PHYSICAL REVIEW A VOLUME 1, NUMBER 4 APRIL 1970

Electromagnetic Fluctuations in a Cavity*

K. M. Case† and S. C. Chiu
Department of Physics, University of Michigan, Ann Arbor, Michigan 48104
(Received 2 June 1969)

The problem of electromagnetic fluctuations in a cavity is treated in some generality using the fluctuation-dissipation theorem. The general line shape of the energy spectrum is derived. This reduces to a Lorentzian when $\epsilon''(\omega)$, the imaginary part of the dielectric constant, is small and varies slowly with frequency. Deviations from the Stefan-Boltzmann law are calculated in detail for a cubical cavity.

I. INTRODUCTION

In this paper, the problem of electromagnetic fluctuations in a cavity is treated in some generality, using the fluctuation-dissipation theorem.[1] We first briefly state the theorem: If a system originally in equilibrium with unperturbed Hamiltonian H_0 and density matrix $\rho=\exp(-H_0/\kappa T)/\mathrm{Tr}\exp(-H_0/\kappa T)$ is under the action of external perturbing forces $F^i(t)$ such that the perturbing Hamiltonian may be written

$$H_1(t)=-\sum_i F_i(t)A_i\ , \tag{1}$$

where A_i are macroscopic internal variables, and if the corresponding macroscopic equation of linear response can be written in the form

$$A_i(\omega)=\alpha_{ik}(\omega)F_k(\omega)\ , \tag{2}$$

where the expression on the right-hand side is summed over k, and

$$A_i(\omega)=\int_{-\infty}^{\infty}dt\, e^{i\omega t}A_i(t)\ ,$$

etc., then the Fourier time components of the correlations of the internal variables $\frac{1}{2}\langle A_iA_j(t)+A_j(t)A_i\rangle$ are given by

$$\langle A_iA_j\rangle_\omega=\frac{i\hbar}{\exp(\hbar\omega/\kappa T)-1}\left[\alpha^*_{ij}(\omega)-\alpha_{ji}(\omega)\right]\ , \tag{3}$$

where $\langle A_iA_j\rangle_\omega$ is defined by

$$\tfrac{1}{2}\langle A_i(\omega)A_j(\omega')+A_j(\omega')A_i(\omega)\rangle = 2\pi\langle A_iA_j\rangle_\omega\delta(\omega+\omega')\ , \tag{4}$$

and κ is the Boltzmann constant. The symbol $\langle\ \rangle$ denotes statistical average with respect to ρ.

II. FLUCTUATIONS IN A CAVITY

Let us consider a cavity of general shape with perfectly conducting boundaries, and assume that the medium inside the cavity is isotropic and homogeneous so that the medium has, in general, a frequency-dependent dielectric constant $\epsilon(\omega)$ and a magnetic permeability $\mu(\omega)$. They are in general complex. Then

$$\vec{D}(\omega)=\epsilon(\omega)\vec{E}(\omega)\ ,\qquad \vec{B}(\omega)=\mu(\omega)\vec{H}(\omega)\ , \tag{5}$$

where, e.g., $\vec{E}(\omega)=\int_{-\infty}^{\infty}e^{i\omega t}\vec{E}(\vec{r},t)\,dt$.

Let $\vec{\phi}_{\alpha\beta}(\vec{r},\lambda_\alpha)$ be a complete set of orthonormal functions satisfying

$$\mathrm{curl\ curl}\ \vec{\phi}_{\alpha\beta}(\vec{r},\lambda_\alpha)=\lambda_\alpha{}^2\vec{\phi}_{\alpha\beta}(\vec{r},\lambda_\alpha)\ ,\qquad \vec{\phi}_{\alpha\beta}(\vec{r},\lambda_\alpha)\times\hat{n}=0\ , \tag{6}$$

where β represents eigenvalues other than λ_α, required to specify an eigenstate uniquely, and $\hat{n}$ is a unit vector normal to the boundary surface. Then we can expand all electric fields in terms of $\vec{\phi}_{\alpha\beta}(\vec{r},\lambda_\alpha)$ and all magnetic fields in terms of the orthonormal set of functions $\mathrm{curl}\,\vec{\phi}_{\alpha\beta}(\vec{r},\lambda_\alpha)/\lambda_\alpha$.

Let there be a perturbing Hamiltonian of the form

$$H_1=-\int[\vec{E}\cdot\vec{K}(\vec{r},t)+\vec{H}\cdot\vec{L}(\vec{r},t)/4\pi]\,d^3r\ , \tag{7}$$

where $\vec{K}(\vec{r},t)$ and $\vec{L}(\vec{r},t)$ are the external forces, and $\vec{E}$ and $\vec{H}$ are electric and magnetic fields considered as internal variables. Then Eq. (5) is to be written[2]

$$\vec{D}(\vec{r},\omega)=\epsilon(\omega)\vec{E}(\vec{r},\omega)+\vec{K}(\vec{r},\omega)\ ,\qquad \vec{B}(\vec{r},\omega)=\mu(\omega)\vec{H}(\vec{r},\omega)+\vec{L}(\vec{r},\omega)\ , \tag{8}$$

while $\vec{D}(\vec{r},\omega)$ and $\vec{B}(\vec{r},\omega)$ satisfy the Maxwell equations

$$\mathrm{curl}\,\vec{E}(\vec{r},\omega)=(i\omega/c)\vec{B}(\vec{r},\omega)\,,$$
$$\mathrm{curl}\,\vec{H}(\vec{r},\omega)=-(i\omega/c)\vec{D}(\vec{r},\omega)\,. \quad (9)$$

Expand the fields in terms of their eigenfunctions:

$$\begin{bmatrix}\vec{E}(\vec{r},\omega)\\ \vec{D}(\vec{r},\omega)\\ \vec{K}(\vec{r},\omega)\end{bmatrix}=\sum_{\alpha,\beta}\begin{bmatrix}E_{\alpha\beta}(\omega)\\ D_{\alpha\beta}(\omega)\\ K_{\alpha\beta}(\omega)\end{bmatrix}\vec{\phi}_{\alpha\beta}(\vec{r},\lambda_\alpha)\,, \quad (10)$$

$$\begin{bmatrix}\vec{H}(\vec{r},\omega)\\ \vec{B}(\vec{r},\omega)\\ \vec{L}(\vec{r},\omega)\end{bmatrix}=\sum_{\alpha,\beta}\begin{bmatrix}H_{\alpha\beta}(\omega)\\ B_{\alpha\beta}(\omega)\\ L_{\alpha\beta}(\omega)\end{bmatrix}\frac{\nabla\times\vec{\phi}_{\alpha\beta}(\vec{r},\lambda_\alpha)}{\lambda_\alpha}\,. \quad (11)$$

Substituting these in Eqs. (9) and utilizing the orthonormal properties of $\vec{\phi}_{\alpha\beta}(\vec{r},\lambda_\alpha)$ and $\nabla\times\vec{\phi}_{\alpha\beta}(\vec{r},\lambda_\alpha)/\lambda_\alpha$, we have

$$\frac{H_{\alpha\beta}(\omega)}{4\pi}=-\frac{\omega^2\epsilon(\omega)L_{\alpha\beta}(\omega)}{4\pi[\omega^2\epsilon(\omega)\mu(\omega)-\lambda_\alpha^2]}+\frac{i\omega\omega_\alpha\mu^{1/2}(\omega_\alpha)\epsilon^{1/2}(\omega_\alpha)K_{\alpha\beta}(\omega)}{4\pi[\omega^2\epsilon(\omega)\mu(\omega)-\lambda_\alpha^2]}\,, \quad (12)$$

$$\frac{E_{\alpha\beta}(\omega)}{4\pi}=-\frac{i\omega\omega_\alpha\mu^{1/2}(\omega_\alpha)\epsilon^{1/2}(\omega_\alpha)L_{\alpha\beta}(\omega)}{4\pi[\omega^2\epsilon(\omega)\mu(\omega)-\lambda_\alpha^2]}-\frac{\omega^2\mu(\omega)K_{\alpha\beta}(\omega)}{4\pi[\omega^2\epsilon(\omega)\mu(\omega)-\lambda_\alpha^2]}\,. \quad (13)$$

The fluctuation-dissipation theorem can be applied to Eqs. (12) and (13) directly. Thus,

$$\langle H_{\alpha\beta}H_{\alpha'\beta'}\rangle_\omega=\frac{i4\pi\hbar}{\exp(\hbar\omega/\kappa T)-1}\delta_{\alpha\alpha'}\delta_{\beta\beta'}\times\left(\frac{\omega^2\epsilon(\omega)}{\omega^2\epsilon(\omega)\mu(\omega)-\lambda_\alpha^2}-\text{c.c.}\right)\,, \quad (14)$$

$$\langle E_{\alpha\beta}E_{\alpha'\beta'}\rangle_\omega=\frac{i4\pi\hbar}{\exp(\hbar\omega/\kappa T)-1}\delta_{\alpha\alpha'}\delta_{\beta\beta'}\times\left(\frac{\omega^2\mu(\omega)}{\omega^2\epsilon(\omega)\mu(\omega)-\lambda_\alpha^2}-\text{c.c.}\right)\,, \quad (15)$$

where c.c. means complex conjugate.

The spatial correlations can be obtained easily. Of particular interest are the correlations at the same spatial point $\vec{r}$:

$$\langle\vec{E}(\vec{r})\cdot\vec{E}(\vec{r})\rangle_\omega=\frac{i4\pi\hbar}{\exp(\hbar\omega/\kappa T)-1}\sum_{\alpha,\beta}\left(\frac{\omega^2\mu(\omega)}{\omega^2\epsilon(\omega)\mu(\omega)-\lambda_\alpha^2}-\text{c.c.}\right)\vec{\phi}_{\alpha\beta}(\vec{r},\lambda_\alpha)\cdot\vec{\phi}_{\alpha\beta}(\vec{r},\lambda_\alpha)\,, \quad (16)$$

$$\langle\vec{H}(\vec{r})\cdot\vec{H}(\vec{r})\rangle_\omega=\frac{i4\pi\hbar}{\exp(\hbar\omega/\kappa T)-1}\times\sum_{\alpha,\beta}\left(\frac{\omega^2\epsilon(\omega)}{\omega^2\epsilon(\omega)\mu(\omega)-\lambda_\alpha^2}-\text{c.c.}\right)\times\frac{\nabla\times\vec{\phi}_{\alpha\beta}\cdot\nabla\times\vec{\phi}_{\alpha\beta}}{\lambda_\alpha^2}\,. \quad (17)$$

The energy spectrum is given by[3]

$$\langle U(\omega)\rangle=\frac{1}{16\pi^2}\left(\frac{d(\omega\epsilon)}{d\omega}\langle\vec{E}\cdot\vec{E}\rangle_\omega+\frac{d(\omega\mu)}{d\omega}\langle\vec{H}\cdot\vec{H}\rangle_\omega\right), \quad (18)$$

where $\langle\vec{E}\cdot\vec{E}\rangle_\omega$ and $\langle\vec{H}\cdot\vec{H}\rangle_\omega$ are given by Eqs. (16) and (17).

We may average over the volume by $1/V\int dV$, denoted by the symbol $\langle\bar{\ }\rangle$. By the orthonormality of the eigenfunctions,

$$\langle\bar{U}(\omega)\rangle=\frac{i\hbar}{4\pi V[\exp(\hbar\omega/\kappa T)-1]}\times\sum_{\alpha,\beta}\left[\frac{d(\omega\epsilon)}{d\omega}\left(\frac{\omega^2\mu}{\omega^2\epsilon\mu-\lambda_\alpha^2}-\text{c.c.}\right)\times\frac{d(\omega\mu)}{d\omega}\left(\frac{\omega^2\epsilon}{\omega^2\epsilon\mu-\lambda_\alpha^2}-\text{c.c.}\right)\right]. \quad (19)$$

If ϵ and μ are real constants, then we have

$$\langle\bar{U}(\omega)\rangle=\frac{1}{V}\sum_{\alpha,\beta}\frac{\hbar\omega_\alpha}{\exp(\hbar\omega_\alpha/\kappa T)-1}\delta(\omega-\omega_\alpha)\,. \quad (20)$$

In the high-frequency limit, we may use the asymptotic expression for the density of eigenstates

$$\sum_{\alpha,\beta}\to\int\frac{Vk_\alpha^2dk_\alpha}{\pi^2}=\int\frac{V(\epsilon\mu)^{3/2}\omega_\alpha^2d\omega_\alpha}{\pi^2c^3}\,. \quad (21)$$

Then we obtain

$$\langle \bar{U}(\omega)\rangle \to \frac{\hbar(\epsilon\mu)^{3/2}\omega^3}{\pi^2 c^3[\exp(\hbar\omega/\kappa T)-1]} .$$

This is the familiar Planck distribution. Similarly, if the volume $V\to\infty$, $\langle U(\omega)\rangle$ becomes the Planck distribution.

In general, if we separate the real and imaginary parts of $\epsilon(\omega)$ and $\mu(\omega)$:

$$\mu(\omega)=\mu'(\omega)+i\mu''(\omega), \qquad \epsilon(\omega)=\epsilon'(\omega)+i\epsilon''(\omega);$$

then Eq. (16) becomes

$$\langle \vec{E}(\vec{r})\cdot\vec{E}(\vec{r})\rangle_\omega = \sum_{\alpha,\beta} \frac{8\pi\hbar\omega^2}{\exp(\hbar\omega/\kappa T)-1} F(\omega)\vec{\phi}_{\alpha\beta}(\vec{r})\cdot\phi_{\alpha\beta}(\vec{r}) , \tag{22}$$

where

$$F(\omega)=\frac{\mu'(\omega)(\epsilon''\mu'+\epsilon'\mu'')\omega^2-\mu''[\omega^2(\epsilon'\mu'-\epsilon''\mu'')-\lambda_\alpha{}^2]}{[\omega^2(\epsilon'\mu'-\epsilon''\mu'')-\lambda_\alpha{}^2]^2+\omega^4(\epsilon''\mu'+\epsilon'\mu'')^2} . \tag{23}$$

For simplicity, let us put $\mu(\omega)=1$. Then we obtain

$$F_1(\omega)=F(\omega)|_{\mu=1}=\omega^2\epsilon''(\omega)\{[\omega^2\epsilon'(\omega)-\lambda_\alpha{}^2]^2+\omega^4\epsilon''^2(\omega)\}^{-1} . \tag{24}$$

Equation (24) gives the general line shape of the energy spectrum I in the limiting case when $\epsilon''(\omega)\to 0$, $F_1(\omega)\to\delta$ function. When $\epsilon''(\omega)$ is small and varies slowly with frequency, we may expand $F_1(\omega)$ around $\omega=\omega_1$, where

$$\omega_1{}^2\epsilon'(\omega_1)-\lambda_\alpha{}^2=0 .$$

Thus, we find

$$\omega^2F_1(\omega)\approx\frac{\epsilon''(\omega)}{K_2{}^2(\omega_1)}\left\{\left(\omega-\omega_1+\frac{1}{2}\frac{\epsilon''(\omega_1)}{K_2{}^2(\omega_1)}\frac{d\epsilon''(\omega_1)}{d\omega_1}\right)^2+\left[\frac{\epsilon''^2(\omega_1)}{K_2{}^2(\omega_1)}-\frac{1}{4}\frac{\epsilon''^2(\omega_1)}{K_2{}^4(\omega_1)}\left(\frac{d\epsilon''(\omega_1)}{d\omega_1}\right)^2\right]\right\}^{-1} , \tag{25}$$

where $K_2{}^2(\omega_1)$ is a constant depending on ω_1, i.e.,

$$K_2{}^2(\omega_1)=\left(\frac{d\epsilon'}{d\omega_1}+\frac{2\lambda_\alpha}{\omega_1{}^2}\right)+\frac{1}{2}\epsilon''(\omega_1)\frac{d^2\epsilon''}{d\omega_1{}^2} .$$

Equation (25) is the usual Lorentzian form which gives approximately the shifts of the peaks from the eigenfrequencies and the linewidths.

Going back to Eq. (16), we may use Carleman's theorem[4] for the asymptotic behavior of $\langle\vec{E}(\vec{r})\cdot\vec{E}(\vec{r})\rangle_\omega$ at high frequencies:

$$\langle\vec{E}(\vec{r})\cdot\vec{E}(\vec{r})\rangle_\omega\approx\int\frac{\lambda_\alpha{}^2d\lambda_\alpha{}^2}{\pi^2c^3}\frac{8\pi\hbar\omega^2}{\exp(\hbar\omega/\kappa T)-1}\frac{[\mu'(\omega)C_2(\omega)-\mu''(\omega)(C_1{}^2(\omega)-\lambda_\alpha{}^2)]}{[C_1{}^2(\omega)-\lambda_\alpha{}^2]^2+C_2{}^2(\omega)} , \tag{26}$$

where

$$C_1{}^2(\omega)=\omega^2[\epsilon'(\omega)\mu'(\omega)-\epsilon''(\omega)\mu''(\omega)], \qquad C_2(\omega)=\omega^2[\epsilon''(\omega)\mu'(\omega)+\epsilon'(\omega)\mu''(\omega)] . \tag{27}$$

The integral in Eq. (26) may be carried out to give

$$\langle\vec{E}(\vec{r})\cdot\vec{E}(\vec{r})\rangle_\omega=\frac{2\hbar\omega^2}{c^3[\exp(\hbar\omega/\kappa T)-1]}\left(\frac{\mu'(\omega)C_2(\omega)}{C_3(\omega)}-2\mu''(\omega)C_3(\omega)\right), \tag{28}$$

where

$$C_3{}^2(\omega)=\tfrac{1}{2}C_1{}^2(\omega)\{-1+[1+C_2{}^2(\omega)/C_1{}^4(\omega)]^{1/2}\} .$$

When $C_2 \ll C_1^2(\omega)$, we may expand the expression in powers of C_2/C_1^2. In the lowest approximation we have

$$\langle \vec{E}(\vec{r})\cdot\vec{E}(\vec{r})\rangle_\omega \approx 4\mu'(\omega)[\epsilon'(\omega)\mu'(\omega)]^{1/2}\frac{\hbar\omega^3}{c^3}\frac{1}{\exp(\hbar\omega/\kappa T)-1} . \tag{29}$$

Similarly, we will find

$$\langle \vec{H}(\vec{r})\cdot\vec{H}(\vec{r})\rangle_\omega \approx 4\epsilon'(\omega)[\epsilon'(\omega)\mu'(\omega)]^{1/2}\frac{\hbar\omega^3}{c^3}\frac{1}{\exp(\hbar\omega/\kappa T)-1} . \tag{30}$$

Letting $\mu' = 1$ and using the expression for energy density, we find the energy spectrum to be

$$\langle U(\omega)\rangle \approx \frac{\epsilon'(\omega)d(\omega\sqrt{\epsilon'})/d\omega}{\pi^2 c^3}\frac{\hbar\omega^3}{\exp(\hbar\omega/\kappa T)-1} . \tag{31}$$

This agrees with Ref. 2 [Eq. (89-2)] for an infinite medium.

III. FLUCTUATIONS IN A CUBICAL CAVITY

Finally, for the simple case of a cubical cavity, explicit calculations are carried out to show the deviations from Stefan-Boltzmann law, and the relative contribution of the lowest modes to the total energy.

Let the length of each side of the cube be L. Then the characteristic frequencies are

$$\omega_\alpha = \pi c(n_1^2 + n_2^2 + n_3^2)^{1/2}/L , \tag{32}$$

where n_1, n_2, and n_3 are integers. The allowable modes are restricted by the equation div $\vec{D} = 0$. Taking into account the degeneracies, the integral of Eq. (20) over frequency, i.e., the total energy density, is

$$\int\langle \bar{U}(\omega)\rangle\, d\omega = \frac{1}{4V}\sum_{n_1,n_2,n_3=-\infty}^{\infty}\frac{\pi\hbar c(n_1^2+n_2^2+n_3^2)^{1/2}/L}{\exp[\pi\hbar c(n_1^2+n_2^2+n_3^3)^{1/2}/L\kappa T]-1}$$

$$-\frac{3}{4V}\sum_{n_1=-\infty}^{\infty}\frac{\pi\hbar c|n_1|/L}{\exp(\pi\hbar c|n_1|/L\kappa T)-1}+\frac{\kappa T}{2V} . \tag{33}$$

The second and third terms are due to the fact that there exists no mode where two or more n elements are zero.

Using the Poisson summation formula and the formula

$$\int_0^\infty \frac{\sin ax\,dx}{e^{2\pi x}-1} = \frac{1}{4}\frac{e^a+1}{e^a-1}-\frac{1}{2a} , \tag{34}$$

Eq. (33) may be transformed to

$$\int\langle \bar{U}(\omega)\rangle\, d\omega = \frac{\pi^2(\kappa T)^4}{15(\hbar c)^3}+\frac{\hbar c}{(4\pi)^2 L}\sum_{\substack{\nu_1,\nu_2,\nu_3=-\infty\\ |\nu|\neq 0}}^{\infty}\frac{(\lambda T)^3}{|\nu|}\left[\left(\frac{1}{\lambda LT|\nu|}\right)^3-\frac{\exp(2\lambda LT|\nu|)+\exp(\lambda LT|\nu|)}{2[\exp(\lambda LT|\nu|)-1]^3}\right]$$

$$-\frac{\kappa\lambda T^2}{16L^2}-\frac{3\kappa}{2\lambda L^4}\sum_{\substack{\nu_1=-\infty\\ \nu_1\neq 0}}^{\infty}\left(\frac{1}{2\nu_1^2}-\frac{(\lambda LT)^2}{2}\frac{\exp(\lambda LT|\nu_1|)}{[\exp(\lambda LT|\nu_1|)-1]^2}\right)+\frac{\kappa T}{2L^3} , \tag{35}$$

where $$|\nu| = (\nu_1^2 + \nu_2^2 + \nu_3^2)^{1/2} , \qquad (36)$$

$$\lambda = \frac{4\pi\kappa}{\hbar c} \approx 0.549 \ (\text{cm }^\circ\text{K})^{-1} . \qquad (37)$$

Note that the first term on the right-hand side of Eq. (35) is the familiar Stefan-Boltzmann law. All other terms are corrections. Also as $T \to 0$, $\int \langle \bar{U}(\omega)\rangle \, d\omega \sim T$ instead of $\sim T^4$, as given by the Stefan-Boltzmann law.

At high temperatures, the first term is the dominant term. We expect that there exists some temperature range in which the exponential terms in the sums are still small, but the other corrections become important. Using the approximations

$$\sum_{\substack{\nu_1, \nu_2, \nu_3 = -\infty \\ |\nu| \neq 0}}^{\infty} \frac{1}{|\nu|^4} \approx 15.7 , \qquad (38)$$

and $$\sum_{\nu_1 = 1}^{\infty} \frac{1}{\nu_1^2} \approx 1.64 , \qquad (39)$$

we have

$$\langle \bar{U}(\omega)\rangle \, d\omega \approx \frac{\pi^2 (\kappa T)^4}{15(\hbar c)^3} - \frac{\pi(\kappa T)^2}{4L^2 \hbar c} + \frac{\kappa T}{2L^3}$$

$$+ 15.7 \frac{\hbar c}{(4\pi)^2 L^4} - 1.64 \frac{3\kappa}{2\lambda L^4} + 0(e^{-\lambda L T}) . \qquad (40)$$

We observe that there is no term depending on $1/L$, which would be the surface-area–dependent correction to the asymptotic form of the density of states given by Eq. (21). Actually, the vanishing of such a term can be proved in general, using the method described in Ref. 4. The ratios of the last four terms and the first term of the right-hand side of Eq. (40) are easily calculated. They are

$$\frac{\text{second term}}{\text{first term}} = \frac{15}{4\pi} \left(\frac{\hbar c}{L\kappa T}\right)^2 \approx \left(\frac{25}{LT}\right)^2 , \qquad (41\text{a})$$

$$\frac{\text{third term}}{\text{first term}} = \frac{15}{2\pi^2} \left(\frac{\hbar c}{\kappa L T}\right)^3 \approx \left(\frac{20.9}{LT}\right)^3 , \qquad (41\text{b})$$

$$\frac{\text{fourth term} + \text{fifth term}}{\text{first term}} \approx \left(\frac{14.0}{LT}\right)^4 . \qquad (41\text{c})$$

Thus at $LT < 20$ cm °K, these terms become important, while $e^{-\lambda LT}$ is still small. Hence, for cavities of average length in the order of centimeters, we expect to find marked deviation from T^4 law for temperatures below about 10°K.

The ratio of the energy in the lowest mode and $\pi^2(\kappa T)^4/15(\hbar c)^3$ is also readily found. It is approximately

$$\frac{1}{\exp(102/LT) - 1} \frac{45\sqrt{2}}{\pi} \left(\frac{\hbar c}{L\kappa T}\right)^4$$

$$\approx \frac{1}{\exp(102/LT) - 1} \left(\frac{48.5}{LT}\right)^4 .$$

This begins to become a significant portion when the other terms of Eq. (40) also become large. Of course, when T is very low, the only significant term is $\kappa T/2V$, which is due to the forbidden mode when all n's are zero.[5]

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*Work supported in part by the National Science Foundation.

†Presently at Rockefeller University, New York, N. Y.

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[2]L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media (Pergamon Press, Inc., New York, 1960), Chap. XIII.

[3]Reference 2, p. 255.

[4]M. Kac, Am. Math. Mon. 73.4, 1 (1966).

[5]The deviations of the energy spectrum from the Stefan-Boltzmann law at low temperatures seem to be known to maser experts. However, to our knowledge, no such detailed calculation has been carried out. See, for example, A. E. Siegman, Microwave Solid State Masers (McGraw-Hill Book Co., New York, 1964), Chap. 8.