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Properties of Dilute Solutions of ^{3}He in Liquid ^{4}He at Low Temperatures

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Using an effective interaction or reaction matrix obtained from a modified Brueckner theory, various low-temperature properties are calculated or estimated for dilute solutions of ^{3}He in liquid ^{4}He. The system is regarded as a low-density Fermi liquid with ^{3}He quasiparticles created by ^{3}He atoms in superfluid ^{4}He. The single-particle energy spectrum is given by an effective mass, and an effective interaction between the ^{3}He quasiparticles is derived. The calculations are done for two different two-body potentials: an Yntema-Schneider potential given by Brueckner and Gammel, and a Frost-Musulin potential given by Bruch and McGee. The Landau f function is estimated from the reaction matrix, and the coefficients of the expansion of the Landau f function in terms of Legendre polynomials are calculated. The estimated values are in reasonably good agreement with experimental results. The exclusion-principle sum rule is also roughly satisfied by the lowest-order coefficients. Low-temperature properties, such as the compressibility, the quasiparticle effective mass or specific-heat ratio, and the magnetic susceptibility, are estimated; results are in fair agreement with experimental values. The various properties are also given as functions of the ^{3}He concentration in the solution, and the maximum solubility of ^{3}He in liquid ^{4}He is estimated in good agreement with the experimental value. Also, transport coefficients – i. e., viscosity, thermal conductivity and spin diffusion – are estimated after identification of the reaction matrix with the scattering amplitude in the formulas developed by Abrikosov and Khalatnikov and by Hone. The agreement with experimental results is surprisingly good, considering that the results depend very much on the value chosen for the effective mass.

I. INTRODUCTION

Measurements by Edwards *et al.*[1] and Anderson *et al.*[2,3] show that ^{3}He atoms in dilute solutions behave like a normal Fermi liquid, as predicted by Landau and Pomeranchuk,[4,5] and that many of the low-temperature properties of ^{3}He in solution should be qualitatively similar to those of pure liquid ^{3}He. The experiments also indicate that there is a weak and predominantly attractive effective interaction between ^{3}He atoms in dilute solutions in liquid ^{4}He. From additional theoretical work,[6–17] one obtains qualitatively an understanding of various properties, but the determination of the Fermi liquid parameters remains phenomenological; and the theories are not really built on a microscopic first-principle basis.

Campbell[18] has done a more microscopic calculation of the ^{3}He-^{4}He solution properties. It is phenomenological in the sense that the experimental values of the volume per atom in the solution and the average nearest-neighbor distance are used. An unknown ^{4}He effective mass is also treated as a phenomenological parameter, and a self-consistent ^{3}He effective mass is chosen at zero ^{3}He concentration. Massey and Woo[19] and Woo *et al.*[20] have tried to calculate from bare atoms and realistic two-body potentials, the effective mass of ^{3}He quasiparticles and the residual interaction between them, using the method of correlated basis functions. Similar work is also done by Davison and Feenberg.[21]

There is an average attractive effective interaction between the ^{3}He quasiparticles in a dilute solution. The effective potential consists of several contributions which partly cancel each other, and the interaction between the ^{3}He quasiparticles is surprisingly weak compared with the interaction of bare particles at the same density, or the interatomic forces in the system. The physical reason for the considerable cancellation between the direct and the induced contributions is just that ^{3}He here is an isotopic impurity. The effective interaction may be defined as a reaction matrix. It includes effects of virtual transitions to states outside the excluded Fermi sea, and will depend to some extent on the concentration or number density, and on the initial state. The effective interaction is here the reaction matrix corresponding to a very low concentration of ^{3}He, i.e., to a small Fermi momentum, and concentration effects or rearrangement terms are small and negligible because they vanish in the low-density limit.

The properties of the system thus can be calculated in a two-body approximation, i.e., by means of Brueckner theory, where the reaction matrix, or G matrix, is obtained by solution of a two-body problem in the system. The physical basis of the Brueckner theory is that when two particles interact they possibly interact strongly, but at the same time the liquid is sufficiently dilute that their interaction with other particles may be considered in an average way. This is certainly the case for low number density and small Fermi momenta. The contribution from three-body correlations can be neglected. It is known that the importance of three-body collisions is not large for nuclear matter.[22] Although liquid helium is a denser system, and the three-body contribution to the binding energy is important[23] in pure liquid ^{3}He, the two-body approximation should be much better here for the mixture, because it is the difference between the scattering of two ^{3}He atoms and the scattering of a ^{3}He and a ^{4}He atom in the medium which is required. The difference from replacing a ^{4}He atom by a ^{3}He atom in an already rare three-body collision then should be quite small.

II. EFFECTIVE INTERACTION AND REACTION MATRIX

We want to calculate the properties of the ^{3}He-^{4}He solutions which are due solely to the presence of the ^{3}He atoms, i.e., which are absent in pure ^{4}He. So we introduce the reaction matrix or G matrix, and the calculations are performed as reported and explained in earlier papers[24, 25] on pure liquid ^{3}He. The contribution from two-body correlations is calculated by means of Brueckner theory, i.e., with a modified Brueckner-Gammel method.[26, 27] The approximation of a reference energy spectrum with an effective mass and a quadratic momentum dependence is used for the input single-particle energy spectrum in the Bethe-Goldstone equation.[28] The intermediate-state potential energies off the energy shell are chosen to be equal to zero, and the outer self-consistency requirement in the Brueckner method is neglected. The Bethe-Goldstone equation is solved numerically as an integral equation by combined matrix inversion and iteration, to give the "true" or perturbed wave function. The matrix inversion is used for the short-range strongly repulsive part of the two-body potential in the numerical integration in coordinate space, and an iteration procedure is used for the outer attractive part of the potential. Afterwards, the G-matrix elements are calculated by numerical integration.

The Bethe-Goldstone equation is written

$$\Psi = \Phi - (Q/e)v\Psi \quad , \tag{2.1}$$

where Φ is the unperturbed free-particle two-body wave function, and Ψ is the corresponding perturbed one. Here v is the two-body potential, which in our case will be an effective potential as explained later. The Pauli exclusion operator Q prevents scattering into occupied intermediate states, and the energy operator e includes potential and kinetic single-particle energies. If k_0 is the initial-state and k is the intermediate-state relative momentum of the two interacting particles, the energy denominator e is defined by a reference energy spectrum

$$e(k) = (\hbar^2/M)(\gamma^2 + k^2) \quad , \tag{2.2}$$

where $\gamma^2 = 2\Delta k_F^2 - k_0^2/m^* \quad , \qquad (2.3)$

for propagation on the energy shell and $k_0 < k_F$. Here, m^* is the effective mass for the hole spectrum or particles in the Fermi sea, and Δ is a measure of the gap between the occupied and the intermediate-state energy spectra.

Equation (2.1) is now written in coordinate space, and, after an expansion and separation into partial waves, we get the Bethe-Goldstone integral equation in the form

$$u_L(k_0, r) = \mathcal{J}_L(k_0 r) + \int_0^\infty \Gamma_L(r, r')v(r')u_L(k_0, r')dr' \quad , \tag{2.4}$$

where we have introduced

$$\mathcal{J}_L(k_0 r) = k_0 r j_L(k_0 r) \quad ,$$
$$u_L(k_0, r) = k_0 r \Psi_L(k_0, r) \quad , \tag{2.5}$$

and the Green's functions $\Gamma_L(r, r')$ are defined by

$$\Gamma_L(r, r') = -(2/\pi)\int_0^\infty dk \times \mathcal{J}_L(kr)\mathcal{J}_L(kr')Q(P, k)/(\gamma^2 + k^2) \ . \tag{2.6}$$

The G_L-matrix elements are finally calculated for each partial wave as

$$\langle k_0 | G_L | k_0 \rangle = (4\pi/k_0^2)(2L+1) \times \int_0^\infty \mathcal{J}_L(k_0 r)v(r)u_L(k_0, r)dr \quad . \tag{2.7}$$

The potential to be used for the mixture is different from the two-body potential for free particles. Campbell[18] has derived an effective coordinate-space potential for the mutual interaction between two ^{3}He atoms by assuming that a ^{3}He atom occupies a volume which is $(1+\alpha)$ times greater than that occupied by a ^{4}He atom. The effective potential is a sum of three terms: The first is the bare ^{3}He-^{3}He potential; the second is the induced potential V_{34} arising from each ^{3}He atom interacting with the ^{4}He background through the bare ^{3}He-^{4}He potential; and the third is the induced potential V_{44} from the ^{4}He background interacting with itself through the bare ^{4}He-^{4}He potential. For small relative distances r, the strongly repulsive core of the bare ^{3}He-^{3}He potential dominates, while the three potential terms cancel each other to order α^2 for large r. The induced potential V_{34} is always repulsive, and the induced potential V_{44} is always attractive.

The total effective potential between two fixed ^{3}He atoms then is

$$v(r) = V(r) - 2(1+\alpha)g(r)V(r_{34}) + (1+\alpha)^2 g^2(r)V(r_{44}) \quad . \tag{2.8}$$

Here, r_{44} is the maximum of r and the ^{4}He-^{4}He average nearest-neighbor distance, which is[29, 30] 3.8 Å. The distance r_{34} is the maximum of r and the ^{3}He-^{4}He average nearest-neighbor distance which, assuming the same local liquid structure, is $\frac{1}{2}[1+(1+\alpha)^{1/3}]$ 3.8 Å ≈ 4.0 Å. The function $g(r)$ is defined by

$$g(r) = \tfrac{3}{4}y - \tfrac{1}{16}y^3, \qquad \text{for } y < 2$$
$$= 1, \qquad \text{for } y > 2 \quad , \tag{2.9}$$

$$\text{where} \quad y = (\tfrac{4}{3}\pi/\omega_3)^{1/3} r \approx r/2.5 \quad . \tag{2.10}$$

The ^{3}He atoms occupy a volume per atom of

$$\omega_3 = (1+\alpha)\omega_4 \quad , \tag{2.11}$$

and we have chosen the value

$$\alpha = 0.30 \tag{2.12}$$

for the molar volume ratio or relative volume increase. Values for α are given as 0.28 by Baym[8]; 0.284 by Ifft *et al.*[31] and by Edwards *et al.*[32]; 0.286 to 0.311 by Davison and Feenberg[21]; 0.303 by Massey and Woo[19]; and 0.308 by Boghosian and Meyer.[33] Bardeen *et al.*[9] obtain 0.31 from a hard-sphere model. So we take the value (2.12) as an approximate "average." The form (2.8) of the effective potential can be expected to persist as long as the superfluid ^{4}He background can deform itself quickly to accommodate the more slowly moving ^{3}He atoms.

The calculations are done for two different potentials since the chosen two-body potential probably would have some influence on the results. The YS potential is the Yntema-Schneider potential defined by Brueckner and Gammel[27] as

$$V_{\text{YS}}(r) = 7250[1200\exp(-4.82r) - 1.24/r^6 - 1.89/r^8] \quad , \tag{2.13}$$

and the FM potential is the Frost-Musulin potential defined by Bruch and McGee[34] as

$$V_{\text{FM}}(r) = -12.54[1 + 8.01(1 - 2.98/r)] \times \exp[8.01(1 - r/2.98)], \quad \text{for } r < 3.5\,\text{Å} \tag{2.14}$$

$$= -7250[1.41/r^6 + 3.82/r^8] \quad , \qquad \text{for } r > 3.5\,\text{Å} \ ,$$

in (°K), with r measured in Å. The calculations are done for four different values of the Fermi momentum k_F, i.e.,

$$k_F = 0.14\,\text{Å}^{-1} \ , \quad k_F = 0.20\,\text{Å}^{-1} \ , \quad k_F = 0.26\,\text{Å}^{-1} \ , \quad k_F = 0.32\,\text{Å}^{-1} \ , \tag{2.15}$$

with corresponding values for number density and ^{3}He concentration. It corresponds to 0.4% ^{3}He, 1.3% ^{3}He, 2.8% ^{3}He, and 5.0% ^{3}He, respectively. For 1.3% and 5.0% ^{3}He, we have experimental results to compare with later.

For the input effective mass of the energy spectrum for particles in the Fermi sea, we choose the value

$$m^* = M^*/M_3 = 2.5 \quad , \tag{2.16}$$

where M_3 is the mass of a bare ^{3}He atom. This is just taken to be an average value. It should vary slightly with the concentration, and more exact values are given by Anderson *et al.*[3] as $m^* = 2.34$ for zero concentration ^{3}He, $m^* = 2.38$ for 1.3% ^{3}He corresponding to $k_F = 0.20\,\text{Å}^{-1}$, and $m^* = 2.46$ for 5.0% ^{3}He corresponding to $k_F = 0.32\,\text{Å}^{-1}$. However, other theoretical and experimental values given during the years are $m^* = 1.51$ to 1.86 by Burke *et al.*[35]; 1.79 to 2.11 by Davison and Feenberg[21]; 1.8 by Parry and Ter Haar[36], 1.85 by Woo *et al.*[20]; 1.9 by Feynman[37]; 1.93 by Luban[38]; 2.0 by King and Fairbank[39] and by Dash and Taylor[40]; 2.33 by Opfer *et al.*[41]; 2.33 to 2.47 by Emery[11]; 2.34 by Baym and Saam[14]; 2.34 to 2.47 by Bardeen *et al.*[9]; 2.35 by Grigorev *et al.*[42]; 2.37 to 2.44 by Campbell[18]; 2.41 by Anderson *et al.*[2]; 2.5 by Husa *et al.*,[43] by Sandiford and Fairbank,[44] and by Ifft *et al.*[31]; 2.7 by Pellam[45]; 2.8 by Ptukha[46]; and $m^* = 3.0$ by Berezniak and Eselson.[47] Because of the small variation of m^* with concentration, we use the value (2.16) as input for all Fermi momenta. For the other parameter Δ in the input energy denominator, we use the value $\Delta = 0.21$, which is the smallest Δ we can choose without having trouble with the energy denominator in our computational method, i.e., with $\gamma^2 > 0$ according to Eq. (2.3).

We define the net effective interaction or reaction matrix as the one we get between ^{3}He quasiparticles, i.e., with $\alpha = 0.30$, minus the one we get with ^{4}He atoms instead of the ^{3}He atoms, i.e., with $\alpha = 0$, in the calculations. Results are shown in Tables I and II. The G-matrix elements are given in dimension (Å), which can be converted to (°K Å^3), the conversion factor being

$$\hbar^2/M = 16.36\ \text{°K Å}^2 \quad . \tag{2.17}$$

TABLE I. Diagonal G_L-matrix elements in Å. Weight of $2L+1$ included. Yntema-Schneider potential O k_0 is varied. $m^* = 2.5$.

k_F(Å^{-1})	k_0/k_F	G_0	G_1	G_2	G_3	G_4	$G_e + 3G_o$	$G_e + G_o$	$G_o - G_e$
0.20	0.001	−5.23	0	0	0	0	−5.23	−5.23	5.23
	0.125	−5.20	−0.03	0	0	0	−5.29	−5.23	5.16
	0.25	−5.10	−0.13	0	0	0	−5.49	−5.22	4.96
	0.375	−4.93	−0.29	−0.01	0	0	−5.81	−5.23	4.64
	0.50	−4.72	−0.51	−0.01	0	0	−6.23	−5.23	4.23
	0.625	−4.45	−0.75	−0.03	0	0	−6.73	−5.22	3.73
	0.75	−4.14	−1.04	−0.06	0	0	−7.30	−5.23	3.16
	0.875	−3.80	−1.33	−0.10	−0.01	0	−7.90	−5.23	2.56
	1.00	−3.43	−1.63	−0.15	−0.01	0	−8.50	−5.23	1.95
0.32	0.001	−6.15	0	0	0	0	−6.15	−6.15	6.15
	0.125	−6.06	−0.08	0	0	0	−6.32	−6.14	5.97
	0.25	−5.82	−0.33	−0.01	0	0	−6.81	−6.14	5.49
	0.375	−5.42	−0.70	−0.03	0	0	−7.54	−6.15	4.76
	0.50	−4.93	−1.14	−0.07	0	0	−8.43	−6.14	3.86
	0.625	−4.36	−1.62	−0.15	−0.01	0	−9.40	−6.14	2.88
	0.75	−3.75	−2.08	−0.27	−0.02	−0.01	−10.37	−6.15	1.93
	0.875	−3.15	−2.50	−0.43	−0.06	−0.02	−11.26	−6.15	1.04
	1.00	−2.55	−2.86	−0.62	−0.11	−0.04	−12.07	−6.17	0.26

TABLE II. Diagonal G_L-matrix elements in Å. Weight of $2L+1$ included. Frost-Musulin potential. k_0 is varied. $m^*=2.5$.

k_F(Å^{-1})	k_0/k_F	G_0	G_1	G_2	G_3	G_4	G_e+3G_o	G_e+G_o	G_o-G_e
	0.001	−6.69	0	0	0	0	−6.69	−6.69	6.69
	0.125	−6.64	−0.05	0	0	0	−6.77	−6.69	6.60
	0.25	−6.51	−0.16	0	0	0	−7.02	−6.69	6.35
	0.375	−6.31	−0.36	−0.01	0	0	−7.42	−6.69	5.96
0.20	0.50	−6.05	−0.63	−0.01	0	0	−7.95	−6.69	5.43
	0.625	−5.70	−0.95	−0.03	0	0	−8.58	−6.69	4.80
	0.75	−5.32	−1.30	−0.07	0	0	−9.29	−6.69	4.08
	0.875	−4.89	−1.68	−0.12	−0.01	0	−10.06	−6.69	3.32
	1.00	−4.43	−2.07	−0.18	−0.01	0	−10.84	−6.69	2.53
	0.001	−7.83	0	0	0	0	−7.83	−7.83	7.83
	0.125	−7.72	−0.11	0	0	0	−8.03	−7.82	7.61
	0.25	−7.41	−0.41	−0.01	0	0	−8.64	−7.82	7.01
	0.375	−6.92	−0.87	−0.03	0	0	−9.57	−7.83	6.08
0.32	0.50	−6.30	−1.44	−0.08	0	0	−10.70	−7.82	4.95
	0.625	−5.59	−2.04	−0.18	−0.02	0	−11.93	−7.83	3.72
	0.75	−4.82	−2.64	−0.33	−0.03	−0.01	−13.18	−7.83	2.49
	0.875	−4.05	−3.18	−0.53	−0.06	−0.03	−14.35	−7.85	1.36
	1.00	−3.28	−3.65	−0.77	−0.12	−0.05	−15.40	−7.86	0.34

III. LANDAU PARAMETERS

Calculations of quasiparticle properties, such as the compressibility, the specific heat and the magnetic susceptibility, are probably best carried out within the Landau theory[48–50] for Fermi liquids. The basic function $f(\vec{k},\vec{k}';\vec{\sigma},\vec{\sigma}')$ is the effective interaction between a pair of quasiparticles, and is defined as the second variational derivative of the total energy with respect to the particle occupation number, i.e.,

$$f(\vec{k},\vec{k}';\vec{\sigma},\vec{\sigma}')=\delta^2E/\delta n(\vec{k},\vec{\sigma})\,\delta n(\vec{k}',\vec{\sigma}') \quad . \tag{3.1}$$

This function is needed for values of its arguments near the Fermi surface, so we write the expansion

$$f(\theta;\vec{\sigma},\vec{\sigma}')\Big|_{k=k'=k_F}=(\pi^2\hbar^2/k_FM^*)\times\sum_L(F_L+\vec{\sigma}\cdot\vec{\sigma}'Z_L)\mathcal{P}_L(\cos\theta) \quad . \tag{3.2}$$

The function $\mathcal{P}_L(\cos\theta)$ is the Legendre polynomial of order L, and θ is the angle between $\vec{k}$ and $\vec{k}'$. The scattering amplitude $a(\theta;\vec{\sigma},\vec{\sigma}')$ is given by a relation analogous to Eq. (3.2), i.e.,

$$a(\theta,\vec{\sigma},\vec{\sigma}')=(\pi^2\hbar^2/k_FM^*)\times\sum_L(B_L+\vec{\sigma}\cdot\vec{\sigma}'C_L)\mathcal{P}_L(\cos\theta) \quad , \tag{3.3}$$

where B_L and C_L are related to F_L and Z_L in a simple way by the connection

$$B_L=F_L/[1+F_L/(2L+1)] \quad , \qquad C_L=Z_L/[1+Z_L/(2L+1)] \quad . \tag{3.4}$$

For a neutral Fermi liquid, we have an exact relationship between the Legendre polynomial expansion coefficients of the Landau f function in the form of a sum rule. It depends on the fact that the forward scattering amplitude for two fermions of the same spin vanishes. We can write

$$\sum_L(B_L+C_L)=\sum_L F_L/[1+F_L/(2L+1)]+\sum_L Z_L/[1+Z_L/(2L+1)]=0 \quad , \tag{3.5}$$

which is the exclusion-principle sum rule for the system.

To make a connection between Brueckner theory and the fundamental equation of the Landau theory, we start from the assumption that the f function can be related to the G matrix through the relation (3.1) as outlined and explained earlier[51] for pure liquid ^{3}He.

The G matrix can be written

$$G=G_D-\tfrac{1}{2}(1+\vec{\sigma}\cdot\vec{\sigma}')G_E \quad . \tag{3.6}$$

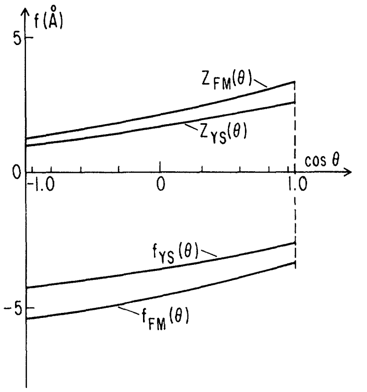

FIG. 1. Functions $f(\theta)$ and $z(\theta)$ in Å. YS is Yntema-Schneider potential, and FM is Frost-Musulin potential. $k_F = 0.20$ Å^{-1}.

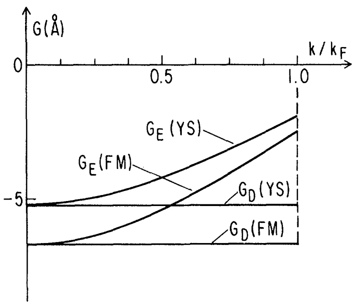

FIG. 3. Functions $G_D(k)$ and $G_E(k)$ in Å. YS is Yntema-Schneider potential and FM is Frost-Musulin potential. $k_F = 0.20$ Å^{-1}.

Here G_D is the "direct" part, and G_E is the "exchange" part which includes only exchange of momenta. The spin exchange is obtained by multiplying G_E by the spin-exchange operator $\frac{1}{2}(1 + \vec{\sigma} \cdot \vec{\sigma}\,')$. Then

$$G_D = G_e + G_o\ , \qquad G_E = G_e - G_o\ , \tag{3.7}$$

$$\text{where}\quad G_e = \sum_{\text{even } L} \langle k_0 | G_L | k_0 \rangle\ ,$$
$$G_o = \sum_{\text{odd } L} \langle k_0 | G_L | k_0 \rangle\ , \tag{3.8}$$

and the G_L-matrix elements are calculated from Eq. (2.7). For a dilute mixture, the rearrangement terms are very small because of the low concentration and low density of ^{3}He, and we get to a first approximation, as functions of the angle θ or the relative momentum k_0:

$$f(k_0) = \tfrac{1}{2}[G_e(k_0) + 3G_o(k_0)]\ ,$$
$$z(k_0) = \tfrac{1}{2}[G_o(k_0) - G_e(k_0)]\ . \tag{3.9}$$

The definition of the function $z(\vec{k}, \vec{k}\,')$ and the coefficients Z_L then differ by a factor of 4 from what is often used by others.

The functions $f(\theta)$ and $z(\theta)$ are now obtained from $G_L(\theta)$, and resulting functions are given in Figs. 1 and 2. Figures 3 and 4 show the functions (3.7), $G_D(k_0)$ and $G_E(k_0)$, which can be compared with similar effective potentials given by Bardeen *et al.*,[7,9] Ebner,[10] Campbell,[18] and Baym and Ebner.[13] Here G_D corresponds to V_0 and G_E corresponds to V_k in the notation of Bardeen *et al.*

Assuming that the total effective interaction can be expanded in a series of Legendre polynomials, we write

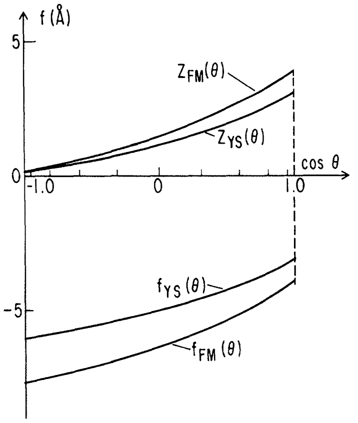

FIG. 2. Functions $f(\theta)$ and $z(\theta)$ in Å. YS is Yntema-Schneider potential and FM is Frost-Musulin potential. $k_F = 0.32$ Å^{-1}.

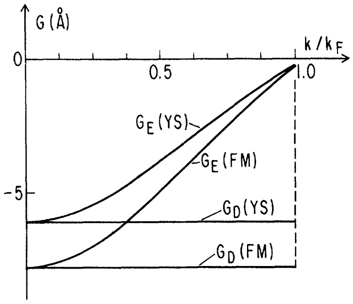

FIG. 4. Functions $G_D(k)$ and $G_E(k)$ in Å. YS is Yntema-Schneider potential and FM is Frost-Musulin potential. $k_F = 0.32$ Å^{-1}.

TABLE III. Landau parameters or expansion coefficients for liquid ^{3}He-^{4}He mixture.

Coef. k_F(Å^{-1})	YS potential 0.14	0.20	0.26	0.32	FM potential 0.14	0.20	0.26	0.32
F_0	−0.065	−0.113	−0.176	−0.251	−0.083	−0.144	−0.224	−0.320
F_1	0.010	0.026	0.049	0.073	0.013	0.033	0.062	0.094
F_2	0	0.002	0.007	0.015	0	0.003	0.008	0.018
F_3	0	0	0.001	0.004	0	0.001	0.002	0.004
Z_0	0.043	0.055	0.063	0.066	0.056	0.071	0.081	0.085
Z_1	0.011	0.026	0.048	0.073	0.013	0.033	0.061	0.093
Z_2	0.001	0.003	0.008	0.016	0.001	0.003	0.009	0.020
Z_3	0	0	0.002	0.003	0	0.001	0.002	0.004
B_0	−0.069	−0.128	−0.214	−0.336	−0.091	−0.169	−0.288	−0.470
B_1	0.010	0.026	0.048	0.072	0.013	0.033	0.061	0.091
B_2	0	0.002	0.007	0.015	0	0.003	0.008	0.018
B_3	0	0	0.001	0.004	0	0.001	0.002	0.004
C_0	0.041	0.052	0.059	0.062	0.053	0.067	0.075	0.078
C_1	0.010	0.026	0.047	0.071	0.013	0.033	0.060	0.090
C_2	0.001	0.003	0.008	0.016	0.001	0.003	0.009	0.020
C_3	0	0	0.002	0.003	0	0.001	0.002	0.004

$$F_L = \tfrac{1}{2}(2L+1)(k_F M^*/\pi^2\hbar^2) \times \int_0^\pi f(\theta)\, \mathcal{P}_L(\cos\theta) \sin\theta\, d\theta \quad ,$$
$$Z_L = \tfrac{1}{2}(2L+1)(k_F M^*/\pi^2\hbar^2) \times \int_0^\pi z(\theta)\, \mathcal{P}_L(\cos\theta) \sin\theta\, d\theta \quad , \qquad (3.10)$$

because

$$\int_{-1}^{-1} \mathcal{P}_L(x)\, \mathcal{P}_{L'}(x) dx = 2\delta_{LL'}/(2L+1) \quad . \qquad (3.11)$$

Afterwards, the coefficients B_L and C_L are calculated according to Eq. (3.4). Results for $L < 4$ are given in Table III. A comparison with other results is given in Table IV. The experimental values are taken from Anderson *et al.*[3] We see that F_0, or the angular average of the spin-averaged effective quasiparticle interaction, is negative, indicating an attractive interaction. F_1 indicates that the effective mass changes very little with the ^{3}He concentration, and the value for Z_0 shows that the magnetic susceptibility is close to the ideal Fermi-Dirac value. It measures the exchange correction to the magnetic susceptibility, and it is small in dilute solutions of ^{3}He in ^{4}He. The agreement with the other theoretical and ex-

TABLE IV. Landau parameters or expansion coefficients for liquid ^{3}He-^{4}He mixture. Comparison with theoretical and experimental results. Experimental values are taken from Anderson *et al.* (Ref. 3).

k_F(Å^{-1})		F_0	F_1	Z_0	B_0	B_1	C_0
0.20	YS potential	−0.11	0.03	0.06	−0.13	0.03	0.05
	FM potential	−0.14	0.03	0.07	−0.17	0.03	0.07
	Bardeen *et al.*[a]	−0.20	0.05	0.09	−0.25	0.05	0.08
	Emery[b]	−0.14	0.05		−0.16	0.05	
	Experimental value		0.05	0.09		0.05	0.08
0.32	YS potential	−0.25	0.07	0.07	−0.34	0.07	0.06
	FM potential	−0.32	0.09	0.09	−0.47	0.09	0.08
	Bardeen *et al.*[a]	−0.42	0.17	0.05	−0.73	0.16	0.05
	Emery[b]	−0.26	0.18		−0.35	0.17	
	Experimental value		0.16	0.08		0.15	0.08

[a] References 7 and 9.

[b] Reference 11.

perimental values is, otherwise, reasonably good.

As a test of the sum rule (3.5) we get from our calculations

$$\sum_{L<4}(B_L+C_L)=-0.007, \quad \text{for } k_F=0.14\,\text{Å}^{-1}$$
$$=-0.019, \quad \text{for } k_F=0.20\,\text{Å}^{-1}$$
$$=-0.042, \quad \text{for } k_F=0.26\,\text{Å}^{-1}$$
$$=-0.093, \quad \text{for } k_F=0.32\,\text{Å}^{-1} \qquad (3.12)$$

for the Yntema-Schneider potential, and

$$\sum_{L<4}(B_L+C_L)=-0.011, \quad \text{for } k_F=0.14\,\text{Å}^{-1}$$
$$=-0.028, \quad \text{for } k_F=0.20\,\text{Å}^{-1}$$
$$=-0.071, \quad \text{for } k_F=0.26\,\text{Å}^{-1}$$
$$=-0.165, \quad \text{for } k_F=0.32\,\text{Å}^{-1} \qquad (3.13)$$

for the Frost-Musulin potential. This indicates that the exclusion-principle sum rule is satisfied to a fairly good approximation, at least for the lowest densities. It has been assumed, or rather hoped, that the Landau coefficients would be small for $L>1$. This assumption would correspond to an effective interaction function of approximately the form

$$f(\theta)\approx A+B\cos\theta \ , \qquad (3.14)$$

and a similar angular dependence for $z(\theta)$. This seems to be a good approximation for our mixture problem, as has already been assumed by Bardeen *et al.*[7, 9]

IV. COMPRESSIBILITY, EFFECTIVE MASS, AND MAGNETIC SUSCEPTIBILITY

The compressibility, the effective mass, and the magnetic susceptibility can be obtained from the Landau coefficients F_0, F_1, and Z_0.

The isothermal compressibility of the ^{3}He system, defined from the change in energy with concentration in the solution, is given by

$$\beta=(\rho M v_s^2)^{-1} \ , \qquad (4.1)$$

or

$$\beta_F/\beta=(1+F_0)/(1+\tfrac{1}{3}F_1) \ , \qquad (4.2)$$

where β_F is the compressibility for the ideal gas. The specific-heat or effective-mass ratio is given by

$$M^*/M^*_0=1+\tfrac{1}{3}F_1 \ , \qquad (4.3)$$

where M^*_0 is the effective mass at zero concentration ^{3}He. The specific heat of the mixture is very close to that of an ideal Fermi-Dirac gas, and the phonon and roton contributions to the specific heat are very small. The energy of a single ^{3}He atom in the ^{4}He is, to a very good approximation, just a kinetic energy with effective mass M^*_0.

The magnetic susceptibility is related to that of an ideal gas χ_F, by

$$\chi_F/\chi=(1+Z_0)/(1+\tfrac{1}{3}F_1) \ . \qquad (4.4)$$

The magnetic susceptibility is very close to the ideal Fermi-Dirac value, and the temperature dependence of the susceptibility per atom is nearly independent of the concentration. The departure of the magnetic susceptibility from the ideal Fermi gas behavior is mainly because of the effective ^{3}He-^{3}He interaction, since ^{4}He has no spin and interacts with spin-up and spin-down ^{3}He atoms in the same way.

Using the results obtained in Sec. IV for the Landau coefficients, which are given in Table III, we get the values given in Table V. These results can be compared with other theoretical and experimental results, as we have done in Table VI. The experimental values are taken from Anderson *et al.*[3] We see that the results are close to those for an ideal Fermi-Dirac gas, and are in fair agreement with other theoretical and experimental values. The various properties are shown as functions of the ^{3}He concentration in Figs. 5 and 6. The effective mass is increasing with the density, while the compressibility and magnetic sus-

TABLE V. Compressibility, effective mass, and magnetic susceptibility for liquid ^{3}He-^{4}He mixture.

Property k_F(Å^{-1})	YS potential 0.14	0.20	0.26	0.32	FM potential 0.14	0.20	0.26	0.32
β_F/β	0.932	0.879	0.811	0.731	0.913	0.847	0.760	0.659
M^*/M^*_0	1.003	1.009	1.016	1.024	1.004	1.011	1.021	1.031
χ_F/χ	1.040	1.046	1.046	1.041	1.052	1.059	1.059	1.052

TABLE VI. Compressibility, effective mass, and magnetic susceptibility for liquid ^{3}He-^{4}He mixture. Comparison with theoretical and experimental results. Experimental values are taken from Anderson *et al.* (Ref. 3).

k_F(Å^{-1})		β_F/β	M^*/M_0^*	χ_F/χ
0.20	YS potential	0.88	1.01	1.05
	FM potential	0.85	1.01	1.06
	Bardeen *et al.*[a]	0.80	1.02	1.09
	Emery[b]	0.86	1.02	1.09
	Campbell[c]		1.01	1.11
	Experimental value		1.02	1.09
0.32	YS potential	0.73	1.02	1.04
	FM potential	0.66	1.03	1.06
	Bardeen *et al.*[a]	0.58	1.06	1.05
	Emery[b]	0.74	1.06	1.05
	Campbell[c]		1.03	1.07
	Experimental value		1.05	1.08

[a] References 7 and 9.

[b] Reference 11.

[c] Reference 18.

ceptibility are decreasing with density. And in pure liquid ^{3}He, F_0 and Z_0 would have opposite signs, positive instead of negative for F_0 and negative instead of positive for Z_0.

We can also estimate the maximum solubility of ^{3}He in ^{4}He. According to thermodynamics, for equilibrium between two phases in the mixture at zero temperature, we have

$$\mu_3(x) = B_3 \quad , \tag{4.5}$$

where $\mu_3(x)$ is the partial chemical potential for ^{3}He at the concentration x for maximum solubility, and B_3 is the chemical potential or binding energy

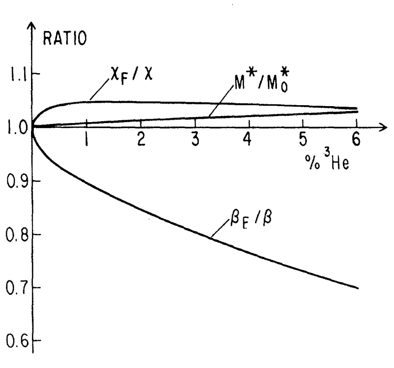

FIG. 5. Properties of ^{3}He-^{4}He solutions as function of concentration ^{3}He. Yntema-Schneider potential.

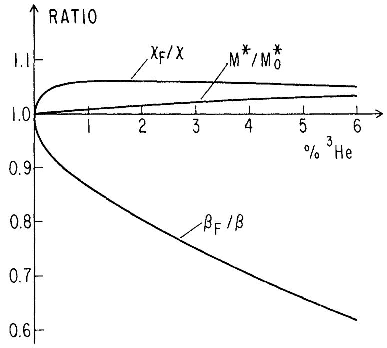

FIG. 6. Properties of ^{3}He-^{4}He solutions as function of concentration ^{3}He. Frost-Musulin potential.

for pure liquid ^{3}He. This means that ^{3}He can dissolve in liquid ^{4}He until Eq. (4.5) is satisfied. For a small concentration of ^{3}He, the dependence of the ^{3}He chemical potential on concentration can be written

$$\mu_3(x) = \mu_3(0) + T_F(x) + \rho_3 V_0 - \mu_{\text{ex}} = B_3 \quad , \tag{4.6}$$

where $\quad T_F = \frac{1}{2}(\hbar^2/M)(k_F^2/m^*) \qquad (4.7)$

is the Fermi energy for an ideal gas of fermions of effective mass m^*,

$$\rho_3 = \tfrac{1}{3} k_F^3/\pi^2 \tag{4.8}$$

is the number density of ^{3}He atoms, and

$$\mu_{\text{ex}} = (2\pi)^{-3} \int_{|k+k_F|<k_F} V_k d\vec{k} = (4\pi^2)^{-1} \times \int_0^{2k_F} k^2[1 - \tfrac{1}{2}(k/k_F)] V_k d\vec{k} \tag{4.9}$$

is an exchange contribution to the chemical potential.

For the difference between the binding energy for zero concentration and the binding energy for pure liquid ^{3}He, we take the value

$$B_3 - \mu_3(0) = 0.28\ {}^\circ\text{K} \quad . \tag{4.10}$$

This difference has been given as 0.24 °K by Wheatley,[52] 0.276 °K by Ebner,[10] 0.284 °K by Edwards *et al.*,[32] 0.287 °K by Ifft *et al.*,[31] 0.295 °K by Bardeen *et al.*,[9] and 0.31 °K by Emery.[11] As mentioned in Sec. III, the functions G_D and G_E given in Eq. (3.7) and Figs. 3 and 4 correspond to V_0 and V_k in the notation of Bardeen *et al.* Extrapolation finally gives that Eq. (4.6) is satisfied for

$$T_F = 0.37\ {}^\circ\text{K} \quad , \quad x = 6.0\% \quad , \tag{4.11}$$

for the Yntema-Schneider potential, and

$$T_F = 0.39\,^\circ\mathrm{K} \quad , \qquad x = 6.5\% \quad , \tag{4.12}$$

for the Frost-Musulin potential.

These results can be compared with other results for the limiting solubility at zero temperature, which are 6% ^{3}He by Bardeen *et al.*,[9] by Ebner,[10] and by Campbell[18]; 6.3% by Vilches and Wheatley[53]; 6.35% by Wheatley[52]; 6.37% by Ifft *et al.*[31]; 6.4% by Edwards *et al.*[32]; and 6.84% ^{3}He by Schermer *et al.*[54]

V. TRANSPORT COEFFICIENTS

The transport coefficients of viscosity, thermal conductivity, and spin diffusion depend directly on the effective interaction in the mixture, and also on concentration and temperature.

The transport coefficients can be estimated or calculated from formulas developed by Abrikosov and Khalatnikov,[55] and by Hone,[56] as we have done[57] for pure liquid ^{3}He. The scattering amplitude a for particles in the liquid is a sum of singlet and triplet amplitudes multiplied with the appropriate spin projection operators, i.e.,

$$\begin{aligned} a &= \tfrac{1}{4}(1 - \vec{\sigma}\cdot\vec{\sigma}\,')a_S + \tfrac{1}{4}(3 + \vec{\sigma}\cdot\vec{\sigma}\,')a_T \\ &= \tfrac{1}{2}(a_e + 3a_o) - \tfrac{1}{2}\vec{\sigma}\cdot\vec{\sigma}\,'(a_e - a_o) \quad . \end{aligned} \tag{5.1}$$

We identify our G matrix with this scattering amplitude. It should be a good approximation, since Table III shows that the coefficients B_L and C_L are quite close to the coefficients F_L and Z_L. And we are then able to include the dependence on the angle φ between the relative momenta before and after a scattering process, while the Landau theory provides no information about nonforward scattering.

The scattering which we consider for the transport coefficients involves particles on the Fermi surface, so the magnitudes of the initial and final relative momenta are equal. The only variable in addition to the angle φ between the scattering planes is the angle θ, defining the total and relative momenta as

$$P = k_F \cos(\tfrac{1}{2}\theta) \quad , \qquad k_0 = k_F \sin(\tfrac{1}{2}\theta) \quad . \tag{5.2}$$

After an expansion of the G matrix, we get

$$a_e = \sum_{\text{even } L} G_L(P, k_0)\,\mathcal{P}_L(\cos\varphi) = \tfrac{1}{2}a_S \quad , \tag{5.3}$$

$$a_o = \sum_{\text{odd } L} G_L(P, k_0)\,\mathcal{P}_L(\cos\varphi) = \tfrac{1}{2}a_T \quad ,$$

where $G_L(P, k_0)$ is given by Eq. (2.7). The scattering amplitude for parallel spins is $2a_o$, and the scattering amplitude for antiparallel spins is $a_e + a_o$.

The transition probability $\omega(\theta, \varphi)$ to be used in the formulas for the viscosity and the thermal conductivity is

$$\begin{aligned} \omega_\eta(\theta, \varphi) &= \omega_\kappa(\theta, \varphi) = (2\pi/\hbar) \\ &\quad \times [\tfrac{1}{2}|a_e(\theta, \varphi)|^2 + \tfrac{3}{2}|a_o(\theta, \varphi)|^2] \quad , \end{aligned} \tag{5.4}$$

which gives the scattering rate for indistinguishable particles with singlet and triplet spin weights. The transition probability for calculating the spin-diffusion coefficient is proportional to the square of the scattering amplitude for particles with opposite spins, i.e.,

$$\omega_D(\theta, \varphi) = (2\pi/\hbar)\,|a_e(\theta, \varphi) + a_o(\theta, \varphi)|^2 \quad , \tag{5.5}$$

which gives the scattering rate for distinguishable particles.

The viscosity η is given by

$$\eta T^2 = \tfrac{64}{45}(\hbar^2/M)^2(k_F^5/m^{*4})(\hbar/2\pi A_\eta) \quad , \tag{5.6}$$

$$\begin{aligned} \text{where}\quad A_\eta &= \{\tfrac{1}{2}[\,|a_e(\theta, \varphi)|^2 + 3\,|a_o(\theta, \varphi)|^2] \\ &\quad \times (1 - \cos\theta)^2 \sin^2\varphi / \cos(\tfrac{1}{2}\theta)\}_{\text{av}} \,. \end{aligned} \tag{5.7}$$

The thermal conductivity κ is given by

$$\kappa T = \tfrac{8}{3}\pi^2(\hbar^2/M)^2(k_F^3/m^{*4})(K^2/2\pi\hbar A_\kappa) \quad , \tag{5.8}$$

$$\begin{aligned} \text{where}\quad A_\kappa &= \{\tfrac{1}{2}[\,|a_e(\theta, \varphi)|^2 + 3\,|a_o(\theta, \varphi)|^2] \\ &\quad \times (1 - \cos\theta)/\cos(\tfrac{1}{2}\theta)\}_{\text{av}} \quad , \end{aligned} \tag{5.9}$$

and K is Boltzmann's constant.

The spin diffusion D is given by

$$DT^2 = \tfrac{32}{3}\pi^2\left(\frac{\hbar^2}{M}\right)^2 \frac{k_F^2}{m^{*5}}(1 + Z_0)\,\frac{K}{2\pi A_D} \quad , \tag{5.10}$$

$$\begin{aligned} \text{where}\quad A_D &= [\,|a_e(\theta, \varphi) + a_o(\theta, \varphi)|^2 \\ &\quad \times (1 - \cos\theta)(1 - \cos\varphi)/\cos(\tfrac{1}{2}\theta)]_{\text{av}} \quad . \end{aligned} \tag{5.11}$$

The expressions for A, or the averages over the angles θ and φ, are calculated numerically by taking the integral

$$(4\pi)^{-1}\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi A(\theta, \varphi) = A(\theta, \varphi)_{\text{av}} \tag{5.12}$$

over the angles θ and φ, where the dependence of the scattering amplitude on the angle θ is given through Eq. (5.2).

The formulas (5.6), (5.8) and (5.10) are probably not very accurate. The determination of the transport coefficients in terms of an effective interaction, is based, in the theoretical expressions given by Abrikosov and Khalatnikov[55] for the viscosity and the thermal conductivity, and by Hone[56] for the spin diffusion. These expressions are derived by solving the linearized Boltzmann transport equation exactly, under the assumption that the excitation energies of the quasiparticles relevant to the transport are small compared with the temperature. Thus, certain factors are approximated by unity in the collision integral. This approximation can be checked by using the resulting solutions for the nonequilibrium quasiparticle distribution functions as trial functions in a variational calculation. Aside from corrections involving Fermi-liquid effects, Baym and Ebner[13] obtain as lower limits on the transport coefficients

$$\eta = \tfrac{3}{4}\eta_{AK} \quad , \quad \kappa = \tfrac{5}{12}\kappa_{AK} \quad , \quad D = \tfrac{3}{4}D_H \quad , \qquad (5.13)$$

where η_{AK} is given by Eq. (5.6), κ_{AK} by Eq. (5.8), and D_H by Eq. (5.10). The variational solutions put lower limits on the exact transport coefficients for a given scattering amplitude, since they are based on the principle of maximizing the entropy production. The usual solutions, on the other hand, overestimate the transport coefficients, since they are derived by effectively neglecting some of the scattering. Brooker and Sykes[58] give both lower and upper limits as

$$\begin{aligned} &0.750\eta_{AK} < n < 0.925\eta_{AK} \ , \\ &0.417\kappa_{AK} < \kappa < 0.561\kappa_{AK} \ , \qquad (5.14) \\ &0.750D_H < D < 0.964D_H \ . \end{aligned}$$

Similar results are also obtained by Emery and Cheng,[15] and by Højgård Jensen *et al.*[59] Since the exact value is somewhat uncertain, we will simply use the values

$$\eta = \eta_{AK} \quad , \quad \kappa = \tfrac{1}{2}\kappa_{AK} \ , \quad D = D_H \quad , \qquad (5.15)$$

in our estimates.

Using the G_L-matrix elements in Tables I and II, we obtain the transport coefficients given in Table VII. A comparison with other theoretical and experimental results is given in Table VIII. The experimental values are taken from Abel *et al.*[60] for the thermal conductivity and from Anderson *et al.*[5] for the spin diffusion. We see that the agreement between our results and the experimental values is not too bad. There is a difference by a factor of approximately 1 to 2 for the thermal conductivity and 1 to 1.5 for the spin diffusion. This is quite good agreement, since the results depend very much on the chosen value for the effective mass. There is also fairly good agreement with other results. The transport coefficients are shown as functions of the ^{3}He concentration in Figs. 7–9. The values are increasing with the density, as one could expect. However, the spin diffusion finally has a lower value for pure liquid ^{3}He.

VI. REARRANGEMENT TERMS

In our calculations in Secs. III and IV, rearrangement terms are neglected. The second variational derivative of the total energy expressed by Brueckner theory, with respect to the particle occupation number, gives the reaction matrix plus rearrangement terms containing first and second derivatives of the reaction matrix with respect to the Fermi momentum. These derivatives are neglected in the definition (3.9) of the Landau functions. Rearrangement terms are important for pure liquid ^{3}He and change, for instance, the effective interaction completely from an average attractive G matrix to an average repulsive Landau f function. However, for the dilute-mixture problem, rearrangement terms are small, and vanish in the low-density limit.

If we include rearrangement terms, we get instead of Eq. (3.9),

$$f(k_0) = \tfrac{1}{2}[G_e(k_0) + 3G_o(k_0)]$$

$$+ \int_0^1 x^2 \Big((2 - x - x^3)k_F \frac{\partial G_e}{\partial k_F}$$

TABLE VII. Transport coefficients for liquid ^{3}He-^{4}He mixture.

Property $k_{\rm F}$(Å^{-1})	YS potential 0.14	0.20	0.26	0.32	FM potential 0.14	0.20	0.26	0.32
ηT^2(P°K^2)	5×10^{-9}	33×10^{-9}	129×10^{-9}	378×10^{-9}	3×10^{-9}	20×10^{-9}	78×10^{-9}	230×10^{-9}
κT(erg cm^{-1} sec^{-1})	3.3	9.3	19.1	32.5	2.0	5.7	11.7	19.9
DT^2(cm^2 sec^{-1} °K^2)	8.6×10^{-6}	22.4×10^{-6}	50.6×10^{-6}	101.9×10^{-6}	5.1×10^{-6}	13.5×10^{-6}	30.6×10^{-6}	61.7×10^{-6}

$$+\left(\tfrac{10}{3}+\tfrac{2}{3}x-\tfrac{13}{3}x^3\right)k_F\,\frac{\partial G_o}{\partial k_F}\Bigg)\,dx\ , \tag{6.1}$$

$$z(k_0)=\tfrac{1}{2}[G_o(k_0)-G_e(k_0)]$$

$$+\int_0^1 x^2\Bigg(\left(-\tfrac{2}{3}+x-\tfrac{1}{3}x^3\right)k_F\,\frac{\partial G_e}{\partial k_F}$$

$$+\left(\tfrac{10}{3}-x-\tfrac{7}{3}x^3\right)k_F\,\frac{\partial G_o}{\partial k_F}\Bigg)\,dx\ ,$$

TABLE VIII. Transport coefficients for liquid ^{3}He-^{4}He mixture. Comparison with theoretical and experimental results. Experimental values are taken from Abel *et al.* (Ref. 60) for the thermal conductivity and from Anderson *et al.* (Ref. 3) for the spin diffusion.

k_F(Å^{-1})		ηT^2(P °K^2)	κT(erg cm^{-1} sec^{-1})	DT^2(cm^2 sec^{-1} °K^2)
0.20	YS potential	33×10^{-9}	9.3	22.4×10^{-6}
	FM potential	20×10^{-9}	5.7	13.5×10^{-6}
	Campbell[a]	35×10^{-9}	11	16×10^{-6}
	Baym and Saam[b]	34×10^{-9}		
	Emery[c]	38×10^{-9}	19.1	
	Emery and Cheng[d]	38×10^{-9}	12.1	
	Baym and Ebner[e]		10	18.6×10^{-6}
	Experimental value		11	17.2×10^{-6}
0.32	YS potential	378×10^{-9}	32.5	101.9×10^{-6}
	FM potential	230×10^{-9}	19.9	61.7×10^{-6}
	Campbell[a]	420×10^{-9}	40	105×10^{-6}
	Baym and Saam[b]	430×10^{-9}		
	Emery[c]	460×10^{-9}	64	
	Emery and Cheng[d]	390×10^{-9}	33.3	
	Baym and Ebner[e]		26	82×10^{-6}
	Experimental value		24	90×10^{-6}

[a] Reference 18.

[b] References 12 and 14.

[c] Reference 11.

[d] References 15 and 16.

[e] Reference 13.

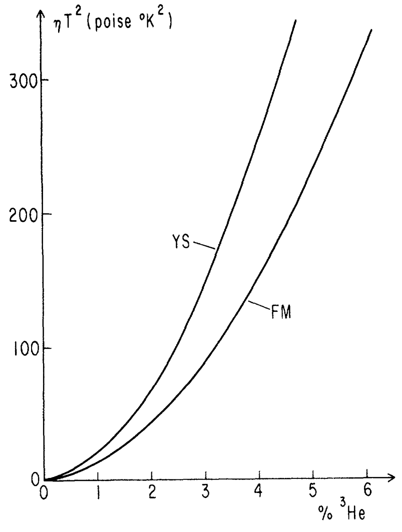

FIG. 7. Viscosity for ^{3}He-^{4}He solutions as function of concentration ^{3}He. YS is Yntema-Schneider potential and FM is Frost-Musulin potential. $m^*=2.5$.

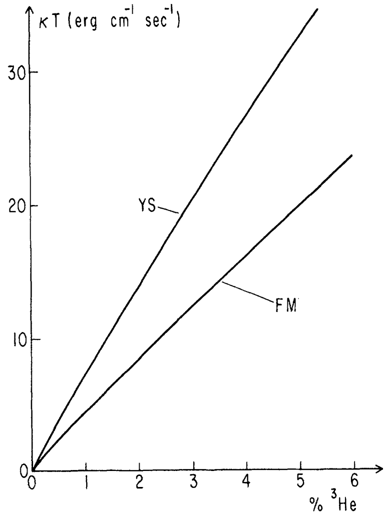

FIG. 8. Thermal conductivity for ^{3}He-^{4}He solutions as function of concentration ^{3}He. YS is Yntema-Schneider potential and FM is Frost-Musulin potential. $m^*=2.5$.

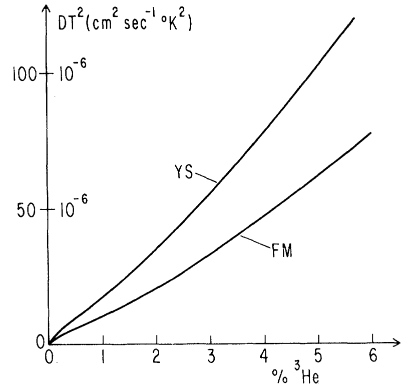

FIG. 9. Spin diffusion coefficient for ^{3}He-^{4}He solutions as function of concentration ^{3}He. YS is Yntema-Schneider potential and FM is Frost-Musulin potential. $m^* = 2.5$.

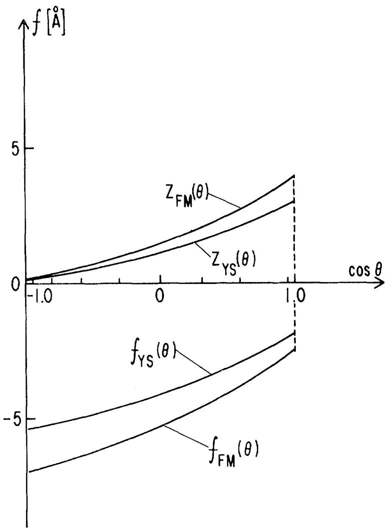

FIG. 10. Functions $f(\theta)$ and $z(\theta)$ in Å. YS is Yntema-Schneider potential and FM is Frost-Musulin potential. Rearrangement terms are included. $k_F = 0.32$ Å^{-1}.

where the second derivatives are neglected. The rearrangement terms are evaluated and calculated as explained earlier[51] for pure liquid ^{3}He. The functions $f(\theta)$ and $z(\theta)$ are obtained from $G(\theta)$, starting from the functions (3.9), and resulting functions are given in Fig. 10 for $k_F = 0.32$ Å^{-1}. The rearrangement terms vanish for lower densities.

The Landau coefficients F_L, Z_L, B_L, and C_L are calculated according to Eqs. (3.4) and (3.10), and results for $L < 4$ are given in Table IX. We see that F_0 and B_0 become less negative, indicating a less attractive interaction. We get larger values for F_1 and B_1, indicating a greater change in the effective mass with the ^{3}He concentration, although it is still very small. Inclusion of rearrangement terms gives, however, better agreement with experimental values. The value for Z_0 does not change, it is small, and indicates that the magnetic susceptibility is close to the ideal Fermi-Dirac value.

As a test of the sum rule (3.5), we get for $k_F = 0.32$ Å^{-1},

$$\sum_{L<4} (B_L + C_L) = 0.002 \qquad (6.2)$$

for the Yntema-Schneider potential, and

$$\sum_{L<4} (B_L + C_L) = -0.034 \qquad (6.3)$$

for the Frost-Musulin potential. Comparison with the results (3.12) and (3.13) shows that the exclusion-principle sum rule is even better satisfied with rearrangement terms included in the calculations.

The compressibility, the effective mass, and the magnetic susceptibility are obtained from the Landau coefficients F_0, F_1, and Z_0, i.e., from Eqs. (4.2), (4.3), and (4.4). Using the results given in Table IX, we get the values given in Table X. These results can be compared with the other the-

TABLE IX. Landau parameters or expansion coefficients for liquid ^{3}He-^{4}He mixture. Rearrangement terms included. $k_F = 0.32$ Å^{-1}. Experimental values are taken from Anderson *et al.* (Ref. 3).

Potential		F_0	F_1	Z_0	B_0	B_1	C_0
YS	Without rearrangement terms	−0.25	0.07	0.07	−0.34	0.07	0.06
	With rearrangement terms	−0.20	0.09	0.07	−0.26	0.09	0.06
FM	Without rearrangement terms	−0.32	0.09	0.09	−0.47	0.09	0.08
	With rearrangement terms	−0.26	0.11	0.09	−0.36	0.11	0.08
	Experimental value		0.16	0.08		0.15	0.08

TABLE X. Compressibility, effective mass, and magnetic susceptibility for liquid ^{3}He-^{4}He mixture. Rearrangement terms included. $k_F = 0.32\,\text{Å}^{-1}$. Experimental values are taken from Anderson *et al.* (Ref. 3).

Potential		β_F/β	M^*/M_0^*	χ_F/χ
YS	Without rearrangement terms	0.73	1.02	1.04
	With rearrangement terms	0.77	1.03	1.04
FM	Without rearrangement terms	0.66	1.03	1.05
	With rearrangement terms	0.71	1.04	1.05
	Experimental value		1.05	1.08

oretical and experimental results, as we have done in Table VI.

VII. DISCUSSION

The Fermi liquid model seems to work quite well for dilute solutions of ^{3}He in liquid ^{4}He. The effective potential is built on the assumption that the superfluid ^{4}He background can deform itself quickly to accommodate the more slowly moving ^{3}He atoms. However, a ^{3}He particle moving through the ^{4}He background has associated with it flowing ^{4}He atoms which can certainly affect the nearby ^{3}He atoms in a rather complicated way.

We have assumed that the effective interaction or Landau f function can be obtained directly from our reaction-matrix elements. The second variational derivative with respect to the particle occupation number of the total energy written in terms of Brueckner theory gives the reaction matrix plus rearrangement terms containing first and second variational derivatives of the reaction matrix. Our approximation that the Landau f function can be related directly to the reaction matrix is then based on the assumption that the energy expression is a reasonable approximation for the total energy, and that the rearrangement terms can be neglected. However, as shown in Sec. VI, inclusion of rearrangement terms would improve the results for F_1 and the corresponding effective mass ratio.

The reaction-matrix elements are calculated as explained, and are rather insensitive to the detailed shape of the input single-particle energy spectrum on the energy shell, and to the corresponding input parameters in the calculations.

We have used the reaction matrix elements to estimate both the Landau f function and the scattering amplitude a in the calculations of the transport coefficients. The function $a(\theta, \vec{\sigma}, \vec{\sigma}')$ may be related to $f(\theta, \vec{\sigma}, \vec{\sigma}')$, but it is not equal to f because, in a, one considers a scattering with no energy change, but with some small momentum transfer. In f, the quasiparticles travel forward, but scatter with some small energy transfer. For low densities, however, the coefficients B_L are approximately the same as F_L, and the coefficients C_L are approximately the same as Z_L.

It has been assumed that the Landau coefficients would be small for $L > 1$. This is true to a good approximation for the dilute-mixture problem. All the coefficients are quite small for $L > 1$, which indicates that the effective interaction function has approximately the form $A + B\cos\theta$.

The restoring force which produces the analog of zero sound in the mixture is $f(\theta)$, which is negative for all momenta $k < k_F$. Zero sound will, therefore, not propagate, and it is the large negative value of F_0 which prevents the observation of zero sound in the ^{3}He-^{4}He mixtures. The restoring force for spin waves is $z(\theta)$, which is positive for all $k < k_F$ in dilute mixtures. Then spin waves should propagate instead of zero sound.

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