle is an unknown quantity,[8] one can denote its average matrix elements by F. Then for plane waves with $p \gg \hbar/\Omega^{1/3}$, Ω being the volume of the ion complex, the Fermi "golden rule" gives[7]

$$R(v)=\frac{\Omega F^2}{4\pi^2\hbar}\int k^2dk\int d\mu\frac{M}{\hbar kp}\delta\left(\frac{\omega_k/k}{p/M}-\mu\right) \tag{A1}$$

for the rate at which quasiparticles are created by the ion complex. As in the main body of this paper, ω_k/k may be identified with the phase velocity $V(k)$ of the quasiparticle excitations. Then in the notation of Eqs. (5.1) and (5.4) for v and k, $R(\vec{v})$ is the function of the ion's velocity given by

$$R(v)=\frac{\Omega|F|^2}{(2\pi\hbar)^2}\frac{1}{v}\int_{V(k)\leq v}k\,dk$$

$$\approx\frac{2\Omega}{vW}\left|\frac{F}{2\pi\hbar}\right|^2\frac{k_0}{V_{\min}}\Delta^{1/2} \tag{A2}$$

for $v > V_{\min}$, and zero otherwise. Here Δ is the relative increase of v above $V_{\min}$. The Landau kinematics contained in the δ function of Eq. (A1) is manifest in the $V(k)\leq v$ condition on the k integration of Eq. (A2) and the well-known result $R(v < V_{\min})=0$. Likewise, $R(v > V_{\min})\propto\Delta^{1/2}$ is also a result of the simple kinematics.

In order to state quantitatively the power dissipated by the roton creation processes, it is necessary to make an estimate of F. Since the quasiparticles in the roton region have to some extent a single-particle nature, the interaction between the ion complex and these quasiparticles is assumed to be roughly that of the interaction between two helium atoms, that is, F is assumed to be within a few orders of magnitude of, say, 10 °K. Using this value, one finds

$$R(v) \approx 5\times10^{17}\Delta^{1/2} \text{ creations/sec} , \quad \text{(A3)}$$

which is quite large unless Δ is vanishingly small. On the basis of this equation for $R(v)$ the electric field $\omega_k R(v)/V_{\min}$ required to supply the power to the ion complex and the mean free path $v/R(v)$ of the ion complex between roton emissions are

$$E(v) \approx 5\times10^{10}\,\Delta^{1/2} \text{ V/cm} \quad \text{(A4)}$$

and $$\lambda(v) \approx 10^{-6}\,\Delta^{-1/2} \text{ Å} . \quad \text{(A5)}$$

This λ is so small that it is physically meaningless. It appears that the rotons are emitted in such rapid succession by the ion complex that considerable energy dissipation is involved and that the relevant quasiparticle states may become macroscopically occupied in the sense that $n \to \infty$ is used in Sec. I.

APPENDIX B

[*Note added in proof*: In a recent letter {Phys. Rev. Letters 23, 1491 (1969)} R. J. Donnelly and P. H. Roberts give a calculation for the probability of a "proto-ring" becoming a "critical fluctuation," namely a piece of vortex ring sufficiently large to become attached to an ion complex. The present Appendix suggests a mechanism for understanding the formation and growth of their "proto-rings."]

The qualitative discussion given here suggests that the retarded dipole flow found for $v \lesssim v_c$ may well be related to the process by which primordial vorticity is created in the superfluid component of helium II. The central concepts of this Appendix are Thomson's theorem,[33] which states that the circulation cannot change around a closed path in a classical nonviscous fluid, and that, while retarded dipole flow goes over into wave radiation as discussed in the text, this dipole flow is qualitatively quite similar to the flow around a vortex ring.[34]

When first considering the possible means by which primordial vorticity can be created in a superfluid, one is tempted to invoke analogy with the known properties of classical fluids for which vorticity can form near a boundary surface and then enter the fluid. However, this growth process occurs only because of a small amount of viscosity in the classical fluid, and it is well known that the limit of zero viscosity in classical hydrodynamic theory gives entirely different flow characteristics from those obtained by starting with zero viscosity assumed initially.[35] In the former case the amount of vorticity is infinite with a wake forming behind any moving object, while in the latter case there is no vorticity at all. One needs to understand how there can be, in spite of Thomson's theorem, a change in circulation around a closed path in the superfluid.

One possibility is that the superfluid flow pattern changes abruptly by a quantum transition,[36] but the experiments with ion complexes do not substantiate this possibility.[5,6] It appears instead that the superfluid flow around an ion complex changes gradually from a curl-free to vortex-ring flow pattern.

The possibility that the transition occurs by a gradual metamorphosis is perhaps related to the perturbation mixed-state idea of Pollack[37] and to the close similarity between dipole flow and the flow around a vortex ring. Figure 5 shows qualitatively the retarded dipole flow discussed in the text for $v \lesssim V_{\min}$. Since the circulation $\int \vec{v}\cdot d\vec{l}$ around a flow line that begins and ends on the surface of the sphere is finite, one needs to understand how this circulation can break away from the surface. For this process to occur some kind of anomalous flow region with nonzero circulation must develop in the vicinity of the surface, this region being indicated by the slashed area in the figure. Then one might suppose that the path around which the circulation is evaluated closes when $\int \vec{v}\cdot d\vec{l} = \hbar/m$, the quantized value characteristic of the superfluid.

Thus, a possible next step beyond asking what happens to Thomson's theorem is to try to justify anomalous flow regions in which $\vec{\nabla}\times\vec{v} \neq 0$ and hence in which the GP equation cannot be used as the basis of a theory. Of course, many alternatives may be suggested, such as localized clouds of roton normal fluid or a timely scattering that could jar the ion out of its position in the dipole flow pattern. The time-retardation effects found in conjunction with the roton wave-radiation field should also be mentioned. For $V \approx v_c$ the dipole flow pattern should be even more retarded than in the flow pattern shown in Fig. 5. With the dipole flow almost "stretched" off the ion complex already just

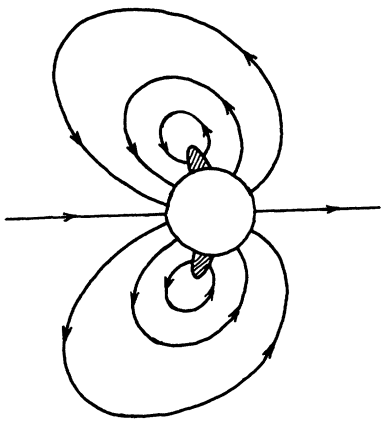

FIG. 5. Retarded dipole flow.

because of time retardation, one has all the more reason to suspect that a fluctuation of any sort could shift flow lines off the surface to make them close in the bulk superfluid.

What is suggested here, then, is that the property of an unchangeable circulation around a closed path in a superfluid should be given at least as much attention in a theory for primordial vortex formation as is given to the quantization condition $\int \vec{v} \cdot d\vec{l} = \hbar/m$. This line integral should be around a path beside the ion complex rather than through or around the complex itself.[5] Time-retardation effects may be instrumental for understanding the change in circulation in the superfluid; and if scattering fluctuations of the ion are involved, the critical velocity for vortex creation need not be a unique function of ion radius.[5]

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[1]L. D. Landau, J. Phys. (USSR) 5, (1941); 11, 91 (1947).

[2]L. Meyer and F. Reif, Phys. Rev. 123, 727 (1961).

[3]The pressure must be greater than 12 atm and the temperature in the range of 0.35–0.55 °K. Below this pressure and temperature range the creation of vortex rings occur at a lower velocity than that needed for roton emission.

[4]Although only the roton part of the excitation spectrum is relevant for the Landau critical velocity, the terms "quasiparticle" and "roton" emission will often be used interchangeably.

[5]G. W. Rayfield, Phys. Rev. Letters 21, 934 (1966); Phys. Rev. 168, 222 (1968).

[6]D. A. Neeper and L. Meyer, Proceedings of the Eleventh International Conference on Low Temperature Physics, edited by J. F. Allen, D. M. Finlayson, and D. M. McCall (University of St. Andrews Printing Department, St. Andrews, Scotland, 1969); D. A. Neeper, Phys. Rev. Letters 21, 274 (1968); D. A. Neeper and L. Meyer, Phys. Rev. 182, 223 (1969).

[7]F. Reif and L. Meyer, Phys. Rev. 119, 4 (1960), Appendix A.

[8]The elastic scattering of phonons by positive- and negative-ion complexes is reasonably understood, and parametrized treatments of elastic roton scattering have been given [G. Baym, R. G. Barrera, and C. J. Pathick, Phys. Rev. Letters 22, 20 (1969); K. W. Schwarz and R. W. Stark, Phys. Rev. Letters 22, 1278 (1969) and 21, 967 (1968)]. Of interest in the present paper is inelastic roton scattering.

[9]The "golden rule" calculation for phonon emission by a moving electron is given by C. Kittel in Quantum Theory of Solids (John Wiley & Sons, Inc., New York, 1963), p. 136, and Chap. 17.

[10]E. P. Gross, Phys. Rev. 106, 161 (1957); Ann. Phys. (N.Y.) 4, 57 (1958); Nuovo Cimento 20, 454 (1961); and L. P. Pitayevski, Zh. Eksperim. i Teor. Fiz. [English transl.: Soviet Physics – JETP 13, 451 (1961)].

[11]E. P. Gross, J. Math. Phys. 4, 195 (1963), hereafter referred to as paper I.

[12]A. Messiah, Quantum Mechanics (Wiley-Interscience Publishers, Inc., New York, 1961), Vol. I, p. 222.

[13]The possibility of Čerenkov radiation was first proposed on the basis of the classical theory by A. Sommerfeld, Amsterdam Roy. Acad. Sci. 8, 346 (1904).

[14]Both minimum and maximum phase velocities can occur for various kinds of water waves. When there is a maximum phase velocity in the wave medium, there can be an associated local maximum in the power dissipation of an object as it moves through the medium at this particular velocity. For this reason sail boats may experience considerable difficulty in maneuvering through shallow water or even through a layer of fresh water over salt water, a situation that can occur at the mouths of rivers and fiords. With the power dissipation going into waves created at the submerged density interface, the latter occurrence is known as "dead water". Although not identified as a meaningful critical-velocity condition until recently, dead water may have been observed by the ancient Greeks around 600 B.C. [V. Walfried Ekman, The Norwegian North Polar Expedition, 1893–1896, Scientific Results, edited by Fridtjof Nansen (Longmans, Green, and Co., New York, 1906), Vol. V, p. 27].

[15]A minimum phase velocity of 23.2 cm/sec occurs for surface water waves at the wavelength 1.73 cm. Although the onset of wave creation by a small object moving over the surface of water at this velocity was observed long before [X. Poncelet and X. Lebros, Ann. Chim. Phys. 116, 386 (1831)], Lord Kelvin noted the phenomenon independently and worked out the first theory [W. Thomson, Phil. Mag., 42, 374 (1871); see Horace Lamb, Hydrodynamics (Cambridge University Press, London, 1932), 6th ed., Sec. 272 for more complete details]. Because of the analogy between this phenomen and the present model given for the roton critical velocity, it is not surprising that the observed water-wave pattern [Scott Russell, Report of the British Association for the Advancement of Science, Fourteenth Meeting, 1844, pp. 311 and 376–81] is quite similar to the results shown here in Fig. 4.

[16]E. P. Gross, Ann. Phys. (N.Y.) 9, 292 (1960), hereafter referred to as Paper II.

[17]N. M. Bogoliubov, J. Phys. (USSR) 11, 23 (1947).

[18]The operator or vector property of a character is denoted by an underline or by an arrow, respectively. Thus the same letter is frequently used both for an operator and for its corresponding c-number wave function, the distinction being made by the line placed under the op-

erator. Vector operators occur occasionally, as in Eqs. (2.11) and (2.19).

[19]The second-quantized formulations of the Schrödinger, Dirac, and Klein-Gordon fields are perhaps best described as Fock space representatives characterized by the choice of the numbers of particles in given individual states as variables. An alternative interpretation is that in which the wave functions are treated in a Lagrangian formulation as "classical matter fields" and "quantized" according to the quantum postulates in the same way as the Dirac treatment of the simple harmonic oscillator. Of course, wave functions are not really classical matter fields and no true quantum mechanics is introduced when they are treated as such in a quantization scheme, but the matter-field approach to second quantization is a well-defined procedure that warrants use whenever it happens to be expedient mathematically. It is, for instance, the only known simple method for introducing interactions between the various relativistic fields and happens also to be relevant to the requirements of the present paper, where the central problem is to determine the wave functions of which the Bogoliubov operators are a second quantization. The one-body wave functions follow from applying to the second-quantized formulation the "dequantization" limit $\hbar \to 0$ with $n \to \infty$ and $\hbar/m$ held constant. Here Bose condensation and interaction effects are incorporated before applying this "dequantization" limit.

[20]The term "particle" is used here to refer to the quanta of the field operator $\underline{\psi}(\vec{x}, t)$, that is, to helium atoms, while the term "quasiparticle" refers to the quanta of the operator $\underline{\alpha}_{\vec{k}}(t)$.

[21]The Bogoliubov coefficients are denoted here by $A_+(\vec{k})$ and $A_-(\vec{k})$, a notation that is in keeping with their interpretation as positive- and negative-frequency coefficients in the expansion of the operator $\underline{\psi}(\vec{x}, t)$ in Eq. (1.8) and as discussed in the paragraphs following Eq. (3.14).

[22]As distinguished in Ref. 20, particle and quasiparticle numbers are noncommuting observables, so there is no reason for their average values to be the same. Although this feature of mutually exclusive particle-number operators appears in all field theories with positive- and negative-frequency coefficients, the distinction does not usually need to be made so explicitly.

[23]Quantitatively in the phonon region where $\xi_2 \gg \xi_1$ the number of particles $A_+^2(\vec{k}) + A_-^2(\vec{k})$ per quasiparticle is approximately $\frac{1}{2}\hbar\omega_k/\epsilon_k = mc/\hbar k = 1\,\text{Å}^{-1}/k$ when $\omega_k = ck$. It is perhaps peculiar that the effective number of particles involved in a phonon excitation varies as $1/k$ rather than remains a constant independent of k. However, the energy of a classical wave is proportional to the square of the amplitude X^2 times the square of the frequency ω_k^2. When the energy of such a classical wave is required to be a discrete quantity proportional to the frequency, according to the rule $\omega_k^2 X^2 \propto n\hbar\omega_k$, where n is an integer, the square of the classical amplitude may roughly be identified as proportional to n/ω_k. $X^2 \propto n$ for fixed ω_k is the usual correspondence between classical wave amplitude and quantum particle number, while $X^2 \propto 1/\omega_k$ for fixed n or $X^2 \propto 1/k$ for a fixed number of phonons gives some feeling for why the effective number of condensate particles involved in the excitation is proportional to $1/k$ rather than being just a constant.

[24]In addition to the ambiguity in defining $\underline{\rho}_E(\vec{x}, t)$, $\underline{\vec{J}}_E(\vec{x}, t)$ is arbitrary to within the curl of a vector. However, these ambiguities do not apply to the average $\vec{J}_{E,\vec{k}}$.

[25]R. P. Feynman and M. Cohen, Phys. Rev. 102, 1189 (1956).

[26]K. A. Brueckner and K. Sawada, Phys. Rev. 106, 1117 (1957); 106, 1128 (1957).

[27]H. Palevsky, K. Otnes, and K. E. Larsson, Phys. Rev. 112, 11 (1959); J. L. Yarnell, G. P. Arnold, P. J. Bendt, and E. C. Kerr, Phys. Rev. 113, 1379 (1959); D. G. Henshaw and A. D. B. Woods in Proceedings of the Seventh International Conference on Low Temperature Physics, Toronto, 1960, edited by G. M. Graham and A. Hollis Hallett (University of Toronto Press, Toronto, 1961), p. 539; D. B. Henshaw and A. D. B. Woods, Phys. Rev. 121, 1266 (1961); and A. D. B. Woods in Quantum Fluids, edited by D. F. Brewer (North-Holland Publishing Co., Amsterdam, 1906), p. 242.

[28]For water-wave problem $2\pi/k_0 = 1.73$ cm, $V_{\min} = 23.2$ cm/sec and $W = \frac{1}{4}$.

[29]The c values are taken from K. R. Atkins, Liquid Helium (Cambridge University Press, London, 1959), Table IV, p. 130; density values from E. F. Burton, H. G. Smith, and J. O. Wilhelm, Phenomenon at the Temperature of Liquid Helium (Reinhold Publishing Corporation, New York, 1940), Fig. 21, p. 70; and a values from B. E. Springett and R. J. Donnelly, Phys. Rev. Letters 17, 364 (1966) and from J. A. Northby and T. M. Sanders, Jr., Phys. Rev. Letters 18, 1184 (1967).

[30]Fields up to 2.5×10^4 v/cm have been used in the temperature range above 1 °K by L. Bruschi, P. Mazzoldi, and M. Santini, Phys. Rev. Letters 21, 1738 (1968).

[31]R. Coelho and P. Sibillot, Nature 221, 757 (1960).

[32]A. Miller, D. Pines, and P. Nozières, Phys. Rev. 127, 1452 (1961), Sec. V.

[33]Fundamentals of Hydro- and Aero-Mechanics by L. Prandtl and O. G. Tietjens (McGraw-Hill Book Company, New York, 1934).

[34]See C. V. Chester in Liquid Helium, edited by G. Careri (Academic Press Inc., New York, 1963), p. 51.

[35]See, for instance, The Feynman Lectures on Physics by R. P. Feynman, R. B. Leighton and M. Sands (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1964), Vol. 2, Secs. 41–5.

[36]W. F. Vinen in Proceedings of the International School of Physics, Enrico Fermi, Volume 21: Liquid Helium, edited by G. Careri (Academic Press Inc., New York, 1963). Such a discontinuous change is basic to the energy and momentum considerations of D. Pines, in Proceedings of the Sussex University Symposium, 1965, edited by D. F. Brewer (North-Holland Publishing Co., Amsterdam, 1966), p. 328.

[37]F. Pollack, Phys. Letters 19, 538 (1965).