over all $l$ save for the absent $^3S_0$ state, and charge independence is valid. In this case, it is useful to write the formulas in terms of $\beta_1{}^J+\beta_2{}^J$, $\beta_1{}^J-\beta_2{}^J$, and $\beta_3{}^J$ because the last two have the same complex phase.

It is interesting to note the connection between this derivation of $M$ and the limitation placed on $M$ by its invariance under time reversal.[1] Terms in $(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{n}$ and $\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2\cdot\mathbf{n}$ do not occur here because change of spin symmetry is forbidden by the assumed charge independence of the interaction. The term in $(\boldsymbol{\sigma}_1\cdot\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{p})+(\boldsymbol{\sigma}_1\cdot\mathbf{p})(\boldsymbol{\sigma}_2\cdot\mathbf{k})$ disappears only because of the symmetric nature of the $S$ matrix. This property of the $S$ matrix can be proved by the invariance of $S$ under time reversal.

The formulas have been checked against those derived by Stapp[5] by a different technique up to $f$ waves with mixing but without Coulomb phase shifts and against Breit and Ehrman[6] formula for $2\,\mathrm{Re}(C^*B)$ without mixing. This work was started after some illuminating discussions the author had with Professor E. Segrè and Professor O. Chamberlain about the analysis of proton-proton scattering experiments.

[5] H. P. Stapp (private communication).

[6] G. Breit and J. B. Ehrman, Phys. Rev. 96, 805 (1954).

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# Low-Energy Theorems for Renormalizable Field Theories

ABRAHAM KLEIN*†
*Harvard University, Cambridge, Massachusetts*

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A general method is described whereby the low-energy behavior of collision processes can be investigated systematically. The basic procedure is the construction of the renormalized Green's functions for the participating particles in the presence of external sources. For processes involving the collision of a boson with a fermion,—photon scattering, meson scattering, and photoproduction of mesons—one requires the propagation function of a nucleon in the presence of external electromagnetic and meson fields, as well as the propagator for a meson in an electromagnetic field. The rather direct relation of these functions to the various transition matrices is derived. The possible forms of the Green's functions are limited by the principles of Lorentz, gauge, and charge conjugation invariance. Their detailed structure is determined in part by the requirement that they describe particles of known mass, charge, magnetic moment, and mesic coupling constant. These conditions are sufficient to imply a number of theorems. For example, it is shown that the amplitude for photon-proton scattering is given correctly up to terms of first order in the photon energy by a Born approximation calculation, if one assigns to the proton its experimental charge and magnetic moment. If one includes the meson-nucleon coupling constant in this set of parameters, then perturbation theory is also valid for the leading terms in a momentum-energy expansion of the $P$-wave in meson-nucleon scattering, the $S$- and $P$-waves associated with the nucleon current in photomeson production, and the entire meson current for the same process.

The methods employed to establish the theorems are extended in order to provide a phenomenological framework for the description of the experiments for energies low enough so that the expansions in boson energy are still valid, but for which the deviations from the theorems are of practical significance. Thus, it is suggested that for the description of photon scattering one should attribute an electric and a magnetic polarizability to the nucleon. The description of the $P$-wave in meson-nucleon scattering also requires the introduction of two additional parameters. From the manner in which these enter the scattering amplitude, it can be concluded that the phase shift in the state of angular momentum and isotopic spin 3/2 is enhanced compared to its Born value, if only the phase shifts in the other states deviate in the opposite direction. Finally, parameters are introduced to describe the $S$-wave in meson-scattering, and the basis for a phenomenological description of photoproduction is indicated.

## I. INTRODUCTION

CONSIDERABLE attention has been devoted recently to the investigation of the low-energy limit of field theories which can be renormalized. In particular, the study of the simplest processes involving the collision of a boson with a fermion—photon scattering,[1] photoproduction of mesons,[2] and scattering of mesons[3]—has yielded a number of theorems which have proved of considerable utility in the interpretation of experiment. For the phenomena involving mesons, the theorems have been of special value in pointing to suitable experiments for measuring the meson-nucleon coupling constant.[2]

In order to investigate the threshold behavior of scattering or production matrices, one expands these, in effect, in power series in the four-momenta of the bosons. One then observes that one or more of the leading powers is of the same form as its Born approximation, expressed, however, in terms of the experimental charge, magnetic moment, and mesic coupling constant of the fermion. The purpose of this note is to report a systematic procedure for the construction of such theorems; in the course of the discussion we shall rederive the results that are already known,[1–3] as well

* Society of Fellows.

† Now at the Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania.

[1] The zero-frequency limit was first treated in full generality by W. Thirring, Phil. Mag. 41, 1193 (1950), the amplitude to first order in the frequency by F. E. Low, Phys. Rev. 96, 1428 (1954) and by M. Gell-Mann and M. L. Goldberger, Phys. Rev. 96, 1433 (1954).

[2] N. M. Kroll and M. A. Ruderman, Phys. Rev. 93, 233 (1954).

[3] Deser, Goldberger, and Thirring, Phys. Rev. 94, 711 (1954).

as a number of additional theorems on photomeson production and meson-nucleon scattering.[4]

The physical properties of a nucleon which underlie these results are most clearly subsumed in the structure of its Green's function in the presence of external boson sources.[5] The rather direct relation of this quantity to the transition amplitudes is derived in Sec. II. The basic physical information at our disposal, to be fitted naturally into a framework that satisfies the requirements of Lorentz, gauge, and charge conjugation invariance, is twofold. First, we know the form and analytic properties of the field-independent part of the renormalized Green's function, $G$, for four-momenta in the neighborhood of the free-nucleon mass shell. Second, we can insist that the propagator describe a particle with the correct experimental electric and mesic charge and magnetic moment, a condition imposed on the terms of $G$ which are linear in the boson sources. The role of the requirement that we deal with a renormalizable theory is merely that the above statements have a meaning, after a suitable scale change, if necessary, for the various functions of the theory.

It is perhaps worth emphasizing that the true significance of the theorems is that they provide a means of measuring parameters which are logically defined by other experiments. Thus, the charge and magnetic moment of the proton is measured, in principle, in the scattering of the proton by a weak, slowly varying external electromagnetic field, weak enough so that only effects linear in the field need be considered and sufficiently slowly varying so that any space-time variation of the *field-strengths* may be neglected. Again, the meson-nucleon coupling constant—in the sense in which the term will be usedin this paper[6]—is measured by the scattering of a nucleon by a weak, slowly varying meson field. Because of the pseudoscalar nature of the meson, the coupling constant so defined is more the analog of the magnetic moment than of the charge, since there is no scattering to first order in the field from a uniform meson field.[7] If we notice the formal identity between the scattering and production phenomena to be considered and the dependence of the scattering of the fermion on terms of second order in the external boson fields, then the theorems may be given an alternative statement. It is that the matrix elements which determine the scattering of a fermion by a weak, slowly varying field also determine—with exceptions to be noted below—the scattering by somewhat stronger but still slowly varying fields.

We have divided the detailed considerations according to phenomena. Section III, which treats of $\gamma$-ray scattering, contains the proof that the scattering amplitude for this process is correctly given up to terms of first order in the frequency of the photon by the Born approximation computed for a nucleon with given charge and Pauli moment.[1] We actually construct the Green's function in greater detail than is required for the proof of the theorem. Thus, we are in a position to specify the number of additional parameters required to determine the scattering amplitude to the second power of the photon momentum. It is suggested that it should be possible to understand experiments up to energies somewhat below the threshold for meson production by assigning to the proton (approximately) constant electric and magnetic polarizabilities in addition to its charge and magnetic moment. The interpretation of existing experiments[8] in terms of single nucleon scattering cross sections is not sufficiently certain to allow a real test of this hypothesis.

In Sec. IV, it is shown that the $P$-wave in meson-nucleon scattering is given by Born approximation except for relative corrections which vanish as *both* the momentum and mass of the meson go to zero. That there is no analogous theorem for the $S$-wave is merely a consequence of the fact that the coupling constant taken as fundamental in the discussion is the strength for the emission of a meson into a $P$ state.[6,7] The $S$-wave near threshold is determined by two parameters,[3] the coefficients of the $S$-wave terms which are independent of and linear respectively in the energy of the meson. The first deviation from the Born amplitude for the $P$-wave can also be expressed in terms of two parameters. Since there are three independent $P$-wave phase shifts ($\alpha_{31}=\alpha_{13}$ in the approximation which neglects nucleon recoil[9]), it is then possible to derive a single relation among them. One qualitative statement of this relation is that if $\alpha_{11}$ and $\alpha_{13}$ are reduced compared to their Born values, then $\alpha_{33}$ is necessarily enhanced. Comparison with experiment yields values for these parameters by no means small compared to the coupling constant, although this is easily understood in terms of the resonance in the scattering.

Three theorems on photomeson production are established in Sec. V. The $S$-wave theorem for the nucleon current leading to charged meson production (Kroll-Ruderman theorem)[2] is shown to be a direct consequence of the definition of the mesonic coupling constant and of gauge invariance. It is then demonstrated that the $P$-wave production from the nucleon is given near threshold by the Born approximation computed with the actual meson-nucleon coupling

[4] The new results to be established in this paper concerning the $P$-wave in meson scattering and the $P$-wave and meson current effects in photoproduction have been established by independent methods by F. E. Low, to whom the author is indebted for illuminating discussions.

[5] In the treatment of photoproduction we also require the Green's function of a meson in an external electromagnetic field.

[6] The alternative definition proposed by Deser, Thirring, and Goldberger, reference 3, in terms of meson scattering at zero energy is less closely related to the original Yukawa hypothesis. In the absence of a satisfactory dynamical theory, however, one can only remark that this definition is of use in the correlation of a different class of phenomena than those considered in this paper.

[7] This is, of course, equivalent to the observation that the emission of a single pseudoscalar meson is into a $P$ state.

[8] Pugh, Frisch, and Gomez, Phys. Rev. **95**, 590 (1954).

[9] We use the notation of Fermi for the $S$ and $P$ phase shifts, Anderson, Fermi, Martin, and Nagle, Phys. Rev. **91**, 155 (1953).

constant and magnetic moment Thirdly, a similar statement is established for the contribution of the meson current which involves the charge and the coupling constant. The latter contribution contains all angular momenta for the meson and multipoles for the photon because of a retardation factor for the motion of the meson. The deviations from these theorems can also be considered. In the present instance, however, it leads only to a type of analysis which has essentially been exploited in the literature;[10] we have therefore been content with noting the general forms of the correction terms.

## II. EXPRESSION FOR SCATTERING AMPLITUDE

We shall derive the scattering amplitude for a boson-fermion collision, taking as a representative case that of photon scattering. As is well known, the transition amplitude from a given state of the system at some initial time to another state at a later time is determined by a suitably expressed matrix element of the Green's function for the system taken between those states. The $S$ matrix is defined as the amplitude for going from a free-particle state in the remote past to another such state in the remote future.[11] For the example chosen, the $S$ matrix is given by the expression

$$\begin{aligned}(p'\sigma', k'\mu|S|p\sigma, k\nu) &= -(\lim x_{10}, \xi_{10}\to+\infty; x_{20}, \xi_{20}\to-\infty)\\ &\times\int d\mathbf{x}_1 d\xi_1 d\mathbf{x}_2 d\xi_2\,(p'\sigma', k'|x_1, \xi_1)_\alpha\\ &\times\tfrac{1}{2}i(\overleftarrow{\partial/\partial\xi_{10}}+\overrightarrow{\partial/\partial\xi_{10}})G_{\alpha\beta,\mu\nu}(x_1,x_2;\xi_1,\xi_2)\\ &\times\tfrac{1}{2}i(\overleftarrow{\partial/\partial\xi_{20}}+\overrightarrow{\partial/\partial\xi_{20}})(x_2,\xi_2|p\sigma,k)_\beta. \qquad (1)\end{aligned}$$

Here $p\sigma$, $p'\sigma'$ are four-momentum and spin variables specifying initial and final nucleon states, $k\nu$, $k'\mu$, are the corresponding momentum and vector component variables of the photon. The amplitude $(x, \xi|p\sigma, k)_\beta$ is the initial state of the combined system:

$$(x, \xi|p\sigma, k)_\beta = (x|p\sigma)_\beta\,(\xi|k), \qquad (2)$$

$$(x|p\sigma) = \left[\frac{1}{(2\pi)^3}\frac{(p^2+m^2)}{m^2}\right]^{\frac{1}{2}} e^{ipx}u_\beta(p\sigma), \qquad (3)$$

$$(\xi|k) = [(2\pi)^3 2k_0]^{-\frac{1}{2}} e^{ik\xi}, \qquad (4)$$

and

$$(p\sigma, k|x, \xi)_\beta = [(x, \xi|p\sigma, k)^\dagger\gamma_0]_\beta, \qquad (5)$$

where $\gamma_0$ is the usual Dirac matrix and the four-momenta are those of free particles. The Green's function of the photon-nucleon system is defined by the relation[12]

$$\begin{aligned}G_{\alpha\beta,\mu\nu}(x_1,x_2;\xi_1,\xi_2) &= -\langle(\psi_\alpha(x_1)\bar\psi_\beta(x_2)A_\mu(\xi_1)A_\nu(\xi_2))_+\rangle\epsilon(x_1-x_2)\\ &= G_+(x_1,x_2)_{\alpha\beta}\mathcal{G}_+(\xi_1,\xi_2)_{\mu\nu}\\ &\quad - i\delta^2 G_+(x_1,x_2)_{\alpha\beta}/\delta J_\mu(\xi_1)\delta J_\nu(\xi_2)|_{J=0}, \qquad (6)\end{aligned}$$

with $J_\mu(\xi)$ standing for the external current which is usefully assumed to be coupled to the electromagnetic field.

To transform Eq. (1), consider the first term of Eq. (6). Since $x_{10}>x_{20}$, we can write

$$\begin{aligned}G_+(x_1,x_2)_{\alpha\beta} &= i\langle 0|\psi_\alpha(x_1)\bar\psi_\beta(x_2)|0\rangle\\ &= i\sum_n e^{iP_n(x_1-x_2)}\langle 0|\psi_\alpha(0)|n\rangle\langle n|\bar\psi_\beta(0)|0\rangle, \qquad (7)\end{aligned}$$

where the sum is over all states of unit "charge," and $P_n$ is the four-momentum of the state $n$,

$$P_n = (\mathbf{p}, E_n(p) = (p^2+m_n^2)^{\frac{1}{2}}).$$

Among the allowed values of $m_n$, there is the isolated lowest point $m_n=m$, the mass of a nucleon. Separating the contribution from this state from all the others, we may write according to the theory of renormalization[13]

$$\begin{aligned}G_+(x_1,x_2)_{\alpha\beta} &= \frac{i}{(2\pi)^3}Z_2\sum_\sigma\int d\mathbf{p}\exp[i\mathbf{p}(\mathbf{x}_1-\mathbf{x}_2)-iE(\mathbf{p})(x_{10}-x_{20})]\\ &\quad\times\langle 0|\psi_\alpha(0)|\mathbf{p}\sigma\rangle\langle\mathbf{p}\sigma|\bar\psi_\beta(0)|0\rangle+\cdots, \qquad (8)\end{aligned}$$

where $\cdots$ indicates the contribution from higher mass states and the matrix element $\langle 0|\psi_\alpha(0)|\mathbf{p}\sigma\rangle$ is now understood to be finite and a solution of the free particle Dirac equation for a real nucleon. From Eq. (8) combined with Eqs. (3) and (5), we then obtain the the relation:

$$\lim(x_{10}\to\infty)\int d\mathbf{x}_1(\mathbf{p}\sigma|x_1)_\alpha G_+(x_1,x_2)_{\alpha\beta} = i(2\pi)^{-\frac{3}{2}}e^{-ipx_2}\bar u_\beta(p\sigma)Z_2. \qquad (9)$$

Equation (9) is true in the sense that all terms which oscillate with infinite rapidity are to be equated to zero, the limiting procedure of Eq. (1) being understood in that sense. We have also required the normalization condition

$$\bar u_\alpha(p\sigma)u_\alpha(p,\sigma') = \delta_{\sigma\sigma'}m/E(\mathbf{p}). \qquad (10)$$

For the photon propagator $\mathcal{G}_+(\xi_1,\xi_2)_{\mu\nu}$, there are

[10] For example, see M. Gell-Mann and K. M. Watson, Ann. Rev. Nuc. Sci. **4**, 219 (1954).

[11] These states are most conveniently considered to describe particles with their experimental masses.

[12] The completely analagous case of the meson-nucleon system has been studied by S. Deser and P. C. Martin, Phys. Rev. **90**, 1072 (1953). For the notation see J. Schwinger, Proc. Natl. Acad. Sci. U.S. **37**, 452 (1951).

[13] F. J. Dyson, Phys. Rev. **85**, 1736 (1949).

similar expressions. From the equation ($\xi_{10}>\xi_{20}$)

$$\mathcal{G}_+(\xi_1,\xi_2)_{\mu\nu} = \frac{i}{(2\pi)^3} Z_3 \sum_\lambda \int \frac{d\mathbf{k}}{2k_0} \exp[i\mathbf{k}\cdot(\xi_1-\xi_2)-ik_0(\xi_{10}-\xi_{20})] \times\langle 0|A_\mu(0)|k\lambda\rangle\langle k\lambda|A_\nu(0)|0\rangle+\cdots, \quad (11)$$

we conclude the relation

$$\lim(\xi_{10}\to\infty)\int d\xi_1(k|\xi_1) \times\tfrac{1}{2}i(\overleftarrow{\partial/\partial\xi_{10}}+\overrightarrow{\partial/\partial\xi_{10}})\mathcal{G}_+(\xi_1,\xi_2)_{\mu\nu} = i[(2\pi)^3 2k_0]^{-\frac{1}{2}}e^{-ik\xi_2}\delta_{\mu\nu}Z_3. \quad (12)$$

Applying Eqs. (9), (12) and corresponding expressions for the limit ($x_{20}$, $\xi_{20}\to-\infty$), to the second term of Eq. (6) as well as to the first, we obtain for Eq. (1) the more immediately useful equation:

$$(p'\sigma', k'\mu|S|p\sigma, k\nu) = Z_2Z_3\delta_{\sigma\sigma'}\delta_{\mu\nu}\delta(\mathbf{p}'-\mathbf{p})\delta(\mathbf{k}'-\mathbf{k})+\frac{iZ_2{}^2Z_3{}^2}{(2k_0'2k_0)^{\frac{1}{2}}(2\pi)^6} \times\int d^4\xi'd^4x'd^4\xi d^4x e^{-ip'x'}e^{-ik'\xi'}\bar{u}(p'\sigma') \times\mathcal{G}_+{}^{-1}(\xi',\xi'')_{\mu\rho}G_+{}^{-1}(x',x'')\delta^2G_+(x'',x''')/ \delta J_\rho(\xi'')\delta J_\epsilon(\xi''')|_{J=0} \times G_+{}^{-1}(x''',x)\mathcal{G}_+{}^{-1}(\xi''',\xi)_{\epsilon\nu}u(p\sigma)e^{ipx}e^{ik\xi}, \quad (13)$$

where summation or integration over repeated variables is understood. Several additional observations are required before Eq. (13) is transformed into final working form. First, we note that we must divide by the product $Z_2Z_3$ to obtain a unitary and indeed finite $S$ matrix. Second, we make a change of variables from the external current to the external electromagnetic potential according to the expression[14] (in which vector indices have been suppressed):

$$\delta^2G/\delta J(\xi)\delta J(\xi')=Z_3(\delta^2G/\delta A_c(\xi'')\delta A_c(\xi''')) \times\mathcal{G}_+{}^{(0)}(\xi'',\xi)\mathcal{G}_+{}^{(0)}(\xi''',\xi'), \quad (14)$$

since $A_c$ and $J$ are related by the expression

$$A_c(\xi)=Z_3{}^{\frac{1}{2}}\mathcal{G}_+{}^{(0)}(\xi\xi')J(\xi'), \quad (15)$$

involving the uncoupled photon Green's function $\mathcal{G}_+{}^{(0)}$. We next note the validity of the equation

$$\int d^4\xi' e^{-ik'\xi'}\mathcal{G}_+{}^{-1}(\xi',\xi'')\mathcal{G}_+{}^{(0)}(\xi'',\xi''')\varphi(\xi''') =Z_3{}^{-1}\int d^4\xi' e^{-ik'\xi'}\varphi(\xi'), \quad (16)$$

for $k'$ the four-momentum of a real photon and $\varphi(\xi)$ any function of $\xi$. Finally, we introduce into Eq. (13) the convergent Green's function, $G_c$, where[15]

$$G=Z_2G_c. \quad (17)$$

By the use of Eqs. (14)–(17) and the accompanying remarks, we are now in a position to state a convergent expression for the $S$ matrix for photon-proton scattering,[16] conveniently expressed in terms of a transition matrix $T$, where

$$\delta(p'+k'-p-k)(p'\sigma', k'\mu|T|p\sigma, k\nu) =-\frac{e^2}{(2\pi)^4}\int d^4\xi'd^4x'd^4\xi d^4x e^{-ip'x'}e^{-ik'\xi'}\bar{u}(p'\sigma') \times(x'|G^{-1}\delta^2G/\delta(eA_\mu(\xi))\delta(eA_\nu(\xi'))|_{A=0}G^{-1}|x) \times u(p\sigma)e^{ipx}e^{ik\xi}, \quad (18)$$

and $T$ is then related to $S$ according to the equation

$$(p'\sigma', k'\mu|S|p\sigma, k\nu) =\delta_{\sigma\sigma'}\delta_{\mu\nu}\delta(\mathbf{p}'-\mathbf{p})\delta(\mathbf{k}'-\mathbf{k})-2\pi i\delta(p'+k'-p-k) \times[(2\pi)^3 2k_0'2k_0]^{-\frac{1}{2}}(p'\sigma', k'\mu|T|p\sigma, k\nu). \quad (19)$$

In Eq. (18) $e$ is the renormalized charge of the proton.

Before proceeding to the proofs of the photon scattering theorems, it is perhaps worthwhile to exemplify the use of Eq. (18) by deriving the Born approximation scattering for a fermion with a charge but no anomalous moment. For the Green's function in the external field, we then take

$$G(x,x'[A])=(x|G[A]|x') =\left(x\left|\left\{\int d^4\xi[\gamma(\xi)(p-eA(\xi))]+m\right\}^{-1}\right|x'\right). \quad (20)$$

The significance of the notation is expressed by the equations

$$(x|\gamma(\xi)|x')=\gamma\delta(\xi-x)\delta(x-x'), \quad (21)$$

$$\int\gamma(\xi)d\xi=\gamma, \quad (22)$$

---

[14] In Eqs. (14) and (15), $A_c(\xi)$ is the renormalized external field, whereas $J(\xi)$ is the unrenormalized current. Consistency between the definition (compare reference 12) $\mathcal{G}_+=\delta\langle A\rangle/\delta J$ and the renormalization scheme, $\mathcal{G}_+=Z_3\mathcal{G}_{+c}$, $\langle A\rangle=Z_3{}^{\frac{1}{2}}\langle A\rangle_c$, requires the renormalization $J=Z_3{}^{-\frac{1}{2}}J_c$.

[15] The renormalization succeeds formally if we assume this relation to obtain even in the presence of the electromagnetic field. The scale factors in Eq. (13) then disappear without any necessity of mentioning vertex renormalization. The proof of equivalence to the more usual statements is easily carried through.

[16] In what follows we discard the subscripts + and $c$, since except briefly at the start of Sec. V, we shall be dealing exclusively with convergent propagation functions of the indicated type.

with $p$ the ordinary four-momentum operator. It then follows directly that

$$G^{-1}\delta^2 G[A]/\delta(eA_\mu(\xi))\delta(eA_\nu(\xi'))|_{A=0}G^{-1}$$
$$=\gamma_\mu(\xi)(\gamma p+m)^{-1}\gamma_\nu(\xi')+\gamma_\nu(\xi')(\gamma p+m)^{-1}\gamma_\mu(\xi), \quad (23)$$

which yields in turn the following form for Eq. (18):

$$(p'\sigma', k'\mu|T|p\sigma, k\nu)$$
$$=-e^2\bar{u}(p'\sigma')\{\gamma_\mu[\gamma(p+k)+m]^{-1}\gamma_\nu$$
$$+\gamma_\nu[\gamma(p-k')+m]^{-1}\gamma_\mu\}u(p\sigma), \quad (24)$$

wherein it is understood that

$$p'+k'=p+k. \quad (25)$$

We have here illustrated and wish further to emphasize the point, which is fundamental to our further discussion, that in any consideration of the second variational derivative of $G[A]$, only that part of it which has a second-order pole on the free-particle energy shell, that is a factor of $G[0]$ standing both on the right and the left, contributes anything to the scattering. Otherwise the factors of

$$\int G^{-1}(x', x[0])e^{ipx}d^4xu(p\sigma), \quad (26)$$

whose vanishing assert the free particle character of initial and final nucleon states, dominate the situation.

## III. PHOTON SCATTERING

We require the structure of the Green's function in the presence of an external electromagnetic field. It is instructive to begin our considerations with the zero-field propagator, since this suffices to obtain the zero frequency limit of the scattering. We thus consider the structure

$$G^{-1}(p;[0])=(\gamma p+m)[1+(2m)^{-1}(\gamma p+m)\mathfrak{F}_1(p^2/m^2)$$
$$+(2m)^{-2}(\gamma p+m)^2\mathfrak{F}_2(p^2/m^2)], \quad (27)$$

as determined by the dual requirements of Lorentz invariance and the condition that

$$G^{-1}(p;[0])(\gamma p+m)^{-1}u(p)=u(p), \quad (28)$$

when $u(p)$ itself satisfies

$$(\gamma p+m)u(p)=0. \quad (29)$$

Equation (28) prescribes the singularity of $G$ itself for four-momenta satisfying $p^2+m^2=0$. It follows that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are analytic functions of their argument in this neighborhood.[17] As a consequence, if we expand $G$ itself about the point $\gamma p+m=0$, we obtain

$$G(p;[0])=(\gamma p+m)^{-1}+(2m)^{-1}\mathfrak{F}_1(0)$$
$$+(2m)^{-2}(\gamma p+m)\mathfrak{F}_2(0)+\cdots$$
$$=(\gamma p+m)^{-1}+\Delta G(\gamma p), \quad (30)$$

with $\Delta G$ again analytic in the domain of interest.

Now to obtain the scattering which is independent of the boson momenta, we can ignore any explicit dependence of $G[A]$ on the field strengths or their derivatives, since for such a dependence we have, for example, the relation

$$\frac{\delta G}{\delta A_\mu(\xi)}=\int\frac{\delta G}{\delta F_{\Lambda\rho}(\xi')}\frac{\delta F_{\Lambda\rho}(\xi')}{\delta A_\mu(\xi)}$$
$$=\int\frac{\delta G}{\delta F_{\Lambda\rho}(\xi')}(\delta_{\rho\mu}\partial_\Lambda'-\delta_{\Lambda\mu}\partial_\rho')\delta(\xi'-\xi), \quad (31)$$

and each derivative with respect to the photon coordinates leads to a factor of the photon momentum upon insertion into Eq. (18). We therefore require only the explicit dependence on the electromagnetic potential, and the form in which this may occur is restricted by the principle of gauge invariance to the combination $\Pi_\mu=p_\mu-eA_\mu$; more explicitly this means the combination

$$\Pi_\mu=p_\mu-e\int d^4\xi 1(\xi)A_\mu(\xi), \quad (32)$$

with the understanding that $\gamma 1(\xi)=\gamma(\xi)$. But Eq. (30) now tells us that $\Delta G(\gamma\Pi)$ is an analytic function of $\gamma\Pi$ so that its derivatives can contribute nothing to any scattering process. Our result is thus that the Green's function, $G_1$, effective for zero energy scattering is[18]

$$G_1(p;[A])=(\gamma\Pi+m)^{-1}. \quad (33)$$

To extend our considerations, we must now include in the Green's function its explicit dependence on electromagnetic field strengths. It is generally true, however, quite independent of the expansion in the photon momentum, that we may ignore any dependence on the external current, $J_\mu(\xi)$, since this is zero for a real photon; that is, $J_\mu(\xi)=-\Box^2 A_\mu(\xi)$ contributes, after computation of the variational derivative, a factor $k^2$ or $k'^2$ both of which vanish. Again, we require the dependence on the field strengths themselves in a very restricted sense, for $G$ need be correct only to the second order of the field. Moreover, terms which depend on the square of the field strength will contribute to the scattering first to the second order in the frequency.

To see these arguments in more detail, we consider the problem of constructing invariants which are linear in $F_{\mu\nu}$. We shall be guided in part by the requirements

[17] This will certainly be the case if we exclude virtual interactions with quanta of vanishing rest mass.

[18] The subscript 1 on $G_1$ will be used to denote that part of the Green's function effective for scattering throughout the varying contexts of this paper.

of charge conjugation invariance, which can be stated in the convenient form

$$CG^{-1}(p,[-A])^T C^{-1}=G^{-1}(p,[A]), \qquad (34)$$

where $C$ is the charge conjugating matrix and the operation of transposition is to be applied to all variables of the nucleon, remembering that $p^T=-p$. For the formation of the required invariants, we have at our disposal the linearly independent Dirac matrices and the vector $\Pi_\mu$. A systematic consideration then shows that essentially the only combination that satisfies Eq. (34) is $\sigma_{\mu\nu}F_{\mu\nu}=\sigma F$; all other possibilities are simply expressible as multiple commutators and anticommutators of $\gamma\Pi$ with $\sigma F$. To the first order in the field then, it is correct to write the series

$$\begin{aligned}G^{-1}(\Pi,[A])=&\gamma\Pi+m-\tfrac{1}{2}\mu'\sigma F+(2m)^{-1}(\gamma\Pi+m)^2\mathfrak{F}_1+\cdots\\&+\tfrac{1}{2}(2m)^{-1}\{\gamma\Pi+m,\sigma F\}\mathfrak{M}_1\\&+\tfrac{1}{4}(2m)^{-2}\{\gamma\Pi+m,\{\gamma\Pi+m,\sigma F\}\}\mathfrak{M}_{21}\\&+\tfrac{1}{4}(2m)^{-2}[\gamma\Pi+m,[\gamma\Pi+m,\sigma F]]\mathfrak{M}_{22}+\cdots \qquad (35)\end{aligned}$$

in terms of the additional set of parameters $\mathfrak{M}_1, \mathfrak{M}_{21}, \cdots$, and $\mu'$. The latter can be correctly identified as the anomalous magnetic moment of the nucleon if we compute from Eq. (35) the scattering of the nucleon to first order in the field, as given by the expression

$$\begin{aligned}&\delta(p'-p-k)(p'\sigma'|T_\mu|p\sigma)\\&=-\frac{e}{(2\pi)^4}\int d^4x'd^4xd^4\xi e^{-ip'x'}e^{ipx}e^{ik\xi}\bar{u}(p'\sigma')\\&\qquad\times(x'|\delta G^{-1}/\delta eA_\mu(\xi)|_{A=0}|x)u(p\sigma)\\&=\delta(p'-p-k)\bar{u}(p'\sigma')\{\gamma_\mu+i(\mu'/e)k_\lambda\sigma_{\lambda\mu}\}u(p\sigma)\\&=\delta(p'-p-k)\bar{u}(p'\sigma')\{[(2p+k)_\mu/2m]\\&\qquad+(i/e)k_\lambda\sigma_{\lambda\mu}(\mu'+e/2m)\}u(p\sigma). \qquad (36)\end{aligned}$$

Equation (35) suffices to compute the scattering to first order in the frequency, since any terms of second order in the field not already contained therein are by the requirements of gauge invariance proportional to the square of the field tensor. Indeed, we wish to establish the theorem that to first order in $k$, the scattering is correctly described by the Green's function

$$G_1=[\gamma\Pi+m-\tfrac{1}{2}\mu'\sigma F]^{-1}. \qquad (37)$$

To see this most clearly, we consider a trivial analytic continuation of the series, Eq. (35), obtained by choosing as independent operators $G_1^{-1}$ and $\sigma F$, rather than $\gamma\Pi+m$ and $\sigma F$. The numerical values of $\mathfrak{M}_1, \cdots$ will then be altered, but without changing notation, the new series reads

$$\begin{aligned}G^{-1}(\Pi,[A])=&G_1^{-1}+G_1^{-2}(2m)^{-1}\mathfrak{F}_1+\cdots\\&+\tfrac{1}{2}(2m)^{-1}\{G_1^{-1},\sigma F\}\mathfrak{M}_1+\cdots. \qquad (38)\end{aligned}$$

Inverting and expanding about $G_1$, we obtain[19]

$$\begin{aligned}G(\Pi,[A])=&G_1+(2m)^{-1}\mathfrak{F}_1+\cdots\\&+\{G_1,(2m)^{-1}\tfrac{1}{2}\sigma F\}\mathfrak{M}_1+\cdots. \qquad (39)\end{aligned}$$

The theorem now follows from the observations that when we compute the second derivative of $G$ at the point $A=0$, the terms involving the coefficients $\mathfrak{F}$ are, as before, analytic on the free-particle energy shell, whereas the new contributions involving the coefficients $\mathfrak{M}$ exhibit at most a pole of the first order. Indeed, the representative terms exhibited in Eq. (38) are the most singular members of their respective classes.

The actual form of the $T$ matrix of Eq. (18) computed to first order in $k$ by means of the $G_1$ of Eq. (37) and expressed as the scalar product of a Pauli spin operator with initial and final polarization vectors is[1]

$$\begin{aligned}\mathbf{e}'\cdot\mathbf{T}\cdot\mathbf{e}=&(e^2/m)\mathbf{e}'\cdot\mathbf{e}-(ie/2m)\mu'2k_0\boldsymbol{\sigma}\cdot(\mathbf{e}'\times\mathbf{e})\\&+(ie/2m)\mu k_0[(\mathbf{n}\cdot\mathbf{e}')(\boldsymbol{\sigma}\cdot\mathbf{n}\times\mathbf{e})+(\boldsymbol{\sigma}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{e}\times\mathbf{e}')\\&-(\mathbf{n}'\cdot\mathbf{e})(\boldsymbol{\sigma}\cdot\mathbf{n}'\times\mathbf{e}')-(\boldsymbol{\sigma}\cdot\mathbf{n}')(\mathbf{n}'\cdot\mathbf{e}'\times\mathbf{e})]\\&+2i\mu^2k_0\boldsymbol{\sigma}\cdot(\mathbf{n}'\times\mathbf{e}')\cdot(\mathbf{n}\times\mathbf{e}). \qquad (40)\end{aligned}$$

Here $\mathbf{n}$ and $\mathbf{n}'$ are unit vectors in the direction of the incident wave and in the direction of observation respectively, and $\mu$ is the total magnetic moment. The differential cross section for unpolarized particles to which Eq. (40) gives rise, may be written as

$$\begin{aligned}d\sigma/d\Omega=&(e^4/m^2)\tfrac{1}{2}(1+\cos^2\theta)+k^2[6\mu^4+(\mu^2e^2/m^2)\\&-(\mu e^3/m^3)+\tfrac{3}{8}(e^4/m^4)]\\&+k^2\cos\theta[-8\mu^3(e/m)+2\mu^2(e^2/m^2)]\\&+k^2\cos^2\theta[-2\mu^4+3\mu^2(e^2/m^2)\\&-\mu(e^3/m^3)-\tfrac{1}{8}(e^4/m^4)]. \qquad (41)\end{aligned}$$

In Eq. (41), we have returned to unrationalized units in which $e^2$ is the fine structure constant. It should be remarked that for $\gamma$ rays with energy of 100 Mev or more scattered from protons, the magnetic terms are of the same order of magnitude as the Thomson cross section, the main contribution arising from the isotropic terms which is proportional to the forth power of the total magnetic moment.

Equation (41) is not yet accurate enough to be compared with experiment even in the energy range for which a frequency expansion of the scattering amplitude might be expected to have approximate validity, the region below the threshold for meson production, since there are additional contributions to the cross section of second order in the photon energy. These come about as a result of interference between spin

[19] The structure of the Green's function, Eq. (39), could have been inferred equally well by considering the proton to be situated in a weak uniform magnetic field, in which it is still possible to prescribe energy values, as determined by the singularities of $G_1$. The term depending on $\mathfrak{M}_1$ is then associated with the first order change in the wave functions.

independent terms in the scattering amplitude, of second order in the photon frequency, and the Thomson amplitude. The terms in question may be divided conveniently into two classes. There are first those which are contained in the Green's function, $G$, itself.[20] The leading effect on the cross section is, at most, proportional to $\mu^2$ and therefore should be small compared to the magnetic scattering. In any case, such additions can be readily calculated although, for reasons to be stated below, we shall not take the trouble.

More important, undoubtedly, are the effects of the transition of the nucleon to excited states after absorption of the incident photon, the Rayleigh scattering. For energies below the threshold for meson production, we may expect to obtain an approximate description of this scattering by adding to the Green's function of Eq. (38) members which are quadratic in the electromagnetic field strengths.

We require only such terms as have nonvanishing matrix elements between solutions of the free-particle Dirac equation and, further, remain linearly independent when such matrix elements are computed. There are in fact only three such forms, which may be taken conveniently as $F_{\mu\nu}^2$, $(\gamma_\mu\Pi_\nu/m)F_{\mu\lambda}F_{\nu\lambda}$, and $\gamma_5F_{\mu\nu}F_{\mu\nu}^*$, where $F_{\mu\nu}^*$ is the tensor dual to $F_{\mu\nu}$. Of these, the last is effectively of third order in the frequency, the second proportional to $E^2$, and the first to $H^2-E^2$. For a nonrelativistic description, therefore, it is sufficient to consider the addition to $G^{-1}$ of the terms

$$-\tfrac{1}{2}\alpha E^2-\tfrac{1}{2}\beta H^2, \tag{42}$$

which yield the not unexpected result that the Rayleigh scattering should be describable by means of an electric and magnetic polarizability, $\alpha$ and $\beta$ respectively.

For a model which includes Thomson but omits magnetic scattering, we then obtain the scattering amplitude:

$$\mathbf{e}'\cdot\mathbf{T}\cdot\mathbf{e}=(e^2/m)\mathbf{e}'\cdot\mathbf{e}-\alpha k_0^2\mathbf{e}'\cdot\mathbf{e}+\beta k_0^2(\mathbf{n}\times\mathbf{e})\cdot(\mathbf{n}'\times\mathbf{e}'), \tag{43}$$

and a cross section

$$d\sigma/d\Omega=[(e^2/m)-\alpha k^2]^2\tfrac{1}{2}(1+\cos^2\theta) -2[(e^2/m)-\alpha k^2]\beta k^2\cos\theta+\beta^2k^4\tfrac{1}{2}(1+\cos^2\theta). \tag{44}$$

The characteristics of Eq. (44) are suited to fit the one existing experiment[8] with the not surprising values for $\alpha$ and $\beta$,[21]

$$\alpha\sim\beta\sim(e^2/mm_\pi^2), \tag{45}$$

to within a factor of two, with $m_\pi$ the mass of the $\pi$ meson. We hesitate to draw any conclusions from this circumstance since the interpretation of the experimental results in terms of individual nucleon scattering cross sections is subject to considerable doubt, especially in view of the fact that no deviation from Thomson scattering was detected for heavy nuclei.

[20] In this sense our method of calculation has greater generality than a direct momentum expansion of the scattering amplitude.

[21] The values of $\alpha$ and $\beta$ in Eq. (44) justify the neglect of the additional contributions of order $k^2$ arising directly from $G_1$.

### IV. MESON-NUCLEON SCATTERING

In direct analogy with Eq. (18), the formula for the renormalized transition amplitude is given by the equation[22]

$$(p'\sigma',q'j|T|p\sigma,qi) =-\frac{g^2}{(2\pi)^4}\int d^4x'd^4\xi'd^4xd^4\xi e^{-ip'x'}e^{-ik'\xi'}\bar{u}(p'\sigma') \times(x'|G^{-1}\delta^2G/\delta(g\phi_i(\xi))\delta(g\phi_j(\xi'))|_{\phi=0}G^{-1}|x) \times u(p\sigma)e^{ipx}e^{iq\xi}. \tag{46}$$

Here $\phi_i(\xi)$ is the external meson field, $i$ and $j$ the isotopic indices of incident and emergent meson.

We proceed immediately to the construction of the Green's function, $G(p,[\phi])$. If we consider first terms linear in $\phi$, we have at our disposal besides the principle of Lorentz invariance, the property corresponding to charge conjugation invariance, as expressed by the condition

$$CG^{-1}(p,[-\phi])^TC^{-1}=G^{-1}(p,[\phi]), \tag{47}$$

where $C$ has the same effect on the Dirac matrices as in the electromagnetic case, and in addition for the isotopic matrices $\tau_i$,

$$C\tau_{iT}C^{-1}=-\tau_i. \tag{48}$$

Again, there is really only one invariant independent of $\gamma p$, that is, the quantity $\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}$, all other invariants being expressible as multiple commutators and anticommutators of the two operators. For example, we have the relation

$$\{\gamma p,\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}\}=i\gamma_5\gamma_\mu\partial_\mu\boldsymbol{\tau}\cdot\boldsymbol{\phi}. \tag{49}$$

It is important to remark that in the choice of a fundamental coupling term to define the constant $g$, we have at our disposal either $-g\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}$, or $i(g/2m)\gamma_5\gamma_\mu\partial_\mu\boldsymbol{\tau}\cdot\boldsymbol{\phi}$, since both lead to the same scattering of the nucleon by $\phi$ to the first order in the momentum transfer.[23] As will become clear gradually, it is convenient to choose the latter form.[24]

Employing the effective Green's function $G_1$,

$$G_1=[\gamma p+m+i(g/2m)\gamma_5\gamma_\mu\partial_\mu\boldsymbol{\tau}\cdot\boldsymbol{\phi}]^{-1}, \tag{50}$$

[22] The symmetry of Eq. (46) in the meson field variables immediately establishes the relation

$$(p'\sigma',q'j|T|p\sigma,qi)=(p'\sigma',-qi|T|p\sigma,-q'j).$$

There was, of course, a similar relation in the case of photon scattering.

[23] We recall that $\bar{u}(p)\gamma_5u(p)=0$.

[24] It is perhaps well to emphasize that this choice has absolutely nothing to do with the fundamental dynamics of the meson-nucleon system.

which will be the subject of a theorem, the expansion to first order in $\phi$ analogous to Eq. (38) can be expressed as

$$G^{-1}(p,[\phi])=G_1^{-1}+G_1^{-2}(2m)^{-1}\mathfrak{F}_1+\cdots \\ +\tfrac{1}{2}\{G_1^{-1},\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}\}\mathfrak{A}_1+\cdots \quad (51)$$

of which we are assured that only the first term contributes anything to the scattering. Indeed, the insertion of Eq. (50) into Eq. (46) yields the Born approximation for the $P$ wave of order $(q^2/\omega)$, where $\omega$ is the meson energy, whereas the $S$ wave behaves as $\omega^2$. If we admit, provisionally, the result that any other $P$ wave in the theory must behave as $q^2$, we should then have established the validity of the Born approximation to the $P$ wave in the limit as $\omega\rightarrow 0$. On the other hand, we shall see that there may be $S$ waves in the theory of zero and first order in $\omega$.

To verify these last assertions, we must consider those contributions to $G$ which are second order in $\phi$ and no more than first order in the derivatives of each of the fields.[25] A systematic procedure for generating those forms which have nonvanishing and linearly independent matrix elements between free-particle nucleon states is to transform the quantities $\boldsymbol{\tau}\cdot\boldsymbol{\phi}(\gamma\Pi)^n\boldsymbol{\tau}\cdot\boldsymbol{\phi}$, $n=0, 1\cdots$ by commuting the factors $\gamma\Pi$ to the outside where they can be replaced by $-m$. By this means we find five independent forms, conveniently expressed as $\boldsymbol{\phi}^2$, $\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi}$, $(\partial_\mu\phi_i)^2$, $\sigma_{\mu\nu}\boldsymbol{\tau}\cdot(\partial_\mu\boldsymbol{\phi}\times\partial_\nu\boldsymbol{\phi})$, and $\{\gamma_\mu\Pi_\nu, \partial_\mu\phi_i\partial_\nu\phi_i\}$. By carrying out the reduction to the nonrelativistic limit, separating $S$ and $P$ wave effects, we find that the $S$ wave up to first order in the meson energy can be represented by adding to $G^{-1}$ the forms

$$\rho_1(g^2/2m)\boldsymbol{\phi}^2+\rho_2(g/2m)^2\boldsymbol{\tau}\cdot\boldsymbol{\phi}\times(\partial\boldsymbol{\phi}/\partial t), \quad (52)$$

whereas the leading $P$ waves are of the form

$$-\lambda_1[g^2/(2m)^3]\nabla\boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi}+\lambda_2[g^2/(2m)^3] \\ \times\epsilon_{\gamma\alpha\beta}\sigma_\gamma\boldsymbol{\tau}\cdot\nabla_\alpha\boldsymbol{\phi}\times\nabla_\beta\boldsymbol{\phi}, \quad (53)$$

where $\epsilon_{\gamma\alpha\beta}$ represents the three-dimensional Levi-Civita tensor density. Equations (52) and (53) provide the justification for the conclusion of a theorem for the $P$ wave, although not for the $S$ wave.

If the parameters $\rho_1$, $\rho_2$, $\lambda_1$, and $\lambda_2$ were all of order unity, it would betoken the validity of the Born approximation for pseudoscalar theory with pseudoscalar coupling, at least in the low-energy region. To understand this, one need only recall the form of the Hamiltonian for this theory which results from the Dyson or Foldy transformations.[26] To see to what extent the actual circumstances differ from this simple one, it is instructive to compare with experiment the phase shifts obtained from a Green's function constructed from Eqs. (50), (52), and (53). Turning first to the $P$ waves, we obtain by a straightforward calculation, the following formula for the transition amplitude, expressed as an operator in isotopic and ordinary spin space,

$$(q'j|T|qi)=-(g/2m)^2\{\tau_j\tau_i[\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{q}'/-\omega(\mathbf{q})] \\ +\tau_i\tau_j[\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{q}'/\omega(q')]\}-2\lambda_1[g^2/(2m)^3]\mathbf{q}\cdot\mathbf{q}'\delta_{ij} \\ +2\lambda_2[g^2/(2m)^3]\epsilon_{jik}\tau_k\boldsymbol{\sigma}\cdot\mathbf{q}'\times\mathbf{q}. \quad (54)$$

If we decompose the scattering into the separate channels labeled by values of total isotopic spin and angular momentum, we obtain for the phase shifts, in the approximation in which $e^{i\delta}\sin\delta\cong\delta$, the formulas

$$\begin{aligned}\alpha_{33}&=(4/3)(g^2/4\pi)(\mu/2m)^2\eta^3(\mu/\omega)\\&\qquad\times[1+\tfrac{1}{4}(\omega/m)(\lambda_1+\lambda_2)],\\ \alpha_{31}=\alpha_{13}&=-(2/3)(g^2/4\pi)(\mu/2m)^2\eta^3(\mu/\omega)\\&\qquad\times[1-\tfrac{1}{2}(\omega/m)(\lambda_1-2\lambda_2)],\\ \alpha_{11}&=-(8/3)(g^2/4\pi)(\mu/2m)^2\eta^3(\mu/\omega)\\&\qquad\times[1-\tfrac{1}{8}(\omega/m)(\lambda_1+4\lambda_2)].\end{aligned} \quad (55)$$

Here $\mu$ is rest mass of the meson and $\eta=q/\mu$. From Eq. (55), we can derive the interesting result that if $\alpha_{31}=\alpha_{13}$ and $\alpha_{11}$ are reduced compared to their Born approximation values, then $\alpha_{33}$ is enhanced relative to its perturbation theoretic value. This is a consequence of the fact $\lambda_1+\lambda_2$ necessarily lies between $\lambda_1-2\lambda_2$ and $\lambda_1+4\lambda_2$, both of which are positive, by hypothesis. Under the same assumptions, $\lambda_1$, itself, is necessarily positive. It is perhaps worth emphasizing that this result holds for the nonrelativistic limit of the relativistic theory and is independent of the actual dynamics of the meson-nucleon system.

Of the $P$-wave phase shifts only $\alpha_{33}$ is known with any accuracy. For meson energies below 100 Mev, roughly, it is represented by the formula[27]

$$\alpha_{33}=0.235\eta^3. \quad (56)$$

To obtain the order of magnitude of $\lambda_1+\lambda_2$ from Eqs. (55), we shall set $\omega\cong\mu$ and assume that[28]

$$(g^2/4\pi)(\mu/2m)^2=0.081.$$

We then find

$$\lambda_1+\lambda_2\cong 4.8M/\mu, \quad (57)$$

a result that hardly inspires confidence in the validity of the energy expansion.[29] Actually, if we look upon Eqs. (55) as the linear approximation to resonance and antiresonance formulas, then Eq. (57) predicts a reso-

[25] Thus, a form such as $\phi\cdot\Box^2\phi$ is of no interest in the present discussion.

[26] F. J. Dyson, Phys. Rev. **73**, 929 (1948); Berger, Foldy, and Osborn, Phys. Rev. **87**, 106 (1952).

[27] H. A. Bethe and F. de Hoffman, Phys. Rev. **95**, 1100 (1954); J. Orear, Phys. Rev. **96**, 176 (1954).

[28] Value quoted at the Fifth Annual Rochester Conference (to be published) by G. Chew and F. E. Low, based on fit to data using an effective range formula. The $S$ wave in photoproduction yields the somewhat smaller value of 0.066. See G. Bernadini and E. L. Goldwasser, Phys. Rev. **95**, 857 (1954).

[29] This value justified, however, the neglect of the corrections to the Born approximation that can be obtained directly from $G_1$.

nance at much too low an energy compared with experiment. It almost certainly determines that the other phase shifts change sign compared to their Born values if we take Eqs. (55) seriously for these also. This means that Eqs. (55) have at most a qualitative validity even in the extreme low-energy region.

We conclude this section by comparing with experiment the $S$ wave represented in Eq. (52). We first record the scattering amplitude

$$(q'j|T|qi)=\rho_1(g^2/m)\delta_{ij}-\rho_2(g/\partial m)^2 \times i\epsilon_{jik}\tau_k[\omega(\mathbf{q})+\omega(\mathbf{q}')], \quad (58)$$

from which we obtain directly the phase shifts for isotopic spin $\frac{3}{2}$ and $\frac{1}{2}$,

$$\begin{aligned}\alpha_3&=-\rho_1(g^2/4\pi)(\mu/M)\eta-\tfrac{1}{2}\rho_2(g^2/4\pi)(\mu\omega/m^2)\eta,\\ \alpha_1&=-\rho_1(g^2/4\pi)(\mu/M)\eta+\rho_2(g^2/4\pi)(\mu\omega/m^2)\eta.\end{aligned} \quad (59)$$

There is perhaps still some question as to whether the $S$ phase shifts behave strictly linearly with momentum near zero energy. For the sake of illustration only, we shall favor such advocacy and select the phase shifts of Orear,[27]

$$\alpha_3=-0.11\eta, \quad \alpha_1=0.16\eta. \quad (60)$$

We then find that

$$\rho_1\cong 0.01, \quad \rho_2\cong 0.56, \quad (61)$$

a rather dramatic expression of the difficulty of the problem of constructing a dynamical theory of the $S$ wave.

## V. PHOTO-MESON PRODUCTION

Since it is our aim to display a formula for the transition amplitude in which the contribution of the meson current alone is separated from the remainder, we must alter somewhat the procedure followed subsequent to Eq. (13). Let $\mathcal{G}_p$ be the photon propagation function and $\mathcal{G}_M$ that for the meson. Let us further suppose that the renormalization constants $Z_2$ and $Z_3$ have the same significance as in Sec. II and that $Z_5$ is the constant that renormalizes $\mathcal{G}_M$. For the *renormalized* transition matrix, it then is a straightforward matter to derive the expression, analogous to Eq. (13),

$$\begin{aligned}&\delta(p'+q'-p-k)(p'\sigma'\tau', q'j|T|p\sigma\tau, k\mu)\\ &=-\frac{Z_2Z_3^{\frac{1}{2}}Z_5^{\frac{1}{2}}}{(2\pi)^x}\int d^4\eta' d^4x' d^4\xi d^4x\bar{u}(p'\sigma'\tau')e^{-ip'x'}\\ &\quad\times e^{-iq'\eta'}\mathcal{G}_M^{-1}(\eta'\eta'')_{jl}G^{-1}(x',x'')\\ &\quad\times\delta^2G(x'',x''')/\delta K_l(\eta'')\delta J_\epsilon(\xi'')|_0G^{-1}(x''',x)\\ &\quad\times\mathcal{G}_p^{-1}(\xi'',\xi)_{\epsilon\nu}u(p\sigma\tau)e^{ipx}e^{ik\xi}, \quad (62)\end{aligned}$$

wherein $K$ is the external source of the meson field, and we have included the charge degree of freedom in designating variables for the nucleon. The various propagation functions remain to be renormalized. It is convenient to change variables from $K$, $J$ to $\langle\phi\rangle$, $\langle A\rangle$, the vacuum matrix elements of the *quantized* fields.[30] In carrying out this transformation, it is well to remember that $K$ and $J$ each depend on *both* matrix elements. We thus find, suppressing indices, and recalling the definitions

$$\mathcal{G}_p=\delta\langle A\rangle/\delta J, \quad \mathcal{G}_M=\delta\langle\phi\rangle/\delta K, \quad (63)$$

that

$$\begin{aligned}&(\delta/\delta J)(\delta G/\delta K)\\ &=\mathcal{G}_M[\delta^2G/\delta\langle\phi\rangle\delta\langle A\rangle]\mathcal{G}_p+(\delta\mathcal{G}_M/\delta\langle A\rangle)(\delta\mathcal{G}/\delta\langle\phi\rangle)\mathcal{G}_p\\ &\quad+\mathcal{G}_M[\delta^2G/\delta\langle\phi\rangle\delta\langle\phi\rangle](\delta\langle\phi\rangle/\delta J)\\ &\quad+(\delta\mathcal{G}_M/\delta\langle\phi\rangle)(\delta G/\delta\langle\phi\rangle)(\delta\langle\phi\rangle/\delta J)\\ &\quad+(\delta/\delta J)[(\delta\langle A\rangle/\delta K)(\delta G/\delta\langle A\rangle)]. \quad (64)\end{aligned}$$

When Eq. (64) is inserted into Eq. (62), only the first two terms survive in virtue of the observation that

$$\begin{aligned}&\int d^4\eta' e^{-iq'\eta'}\mathcal{G}_M^{-1}(\eta',\eta'')\delta\langle A\rangle/\delta K(\eta'')\\ &\qquad=\int d^4\xi(\delta\langle\phi\rangle/\delta J(\xi'))\mathcal{G}_p^{-1}(\xi'\xi)e^{ik\xi}=0, \quad (65)\end{aligned}$$

since the functions $(\delta\langle A\rangle/\delta K)=\delta\langle\phi\rangle/\delta J$ are nonsingular. The renormalization goes through smoothly if first we remember Eq. (17), and second we recall the relations

$$\begin{aligned}\langle A\rangle&=Z_3^{\frac{1}{2}}A+O_1(\Box^2)A+Q_1(\Box^2-\mu^2)\phi,\\ \langle\phi\rangle&=Z_5^{\frac{1}{2}}\phi+Q_2(\Box^2-\mu^2)\phi+O_2(\Box^2)A,\end{aligned} \quad (66)$$

where $A$ and $\phi$ are the prescribed fields and the operators $O$ and $Q$ possess the property

$$O(0)=Q(0)=0. \quad (67)$$

Since it is only the latter value of the operators that enters Eq. (62), the renormalization constants cancel out and we obtain finally the desired formula:

$$\begin{aligned}&\delta(p'+q'-p-k)(p'\sigma'\tau', q'j|T|p\sigma\tau, k\mu)\\ &=-\frac{eg}{(2\pi)^4}\int d^4\eta' d^4x' d^4\xi d^4x e^{-ip'x'}e^{-iq'\eta'}\bar{u}(p'\sigma',\tau')\\ &\quad\times\{(x'|G^{-1}\delta^2G/\delta g\phi_j(\eta')\delta eA_\mu(\xi)G^{-1}|x)\\ &\quad-(x'|\mathcal{G}_M^{-1}(\eta',\eta'')\delta\mathcal{G}_M(\eta'',\eta''')/\delta eA_\mu(\xi)\\ &\quad\times\delta G^{-1}/\delta g\phi_l(\eta''')|x)\}_{A=\phi=0}u(p\sigma\tau)e^{ipx}e^{ik\xi}, \quad (68)\end{aligned}$$

which is understood to be expressed completely in terms of finite propagation functions. Of the two terms of Eq. (68), the first has the same structure as encountered in the previous sections, and will be considered below. We turn now to the second term, which describes the effect of the meson current.

The general form of the renormalized meson propa-

[30] See J. Schwinger, reference 12.

gator in the zero-field limit is given by the expression:

$$\mathcal{G}_M^{-1}(q, [0]) = (q^2+\mu^2)\{[1+(q^2+\mu^2)/m^2]\mathcal{K}(q^2/m^2)\}, \quad (69)$$

where $q$ is the meson-momentum operator and $\mathcal{K}(q^2/m^2)$ is an analytic function of its variable. The dependence of $\mathcal{G}_M$ on the electromagnetic potential is given by the replacement

$$q \rightarrow q-eA. \quad (70)$$

To construct invariants depending explicitly on the field strengths and currents, we have at our disposal solely the vector $q_\mu$. The only invariants permissible in a power series expansion are then of the form $(\Box^2)^n q_\mu J_\mu$, for integral $n$, and such terms can contribute nothing to processes involving real photons. In the present instance, therefore, it suffices to assume the relation

$$\mathcal{G}_M^{-1}(q, [A]) = \mathcal{G}_M^{-1}(q-eA, [0]). \quad (71)$$

The combination that occurs in Eq. (68) can now be evaluated using Eqs. (69) and (71) as follows:

$$\mathcal{G}_M^{-1}(\delta\mathcal{G}_M/\delta eA_\mu)|_0 = -(\delta\mathcal{G}_M^{-1}/\delta eA_\mu)|_0\mathcal{G}_M = 2q_\mu[1+O(q^2/m^2)](q^2+\mu^2)^{-1}[1+O(q^2/m^2)]. \quad (72)$$

Consequently, the relative correction to the Born approximation is of the second order of smallness in $q$. Likewise, the matrix element for the emission of a meson, $\delta G^{-1}/\delta(g\phi)$, taken between free-particle states is by definition given by its Born approximation with relative corrections once again of the second order. Taken together, these statements constitute a proof that the contribution to photoproduction arising from the meson current is given correctly to the leading order in the boson momenta by perturbation theory, with the relative corrections as specified.[31] The actual form of the transition amplitude for charged production is

$$(q'\pm|T_M|k\mu) \cong -ie\sqrt{2}(g/2m)\frac{\boldsymbol{\sigma}\cdot(\mathbf{k}-\mathbf{q})}{\mathbf{k}\cdot\mathbf{q}-k_0\omega(\mathbf{q})}(q-k)_\mu, \quad (73)$$

wherein the isotopic matrix element has already been evaluated [compare Eq. (79) below].

We turn finally to the consideration of the nucleon current, for which we assert that the leading contributions to both the $S$ and $P$ wave are contained in the Green's function (which ignores the neutron-proton mass difference)

$$G_1^{-1}(p, [A], [\phi]) = \gamma[p-e\tfrac{1}{2}(1+\tau_3)A]+m-\tfrac{1}{2}\mu_p'\sigma F\tfrac{1}{2}(1+\tau_3) -\tfrac{1}{2}\mu_n'\sigma F\tfrac{1}{2}(1-\tau_3)+i(g/2m)\gamma_5\gamma_\mu\partial_\mu\boldsymbol{\tau}\cdot\boldsymbol{\phi} +(eg/2m)\gamma_5\gamma_\mu A_\mu\tfrac{1}{2}[\tau_3, \boldsymbol{\tau}\cdot\boldsymbol{\phi}]. \quad (74)$$

[31] The corrections to the meson current are actually only one order smaller, when one takes into account the nucleon current.

Here $\mu_p'$ and $\mu_n'$ are anomalous moments of proton and neutron respectively. The familiar last term of Eq. (74) is a consequence of the requirements of gauge invariance for the interaction of charged mesons with the electromagnetic field, as expressed by the replacement

$$\partial_\mu\delta_{ij} \rightarrow \partial_\mu\delta_{ij} - ieA_\mu(\mathcal{T}_3)_{ij}, \quad (75)$$

where $\mathcal{T}_3$ is the matrix

$$\mathcal{T}_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (76)$$

It follows that

$$\tau_i(\mathcal{T}_3)_{ij}\phi_j = \tfrac{1}{2}[\tau_3, \boldsymbol{\tau}\cdot\boldsymbol{\phi}]. \quad (77)$$

Alternatively we could have remarked that

$$\partial_\mu\boldsymbol{\tau}\cdot\boldsymbol{\phi} = -i[p_\mu, \boldsymbol{\tau}\cdot\boldsymbol{\phi}] \rightarrow \partial_\mu\boldsymbol{\tau}\cdot\boldsymbol{\phi} - ie\tfrac{1}{2}[(1+\tau_3)A_\mu, \boldsymbol{\tau}\cdot\boldsymbol{\phi}]. \quad (78)$$

Either Eqs. (75)–(77) or Eq. (78) reminds us of the existence of the last term in Eq. (74).

Now this term contributes to charged photo-meson production an $S$ wave which is energy independent. We have only to observe that there is no other energy-independent $S$ wave in the theory to complete the proof of the Kroll-Ruderman theorems. Direct computation establishes that the remainder of $G_1$ contributes an $S$ wave which is linear in the photon (or meson) energy. Any other $S$ wave must appear in $G^{-1}$ as a term bilinear in $A$, $\phi$ and will be at least first order in the photon energy, according to the requirements of gauge invariance. Such a term will be exhibited below.

To separate photoproduction processes according to the charge of the meson produced, we introduce the usual definitions

$$\phi = -(2)^{-\frac{1}{2}}(\phi_1+i\phi_2), \quad \phi^* = (2)^{-\frac{1}{2}}(\phi_1-i\phi_2), \quad \tau_+ = \tfrac{1}{2}(\tau_1+i\tau_2), \quad \tau_- = \tfrac{1}{2}(\tau_1-i\tau_2). \quad (79)$$

Up to terms of first order in the meson energy, $G_1$ yields for positive and negative meson production, from protons and neutrons respectively,

$$\mathbf{T}_\pm\cdot\mathbf{e} = \sqrt{2}(eg/2m)i\boldsymbol{\sigma}\cdot\mathbf{e}\{1\mp[\omega(q)/2m]\}. \quad (80)$$

Equation (80) predicts a threshold yield of negative to positive mesons:

$$\sigma(-)/\sigma(+) \cong (1+2\mu/m) = 1.3, \quad (81)$$

not in essential disagreement with experiment.[32]

As the final theorem of this paper, we have the statement that except for relative corrections which vanish as $\omega \rightarrow 0$, the $P$ wave is given by Born approximation computed for a nucleon characterized by its experimental charge, magnetic moment, and mesic

[32] Sands, Teasdale, and Walker, Phys. Rev. **95**, 592 (1954).

coupling constant, that is, by the Green's function $G_1$. By analogy with the meson scattering case, $G_1$ generates a $P$ wave of order $(kq/\omega)$, whereas any other $P$ wave in the theory behaves at least as $kq$.

We may record here for convenience the leading $P$-wave terms for positive production from protons, negative production from neutrons, and neutral production from protons, respectively:

$$\mathbf{T}_+\cdot\mathbf{e}=\sqrt{2}\frac{g}{2m}\left[\mu_n\frac{(\boldsymbol{\sigma}\cdot\mathbf{k}\times\mathbf{e})(\boldsymbol{\sigma}\cdot\mathbf{q})}{\omega}-\mu_p\frac{(\boldsymbol{\sigma}\cdot\mathbf{q})(\boldsymbol{\sigma}\cdot\mathbf{k}\times\mathbf{e})}{k_0}\right], \quad (82)$$

$$\mathbf{T}_-\cdot\mathbf{e}=\sqrt{2}\frac{g}{2m}\left[\mu_n\frac{(\boldsymbol{\sigma}\cdot\mathbf{q})(\boldsymbol{\sigma}\cdot\mathbf{k}\times\mathbf{e})}{k_0}-\mu_p\frac{(\boldsymbol{\sigma}\cdot\mathbf{k}\times\mathbf{e})(\boldsymbol{\sigma}\cdot\mathbf{q})}{\omega}\right], \quad (83)$$

$$\mathbf{T}_0\cdot\mathbf{e}=\frac{g}{2m}\mu_p\left[\frac{(\boldsymbol{\sigma}\cdot\mathbf{q})(\boldsymbol{\sigma}\cdot\mathbf{k}\times\mathbf{e})}{k_0}-\frac{(\boldsymbol{\sigma}\cdot\mathbf{k}\times\mathbf{e})(\boldsymbol{\sigma}\cdot\mathbf{q})}{\omega}\right], \quad (84)$$

in terms of the total magnetic moments of proton and neutron.

As with the processes of photon and meson scattering, our methods provide a basis for the phenomenological treatment of photoproduction as long as the experimental behavior of the cross section is "normal," that is, as long as the $S$ wave goes linearly and the $P$ wave cubically with the meson momentum. We prefer not to enter into the details of a numerical analysis since such an analysis would not differ substantially from considerations that have already been recorded in the literature.[10] We shall, conclude this work, however, by providing the framework for such an analysis, at the same time essentially filling in the details of proofs given in the foregoing.

It is amusing to remark first that in the formation of invariants bilinear in $A$, $\phi$ we are *not* constrained to terms which are explicitly gauge-invariant. Thus a term in $G^{-1}$ of the form $(2m)^{-1}i\gamma_5\gamma_\mu\Box^2\partial_\mu\boldsymbol{\tau}\cdot\boldsymbol{\phi}$, not hitherto considered, gives rise by the considerations of gauge invariance discussed in connection with Eq. (78) and preceding equations, to an additional term of the form

$$[e/(2m)^3]\gamma_5\gamma_\mu\tfrac{1}{2}[\tau_3,\tau_i][A_\mu\Box^2+\tfrac{1}{2}(\Box^2A_\mu)+2A_\lambda\partial_\lambda\partial_\mu+(\partial_\lambda A_\mu)\partial_\lambda+(\partial_\mu A_\lambda)\partial_\lambda]\phi_i, \quad (85)$$

of which the first three members essentially describe electric dipole absorption with emission of charged mesons into an $S$ state and the last two members electric quadrupole absorption with emission in a $P$ state; both effects are quadratic in the energy-momentum vectors of the bosons.

Turning to forms that are explicitly gauge invariant, we have three possible types of isotopic dependence, $\boldsymbol{\tau}\cdot\boldsymbol{\phi}$, $(\boldsymbol{\tau}\times\boldsymbol{\phi})_3$, and $\phi_3$. If $\Phi$ stands for a linear combination of these, the permissible, independent Dirac invariants at most of second order in the momenta are four in number, $\gamma_5\Phi\sigma F$, $\gamma_5\gamma_\lambda\partial_\mu\Phi F_{\mu\lambda}$, $\{\gamma_5\gamma_\mu\partial_\mu\Phi,\sigma F\}$, and $\{\gamma_5\Pi_\mu\gamma_\Lambda\gamma_\rho,\partial_\mu\Phi F_{\lambda\rho}\}$. Passing to the nonrelativistic limit, we obtain as the leading $S$ wave $\Phi\boldsymbol{\sigma}\cdot\mathbf{E}$, representing electric dipole absorption, and as the leading $P$ waves $\nabla\Phi\cdot\mathbf{H}$ and $\boldsymbol{\sigma}\cdot\nabla\Phi\times\mathbf{H}$, giving magnetic dipole absorption. To obtain additional electric quadrupole absorption beyond that contained in Eq. (82) we must include terms of still one higher power in the energy of the photon. We have therefore verified the assertions about the minimal energy-momentum dependence of the invariants bilinear in electromagnetic and meson fields.

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