

Some Properties of Unstable Particle Gases*

Michael D. Delano

Belfer Graduate School of Science, Yeshiva University, New York, New York 10033

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The elastic scattering of two particles is interpreted as the formation and decay of an unstable particle. The collision cross section and the reciprocal of the scattering delay time are used to determine rates of production and destruction in a gas of interacting stable particles. By equating the two rates, a general expression for the equilibrium number density of unstable particles is obtained. This is tested for two specific types of scattering, and it is found that in resonance scattering the expression, in the limit as $\Gamma \rightarrow 0$ (stable particles), reduces to that predicted by the equations of chemical and nuclear statistical equilibrium. Dividing a real gas into an ideal stable particle gas and an ideal unstable particle gas, an equation of state for the latter is inferred from the form of the second virial coefficient. The physical and chemical "reality" of unstable particle gases is examined. It is shown that their existence implies small corrections to the equations of chemical and nuclear statistical equilibrium, as they apply to stable particle abundances.

1. INTRODUCTION

Since its introduction by Eisenbud,¹ the interpretation of the energy derivative of the phase shift $\hbar d\eta/dE$ as the time delay induced in the elastic scattering of two particles, has been rederived in terms of the collision lifetime matrix,² has been reexpressed in operator form,³ and has also been rediscovered.⁴ It is nevertheless surprising that such a simple and intrinsically interesting formula has found practically no application outside the field of quantum scattering theory. This, in spite of a rather natural application of the formula.

Throughout this work, the words "unstable particle" shall always mean an *unbound* two-body system which subsequently decays into the two bodies composing it. This is in contradistinction to bound systems, which are unstable to, say β or α decay. Interpreting the elastic scattering of two particles as the formation and decay of an unstable particle, where the unstable particle is "formed" the instant the scattering pair "start to interact" and has "decayed" the instant the scattering pair "stop interacting," it is clear that the energy derivative of the phase shift provides the needed expression for the mean life of the state:

$$\tau(E) = \hbar d\eta/dE. \quad (1)$$

As a definition of "unstable particle" the preceding paragraph and Eq. (1) need some qualification. It is important to recognize that an unstable particle is not just a "stable particle" with a finite lifetime; they are fundamentally different. In

characterizing a particle as unstable, there is the explicit statement that the particle has a substructure which no order of approximation can neglect. For a stable particle, it is tacitly assumed that the particle will be specularly reflected at a plane boundary, and hence that a wave function, normalized in some region of space, can be written for it. The assumption is justified as long as the energies involved are below the threshold for exciting or disrupting the substructure (if any) of the particle. As emphasized above, in this paper unstable means unbound, and it becomes obvious that for this type of individual, which may not even exist long enough to traverse the dimensions of some spatial volume, it is doubtful whether it can undergo specular reflection at a plane boundary and remain intact. Furthermore, it is uncertain as to whether an unstable particle moving in even the weakest of external potentials will not be disrupted by the potential. In short then, the major distinction between stable and unstable particles is that for the latter, any interaction greater than no interaction at all is capable of altering the identity of the particle experiencing the interaction.

The fact that an unstable particle may not survive an interaction should not suggest that there is something lacking in its definition or that it does not interact in a physically meaningful way. A good example of an unstable particle interacting with a stable particle is that of a nuclear reaction with three particles in the exit channel. If there is a final-state interaction between two of the three emitted particles it may be interpreted as an unstable particle. This gives rise to measurable effects in the kinetic energy spectra of all three par-

ticles and, in some instances, these spectra clearly indicate that the exit channel is two-body rather than three, where one of the bodies is an unstable particle which subsequently decays into two stable particles.⁴

In the course of the presentation here, there will be two situations where an exact knowledge of how unstable particles interact would be helpful. While no direct attempt shall be made to ascertain this information, it will still be possible to discover some general properties of how a group of unstable particles behave as a gas.

In general, all qualitative remarks concerning Eq. (1) and other relations involving the phase shifts shall be confined to positive phase shifts or, equivalently, to potentials which tend to trap particles. Assuming the scattering particles to be infinitely long plane waves outside the region of their mutual interaction, the energy stored in the unstable system is known precisely; consequently, unstable particles formed at different incident energies are distinguishable from each other because they have different rest masses. This fact is also reflected by the indicated energy dependence of the mean lifetime.

The reciprocal of Eq. (1)

$$\lambda(E) = (1/\bar{n}) (d\eta/dE)^{-1} \quad (2)$$

is the mean rate of decay for unstable particles with internal energy (formation energy) E . Regarding the elastic scattering cross section for two particles as the formation cross section for unstable particles, it should be evident that in a gas of interacting particles, this, along with Eq. (2), will determine rates of destruction and production; and by equating the two, the dynamical equilibrium abundance of unstable particles may be calculated. This simple idea, well known in the fields of chemical kinetics and nuclear astrophysics, seems to have been overlooked as far as unstable constituents are concerned.

The general abundance relation (equilibrium number density), derived in Sec. 2 and applied to two different types of scattering in Secs. 3 and 4 suggests that a gas of interacting particles may be divided into two gases, one composed of free noninteracting stable particles, an ideal gas, and the other composed of free noninteracting unstable particles, an ideal unstable particle gas. In Sec. 5, the simplest possible equation of state for an unstable particle gas is inferred from the form of the second virial coefficient, and in Sec. 6, the possible effects of unstable particle gases on the equations of chemical and nuclear statistical equilibrium are discussed.

2. ABUNDANCE RELATIONS FOR UNSTABLE CONSTITUENTS

The foregoing discussion is best summarized by the following set of reactions:



Here, k represents the unstable particle, and i and j the scattering particles, it being noted earlier that k exists as long as particles i and j are interacting. In a gas containing constituents i and j , the equilibrium number density of k may be related to the number densities of i and j if two assumptions are made:

(i) The predominant mechanism for the production of k is the direct elastic scattering of particles i and j .

(ii) The predominant mechanism for the destruction of k is its instability against direct decay into particles i and j .

In general, assumptions (i) and (ii) are fulfilled for large scattering cross sections and small lifetimes, but they will hold up as long as there are no processes competing with the reactions of Eq. (3).

In a gas, the differential collision rate per unit volume between constituents i and j is

$$dR_{ij} = \frac{1}{1 + \delta_{ij}} f_i(\vec{V}_i) f_j(\vec{V}_j) |\vec{V}_i - \vec{V}_j| \times \frac{\partial \sigma_{ij}}{\partial \Omega_{ij}} d\Omega_{ij} d^3 \vec{V}_i d^3 \vec{V}_j, \quad (4)$$

where $\partial \sigma_{ij} / \partial \Omega_{ij}$ is the differential scattering cross section, and $f_i(\vec{V}_i)$ is the velocity distribution function for particles of type i ; $f_i(\vec{V}_i) d^3 \vec{V}_i$ giving the number of type- i particles per unit volume whose velocity vectors lie in $d^3 \vec{V}_i$ about \vec{V}_i . The Kronecker δ factor $1/(1 + \delta_{ij})$ introduces a factor of $\frac{1}{2}$ in the event that the colliding particles are identical. For Maxwellian velocity distribution functions,

$$f_i(\vec{V}_i) = n_i \left(\frac{m_i}{2\pi K T} \right)^{3/2} e^{-m_i \vec{V}_i^2 / 2 K T} \quad (5)$$

a transformation to c. m. and relative coordinates will allow the angular coordinates and c. m. motion to be integrated out of Eq. (4); and the production rate per unit volume for unstable particles with internal energies in the range E to $E + dE$ may be defined as

$$R_k(E) dE = \frac{n_i n_j}{1 + \delta_{ij}} \left(\frac{8}{\pi \mu_{ij}} \right)^{1/2} (K T)^{-3/2} \times E \sigma_{ij}(E) e^{-E/K T} dE. \quad (6)$$

Here, $\mu_{ij} = m_i m_j / (m_i + m_j)$ is the reduced mass; σ_{ij} is the total scattering cross section; and E is the relative kinetic energy.

Defining $N_k(E) dE$ to be the number density of unstable particles with internal energies in the range E to $E + dE$, it will be noted that multiplication with the decay rate of Eq. (2) will yield the destruction rate per unit volume of unstable particles:

$$D_k(E) dE = \lambda(E) N_k(E) dE = \frac{N_k(E)}{\hbar} \left(\frac{d\eta}{dE} \right)^{-1} dE. \quad (7)$$

It should be evident that under conditions of thermodynamic equilibrium for the gas, a state of dynamic equilibrium will obtain rates of production [Eq. (6)] and destruction [Eq. (7)], the dynamical equilibrium abundance or number density may be solved for:

$$N_k(E) = \frac{n_i n_j}{1 + \delta_{ij}} \left(\frac{8\hbar^2}{\pi \mu_{ij}} \right)^{1/2} (KT)^{-3/2} \times E \frac{d\eta}{dE} \sigma_{ij}(E) e^{-E/KT}. \quad (8)$$

For gas particles i and j obeying Boltzmann statistics, Eq. (8) is a general expression relating the number $N_k(E)$ of unstable particles k (at internal energy E) per unit volume per unit energy to the number densities of the scattering particles i and j and the gas temperature T . In the case of uncharged spinless particles, the well-known relation

$$\sigma_i = (4\pi/K^2) (2l+1) \sin^2 \eta_l, \quad (9)$$

or, for identical bosons or fermions,

$$\sigma_i = [1 \pm (-1)^l] (4\pi/K^2) (2l+1) \sin^2 \eta_l, \quad (9')$$

may be substituted for the scattering cross section yielding an expression which depends only on the l_{+h} (angular momentum) partial-wave phase shift η_l :

$$N_k^i(E) = n_i n_j (2\pi\hbar^2/\mu_{ij}KT)^{3/2} (2/\pi) \times (2l+1) (d\eta_l/dE) \sin^2 \eta_l e^{-E/KT}. \quad (10)$$

While not explicitly shown in Eq. (10), it should be kept in mind that by Eq. (9') identical bosons can interact only in even angular momentum states and identical fermions only in odd angular momentum states. To find the total number density n_k^i of unstable particles at all internal energies, the distribution $N_k^i(E)$ must be integrated over all en-

ergies:

$$n_k^i = \int_0^\infty N_k^i(E) dE. \quad (11)$$

If many different angular momenta are to be considered, then Eq. (11) must be summed over these states.

3. CHEMICAL EQUILIBRIUM AND UNSTABLE RESONANT STATE PARTICLES

For stable components, the equations of chemical⁵ or nuclear⁶ statistical equilibrium may be derived by demanding that the free energy of the system be a minimum at equilibrium. In the chemical case, the equations relate the equilibrium concentrations to each other. The nuclear statistical equilibrium equations, which are of a similar form, relate the number density $n(A, Z)$ of the nucleus with mass number A and atomic number Z to the free neutron and proton number densities n_n and n_p . If two spinless particles of masses m_i and m_j come together with angular momentum l to form an unbound state of positive energy E_r , both sets of equations estimate the equilibrium number density of the state to be

$$n_k = (2l+1) n_i n_j (2\pi\hbar^2/\mu_{ij}KT)^{3/2} e^{-E_r/KT}, \quad (12)$$

where μ_{ij} is the reduced mass.

This equation, sometimes referred to as the "law of mass action," has on occasion been used to find the abundances of unstable particles in stellar interiors.⁷ This is not strictly correct, and it is quite revealing to see how Eq. (12) compares with the formalism developed in this paper. This can only be done by assuming a specific form for the scattering phase shift appearing in Eq. (10). The remainder of this section will be devoted to resonance scattering, and Sec. 4 will investigate scattering through a virtual state.

In the scattering of two uncharged spinless particles through a single isolated resonance, if it is assumed that the potential causing the scattering has no bound states for relative angular momentum l , then the l_{+h} partial-wave phase shift may be written

$$\eta_l(E) = \tan^{-1} \left(\frac{\frac{1}{2} \Gamma}{E_r - E} \right) - \tan^{-1} \left(\frac{\frac{1}{2} \Gamma}{E_r} \right), \quad (13)$$

where E_r is the resonance energy and Γ is the width of the state. The second term on the right-hand side ensures that the phase shift vanishes at zero energy, and, for narrow resonances far from the energy origin, it may be neglected, leaving only

the first term for substitution into Eq. (10):

$$N_k(E) = (2l+1)n_i n_j \left(\frac{2\pi\hbar^2}{\mu_{ij}KT} \right)^{3/2} \times \frac{2}{\pi} \frac{(\frac{1}{2}\Gamma)^3}{[(E-E_r)^2 + (\frac{1}{2}\Gamma)^2]^2} e^{-E/KT}. \quad (14)$$

It is interesting to note that the usual resonance form factor $[(E-E_r)^2 + (\frac{1}{2}\Gamma)^2]^{-1}$ is here raised to the power of 2. Given the gas temperature and the number densities n_i and n_j , Eq. (14) indicates how the unstable particles k in the gas have their internal energies distributed over the resonance level.

Integration of Eq. (14) over all energies yields the total unstable particle number density n_k and involves an integral of the form

$$I(E_r, \Gamma, T) = \frac{2}{\pi} (\frac{1}{2}\Gamma)^3 \times \int_0^\infty \frac{e^{-E/KT}}{[(E-E_r)^2 + (\frac{1}{2}\Gamma)^2]^2} dE. \quad (15)$$

For narrow resonances or whenever $\Gamma \ll 2KT$, the exponential may be expanded about the resonance energy E_r and the lowest-order terms saved. To second order in $(\Gamma/2KT)$, the integral is

$$I(E_r, \Gamma, T) = e^{-E_r/KT} F(E_r, \Gamma, T), \quad (16)$$

where $F(E_r, \Gamma, T) \equiv G(E_r, \Gamma) \left[1 + \frac{1}{2} \left(\frac{\Gamma}{2KT} \right)^2 \right]$

$$- \frac{1}{\pi} \left(\frac{\Gamma}{2KT} \right) \frac{(\frac{1}{2}\Gamma)^2}{E_r^2 + (\frac{1}{2}\Gamma)^2}, \quad (17a)$$

and
$$G(E_r, \Gamma) \equiv \frac{2}{\pi} \left[\frac{\pi}{4} + \frac{E_r(\frac{1}{2}\Gamma)}{2[E_r^2 + (\frac{1}{2}\Gamma)^2]} + \frac{1}{2} \tan^{-1} \left(\frac{E_r}{\frac{1}{2}\Gamma} \right) \right]. \quad (17b)$$

The total number density is therefore

$$n_k = (2l+1)n_i n_j \left(\frac{2\pi\hbar^2}{\mu_{ij}KT} \right)^{3/2} \times e^{-E_r/KT} F(E_r, \Gamma, T). \quad (18)$$

Comparing this with Eq. (12), it is seen that they

differ by the factor of $F(E_r, \Gamma, T)$. The function $F(E_r, \Gamma, T)$ accounts for the finite width Γ of the resonance. It is easily demonstrated that in the limit as $\Gamma \rightarrow 0$ (stable but unbound particles) $F(E_r, \Gamma, T) \rightarrow 1$, and Eqs. (12) and (18) become identical. The agreement is important because it establishes the limit in which the equations of chemical or nuclear statistical equilibrium may be used to make predictions about unstable *resonant* state particles. This region is best defined by the inequality

$$KT \gg \frac{1}{2}\Gamma \ll E_r. \quad (19)$$

It should be pointed out that assumptions (i) and (ii) are not unreasonable, there being a broad range of physical situations where they are satisfied. To illustrate with an example from astrophysics, attention is called to the graphs of Fig. 1, where the curves plot the fraction of α particles expected to be in the form of the unstable He^5 nucleus ($J = \frac{3}{2}$; $E_r = 0.92$ MeV; $\Gamma = 1$ MeV) when the α particles exist in a hot dense gas of neutrons. The plots, as a function of temperature, are for several different densities, and also show the predictions of chemical and nuclear statistical equilibrium. This range of temperatures and densities obtains for the neutron-rich material believed to be ejected in supernova explosions. In the numerical integration of Eq. (14), the zero-spin statistical factor $(2l+1)$ was replaced by the spin statistical weight $(2S+1)/(2S+1)(2l+1)$, where $J = \frac{3}{2}$, $S = \frac{1}{2}$, and $l = 1$.

These curves demonstrate quite vividly the extent of the discrepancy between the two formalisms in the low-temperature region, where the inequality $KT \gg \frac{1}{2}\Gamma$ is no longer valid. It will also be noted that even in the high-temperature limit, the two estimates will still differ since $\frac{1}{2}\Gamma \sim E_r$, instead of $\frac{1}{2}\Gamma \ll E_r$.

4. UNSTABLE VIRTUAL STATE PARTICLES

If the chemical equilibrium approach to unstable particles runs into difficulties for resonance scattering, it is safe to say that it does not work at all for scattering through a virtual state.

For the low-energy *s*-wave scattering of uncharged spinless particles, the cross section in the zero-range approximation may be expressed in the form

$$\sigma = 4\pi/(k^2 + 1/a^2), \quad (20)$$

where a is the scattering length, and k is the wave number. If the attractive potential inducing the scattering has no bound *s* states, a is an intrinsic negative quantity, and the phase shift

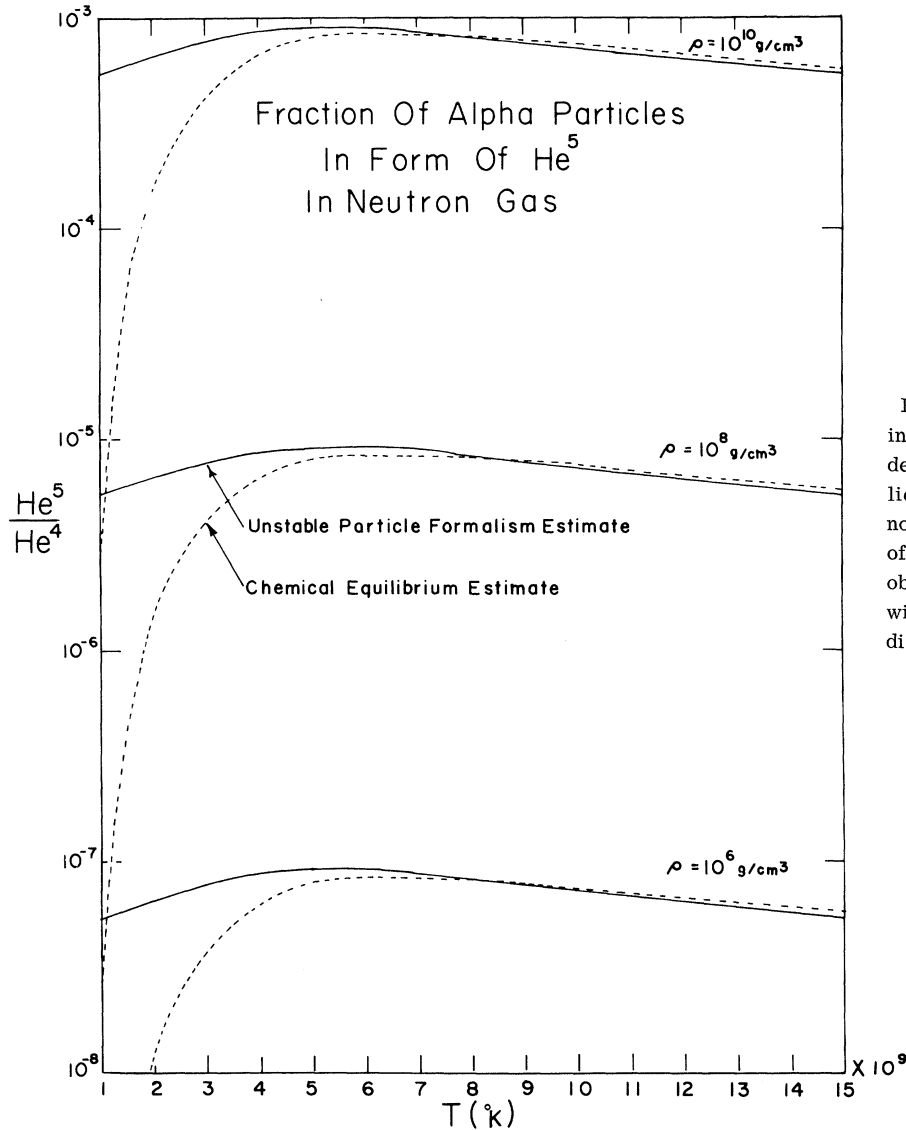


FIG. 1. Fraction of α particles in the form of He^5 in the hot, dense neutron-rich gases believed to be ejected in supernova explosions. The effects of finite resonance widths may be observed by making comparisons with the chemical equilibrium predictions (broken line).

$$\eta = -\cot^{-1}(1/ka) \tag{21}$$

is restricted to positive values between 0 and π .

With (20) and (21) in Eq. (8), the energy distribution function for unstable "virtual-state particles" is

$$N(E) = n_i n_j \left(\frac{2\pi\hbar^2}{\mu_{ij}KT} \right)^{3/2} \frac{2\gamma}{\pi} \frac{\sqrt{E}}{(E+\gamma^2)^2} e^{-E/KT}, \tag{22}$$

where $\gamma^2 = \hbar^2/2\mu_{ij}a^2$. (23)

The integral of (22), which cannot be written in any simple form, yields the total number density of unstable virtual-state particles.

If, as is frequently done, γ^2 is interpreted as

playing the role of the "virtual-state energy," then it may be used in the exponential term of Eq. (12) to arrive at the chemical equilibrium estimate for the total number density. McVoy⁹ has recently discussed the whole question of virtual states and resonances, including the above misleading description of $(+\gamma^2)$. To be sure, γ^2 is an energy parameter characterizing the virtual state, but it is improper to designate all or even part of the particles in this state as having an energy $+\gamma^2$. Without this designation, however, it is impossible to use the equations of chemical and nuclear statistical equilibrium, since they require definite energy values in their exponential terms.

The point being emphasized here is that when considering the totality of unstable particles in a gas, the idea of an exact internal energy must be aban-

done in favor of energy distribution functions, from which average properties may be calculated. This is true regardless of the form of the scattering cross section. While, in the case of very narrow resonances, it is certainly easier to give an *a priori* estimate of the average energy stored in an unstable state, this facility should not be construed as implying a greater degree of reality for unstable "resonant-state particles" over, say, unstable virtual-state particles.

The normalized energy distribution for unstable virtual-state particles

$$F(E) = C \frac{\sqrt{E}}{(E + \gamma^2)^2} e^{-E/KT}, \quad (24)$$

where C is the normalization constant, peaks at the energy

$$E_p = \frac{1}{2} \left\{ \left[\left(\frac{3}{2}KT + \gamma^2 \right)^2 + 2KT\gamma^2 \right]^{1/2} - \left(\frac{3}{2}KT + \gamma^2 \right) \right\}. \quad (25)$$

This distribution function is displayed in Fig. 2, for different γ^2 , as a function of energy where all

energies are measured in units of KT . The normalization was determined numerically for each γ^2 value.

These curves, which are asymmetrical about their maximums (broken line in the figure), have long exponential tails. An interesting feature is that, as γ^2/KT becomes smaller, the functions become more peaked about their maximums. For $\gamma^2/KT \ll 1$, Eq. (25) is

$$E_p \approx \frac{1}{3} \left(\frac{3}{2}KT + \gamma^2 \right) \gamma^2 / \frac{3}{2}KT, \quad (26)$$

and in the limit as $\gamma^2/KT \rightarrow 0$ the distribution of energies becomes infinitely sharp at $E = \frac{1}{3}\gamma^2$. The energy integral of Eq. (22) in this limit is

$$n = n_i n_j \left(\frac{2\pi\hbar^2}{\mu_{ij}KT} \right)^{3/2} e^{-\gamma^2/3KT}, \quad (27)$$

which is identical with the chemical equilibrium estimate, Eq. (12), if the energy of the state is set equal to $\frac{1}{3}\gamma^2$. This agreement is fortuitous,

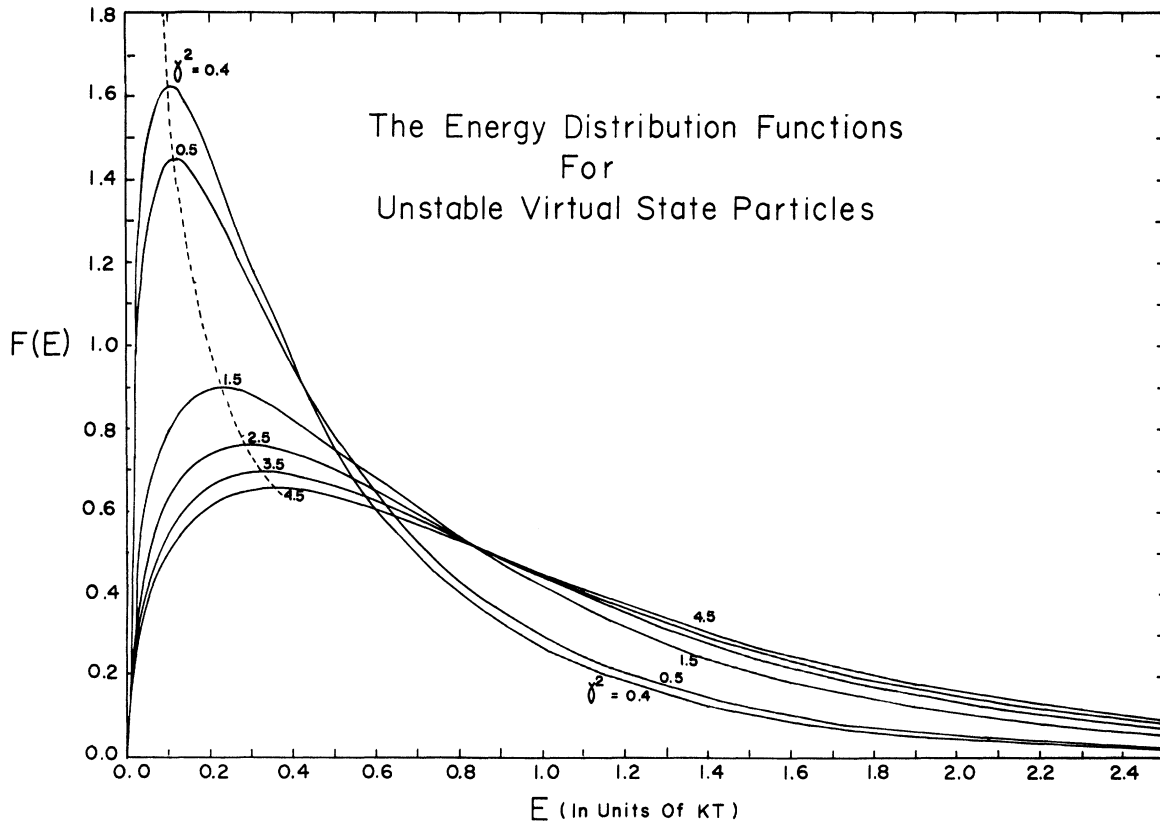


FIG. 2. Normalized distribution functions for the internal energy of unstable virtual-state particles. All energies, including γ^2 , are measured in units of KT . Note the strong peaking of these functions as $\gamma^2/KT \rightarrow 0$.

however, and it arises from the mathematical peculiarities of the virtual-state distribution function [Eq. (24)] and the fact that the underlying assumptions made in the unstable particle formalism are consistent with equilibrium statistical mechanics for "states" of definite energy. Physically, there is nothing unique about the $\frac{1}{3}\gamma^2$ energy value. As an energy state it is still unbound and unstable with a finite lifetime of $\tau = \frac{3}{8}\sqrt{3}\hbar/\gamma^2$. Moreover, categorizing $\frac{1}{3}\gamma^2$ as an energy state may be ill advised. Without interpretation, it may be simply stated that $\frac{1}{3}\gamma^2$ is the internal energy value that all the unstable virtual-state particles in the gas will possess in the limit as $\gamma^2/KT \rightarrow 0$.

5. SECOND VIRIAL COEFFICIENT AND EQUATION OF STATE FOR AN IDEAL UNSTABLE PARTICLE GAS

Consider binary collisions in a monatomic gas whose particles obey the Maxwell velocity distribution law. It is a property of the Maxwellian that the c. m. of all colliding pairs also obey the Maxwell velocity distribution law. Consequently, in assuming that unstable particles do not interact with the gas in which they exist [assumption (ii)], it must be concluded that unstable particles have a Maxwellian velocity distribution. Furthermore, because of their noninteracting nature (each individual decays before it can interact), it would appear that as a gas of particles their behavior is similar to that of an ideal Boltzmann gas. A gas of unstable particles, of course, will not exert the same pressure on the surfaces of a container as an ideal Boltzmann gas. The reason is that, in colliding with a wall, an unstable particle has very little chance of remaining intact, and therefore the momentum transfer to the wall is not simply that of specular reflection of the c. m. motion.

Whatever the momentum transfer is, however, it should be evident that the above interpretation allows the total pressure P_t to be expressed as the sum of the partial pressures of the ideal stable P_s and ideal unstable P_u particle gases:

$$P_t = P_s + P_u. \quad (28)$$

If an unstable particle gas behaved like an ideal Boltzmann gas, its equation of state would be

$$P_u/KT = N_u, \quad (29)$$

where the total number density of unstable particles, from Eqs. (10) and (11), is

$$N_u = n_s^2 \left(\frac{2\pi\hbar^2}{mKT} \right)^{3/2} \sum_l \frac{2^{3/2}}{\pi} (2l+1)$$

$$\times \int_0^\infty 2 \frac{d\eta_l}{dE} \sin^2 \eta_l e^{-E/KT} dE. \quad (30)$$

Here, n_s is the total number density of *free* stable particles. Defining

$$f \equiv \left(\frac{2\pi\hbar^2}{mKT} \right)^{3/2} \frac{2^{3/2}}{\pi} \sum_l (2l+1) I_l, \quad (31)$$

$$\text{and } I_l \equiv \int_0^\infty 2 \frac{d\eta_l}{dE} \sin^2 \eta_l e^{-E/KT} dE, \quad (32)$$

N_u may be written

$$N_u = n_s^2 f. \quad (33)$$

The actual pressure exerted by the unstable particles may be expressed in the same form as Eq. (29), if N_u is replaced by an effective number density \bar{N}_u , defined by

$$\bar{N}_u \equiv n_s^2 \bar{f}, \quad (34)$$

and where

$$\bar{f} \equiv \left(\frac{2\pi\hbar^2}{mKT} \right)^{3/2} \frac{2^{3/2}}{\pi} \sum_l (2l+1) \bar{I}_l. \quad (35)$$

The only difference between Eqs. (31) and (35) is in the integral term I_l .

If in the expansions for stable particle gases, terms higher than third order in the thermal wavelength $\lambda = (2\pi\hbar^2/mKT)^{1/2}$ are neglected (high-temperature limit), the distribution functions will still be Maxwell-Boltzmann while the equations of state are

$$\frac{P_s}{KT} = n_s \pm \frac{n_s^2}{2^{5/2}} \left(\frac{2\pi\hbar^2}{mKT} \right)^{3/2}, \quad \left(\begin{array}{l} \text{F. D.} \\ \text{B. E.} \end{array} \right). \quad (36)$$

This may be added to Eq. (29) (with \bar{N}_u replacing N_u) to give the total gas pressure.

$$\frac{P_t}{KT} = n_s + n_s^2 \left[\bar{f} \pm \frac{1}{2^{5/2}} \left(\frac{2\pi\hbar^2}{mKT} \right)^{3/2} \right], \quad \left(\begin{array}{l} \text{F. D.} \\ \text{B. E.} \end{array} \right). \quad (37)$$

In Eq. (37), it is important to realize that n_s is the number density of *free* stable particles. That is, n_s only includes stable particles *not* involved in an interacting pair (unstable particle). The total number density of all stable particles, free or interacting, in the gas is

$$n = n_s + 2N_u = n_s + 2n_s^2 f,$$

which may be solved for n_s :

$$\begin{aligned} n_s &= [-1 + (1 + 8fn)^{1/2}] / 4f \\ &= +n - 2fn^2 + 8f^2n^3 - 40f^3n^4 + \dots \end{aligned}$$

Since f is proportional to λ^3 , and only terms to third order in λ are to be saved, n_s reduces to

$$n_s \approx n - 2fn^2. \quad (38)$$

To the same order in f , this may be used to re-

$$\begin{aligned} \frac{P_t}{KT} &= n - n^2 \left(\frac{2\pi\hbar^2}{mKT} \right)^{3/2} \left(\mp \frac{1}{2^{5/2}} + \frac{2^{3/2}}{\pi} \sum_l (2l+1) \int_0^\infty \frac{d\eta_l}{dE} e^{-E/KT} dE \right) \\ &+ n^2 \left(\frac{2\pi\hbar^2}{mKT} \right)^{3/2} \frac{2^{3/2}}{\pi} \sum_l (2l+1) \int_0^\infty (\cos^2\eta_l - \sin^2\eta_l) \frac{d\eta_l}{dE} e^{-E/KT} dE, \quad \left(\begin{array}{l} \text{F. D.} \\ \text{B. E.} \end{array} \right), \end{aligned} \quad (40)$$

where use has been made of Eq. (32) and the identity $\sin^2\eta_l = 1 - \cos^2\eta_l$. The first and second terms will be recognized as just the general two-body (no bound states) contributions of the first and second virial coefficients to the equation of state.¹⁰ The third term must therefore represent the overpressure error made in assuming that an unstable particle will remain intact and will be specularly reflected by the walls of a container. This illustrates the point raised in the Introduction, namely, that the interaction properties of unstable particles are quite different than those of particles with bound substructures. The inadequacy of the approximation that both types of particles interact in an identical manner with the containing walls ($I_l = \bar{I}_l$) is most strikingly revealed in the presence of this unwanted overpressure term.

It is to be noted that the general solution to the momentum transfer problem of an unstable particle colliding with a plane interface would constitute an independent derivation of the Beth-Uhlenbeck expression for the second virial coefficient.^{11,12} The intention here, however, is to determine \bar{I}_l , and this may be accomplished by demanding that Eq. (39) be consistent with the form of the second virial coefficient. The only way this is possible is if

$$2I_l - \bar{I}_l = \int_0^\infty \frac{d\eta_l}{dE} e^{-E/KT} dE,$$

or, by Eq. (32),

write Eq. (37) in terms of n :

$$\begin{aligned} \frac{P_t}{KT} &= n - n^2 \left(\frac{2\pi\hbar^2}{mKT} \right)^{3/2} \left(\mp \frac{1}{2^{5/2}} + \frac{2^{3/2}}{\pi} \right. \\ &\times \left. \sum_l (2l+1) (2I_l - \bar{I}_l) \right), \quad \left(\begin{array}{l} \text{F. D.} \\ \text{B. E.} \end{array} \right). \end{aligned} \quad (39)$$

Here, Eqs. (31) and (35) have been substituted for f and \bar{f} ; I_l is given by Eq. (32); and \bar{I}_l is yet to be determined.

It is instructive to investigate Eq. (39) in the specular reflection approximation $I_l = \bar{I}_l$. It becomes

$$\bar{I}_l = \int_0^\infty \frac{d\eta_l}{dE} (4 \sin^2\eta_l - 1) e^{-E/KT} dE. \quad (41)$$

Using Eqs. (29), (34), (35), (39), and (41), one obtains for the equation of state of an ideal unstable particle gas

$$\begin{aligned} \frac{P_u}{KT} &= n^2 \left(\frac{2\pi\hbar^2}{mKT} \right)^{3/2} \frac{2^{3/2}}{\pi} \sum_l (2l+1) \\ &\times \int_0^\infty \frac{d\eta_l}{dE} (4 \sin^2\eta_l - 1) e^{-E/KT} dE. \end{aligned} \quad (42)$$

This is the simplest (lowest order in λ^3) possible equation of state, and it is valid in the low-density high-temperature limit for the case where the interaction potential between two stable particles is attractive but has no bound states.

While of passing interest, Eq. (42) would seem to be of little practical value in calculating equations of state, since this may be done directly through Eq. (39). The motivation for deriving (42), however, was not to exhibit any usefulness it might possess but rather to demonstrate its existence. The existence of an unstable particle gas pressure along with well defined number densities and Maxwellian velocity distributions should present convincing evidence that, within the context of the assumptions which have been made, a gas of unstable particles behaves very much like an ideal gas of stable particles. The question then arises as to just how much "reality" can be as-

signed to an unstable particle gas? That is, are the words unstable particle gas just a concept or are they physically manifested by measurable effects? This question will be dealt with in Sec. 6.

Before closing this section, it should be mentioned that the form of the second virial coefficient referred to, and the equation of state (39), were derived under the assumption that the interacting particles, regardless of the statistics they obeyed, are spinless. The generalization to the case of spin is straightforward and may be found in Ref. 13.

6. CORRECTIONS TO EQUATIONS OF CHEMICAL AND NUCLEAR STATISTICAL EQUILIBRIUM

As evidenced in Secs. 3 and 4, the chemical equilibrium approach to unstable particles is only valid under very special circumstances. In this section, it will be argued that if one accepts the reality of unstable particle gases, then one must also admit the possibility of corrections to the equations of chemical and nuclear statistical equilibrium, as they apply to stable particle abundances. In order to keep the mathematical notation at a minimum, a simplified version of the general problem shall be studied. Generalization to the complete case will be obvious.

Consider a gas in chemical equilibrium where the different chemical or nuclear species are just 2, 3, 4, ..., r -body bound states of some fundamental constituent to be called the one-body particle. Assuming all constituents to be spinless and to have no bound excited states, Q_r is defined as the energy needed to break up an r -body particle into the r one-body particles composing it. Without specifying the different chemical or nuclear reactions which transmute one species into another, it is possible, through Eq. (12), to express the equilibrium number density of any constituent in terms of the one-body particle number density:

$$N_r = (N_1)^r \left(\frac{2\pi\hbar^2}{mKT} \right)^{3(r-1)/2} (r)^{3/2} e^{Q_r/KT}, \quad (43)$$

$r = 1, 2, \dots$

Equation (43) is just a special case of the equations of chemical and nuclear statistical equilibrium and the number densities appearing in it are always termed "free." The term "free" in the chemical equilibrium sense means not *bound* up in some other form, whereas throughout this work it has been used to mean *not interacting*. The distinction is important because a certain fraction of the two-body, three-body, and r -body particles will form unstable particles with each other and with the one-

body particles, and it is not immediately clear whether Eq. (43) includes particles in this form.

To illustrate, recall Fig. 1 in Sec. 3, which displayed the fraction of α particles in the form of He^5 when the α particles were imbedded in a hot, dense neutron gas. Do α particles which are temporarily interacting with a neutron participate in nuclear reactions at the same rate and in the same way that free α particles do? For that matter are free neutrons chemically indistinguishable from neutrons in temporary unions with α particles? Stated in these terms the answer would in general appear to be *no*. We should not, however, be too emphatic in this "no" since it is not difficult to imagine situations where the energies of a chemical or nuclear reaction are such as to completely neglect the presence of one of the participants in an unstable particle.

If it is assumed that unstable particles are chemically distinguishable from stable particles, then it is clear that Eq. (43) can only refer to particles which are free in the noninteracting sense. To find the total number density of a particular constituent, one needs to add its free and interacting number densities. If the r th constituent can form unstable particles with itself and all the other constituents in the gas then its total number density is

$$n_r = N_r + \sum_i \mathfrak{N}_{ir} (1 + \delta_{ir}), \quad (44)$$

where N_r is given by Eq. (43), and, from Eqs. (10) and (11),

$$\mathfrak{N}_{ir} = N_i N_r \left(\frac{2\pi\hbar^2}{\mu_{ir}KT} \right)^{3/2} \sum_l \frac{(2l+1)}{\pi} \times \int_0^\infty 2 \frac{d\eta_l}{dE} \sin^2 \eta_l e^{-E/KT} dE. \quad (45)$$

\mathfrak{N}_{ir} is the total number density of unstable particles that constituent r forms with constituent i for all values of the angular momentum l .

§ The difference between Eqs. (43) and (44)

$$\Delta n_r = \sum_i \mathfrak{N}_{ir} (1 + \delta_{ir}), \quad (46)$$

are corrections to the equations of chemical and nuclear statistical equilibrium. While these corrections are in general small it still might be possible to detect them experimentally. For example, any sampling process which made high-precision measurements of the total number densities would be adequate. This could be achieved by, say, a chemical analysis of the molecules escaping from a small hole in a container filled with a gas in chemical equilibrium. Any systematic difference

between these measurements and the predictions of Eq. (43) which could be accounted for, at least in part, by Eq. (46) would be a confirmation of the chemical and physical presence of unstable particle gases.

Attention must be called to the fact that Eq. (46) is an approximation to the actual corrections. That a particle involved in an interacting pair (unstable particle) will participate in chemical reactions differently than if it were free should be clear. That it will not participate at all, as was assumed in deriving (46), is a question which can only be answered by a deeper understanding of the interaction properties of unstable particles.

7. CONCLUSIONS

A word of caution is in order here. Up to now the discussion has been restricted to the positive phase shifts expected in the scattering from attractive potentials. Actually, the restriction was really to positive collision lifetimes or by Eq. (1)

to positive values for the energy derivatives of the scattering phase shifts; $d\eta/dE \geq 0$. This should be kept in mind for all the equations appearing in this work, otherwise, seemingly bizarre phenomena like negative number densities for unstable particles may occur.

The possibility of extending the unstable particle formalism into the region $d\eta/dE < 0$ by means of a gas of unstable "quasiholes" with negative lifetimes and negative particle number densities is presently under consideration.

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