Pion-Nucleon Scattering at High Energies*

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The pseudoscalar theory in the Tamm-Dancoff approximation is applied to high-energy pion-nucleon scattering with a view to explaining the observed maximum in π^--p scattering around 1 Bev as a resonant interaction. The integral equation is solved for the $D_{\frac{3}{2}}$ state of $T=\frac{1}{2}$ which is attractive. The various singularities in the kernel and the wave function are taken into account in a semianalytical fashion. A short derivation is given of the complex intefral equation which takes the effects of meson production into account, and it is solved in an approximate manner.

It is found that, contrary to the original expectation of a resonant interaction, the increase of $D_{\frac{3}{2},\frac{1}{2}}$ phase shift over the Born approximation is not large. Also the effects of meson production are not found to be very significant over the energy range 1 Bev to 2 Bev. This last is contrary to the observations in this energy range.

1. INTRODUCTION

THE early Brookhaven experiments on the total cross section for π^--p scattering indicated a maximum near 1 Bev (laboratory system) in addition to the (now famous) maximum around 200 Mev which had been observed before. The latter one had been successfully explained by means of the Tamm-Dancoff (T.D.) approximation in pseudoscalar theory,[1] as a resonant interaction in the state $P_{\frac{3}{2},\frac{3}{2}}$ ($J=\frac{3}{2}$, $T=\frac{3}{2}$) of the meson-nucleon system. The success of the T.D. approximation encouraged the belief that a similar explanation of the maximum around 1 Bev might not be unreasonable.

The basis for this expectation was, of course, somewhat weaker than in the case of the first maximum around 200 Mev. Firstly, the data were rather few (and not too accurate), so that any detailed analysis (of the type carried out for the "low"-energy case) had not been feasible in this region. Thus the interpretation of the high-energy region had to be based on rather general considerations instead of more positive experimental facts (which were available in the "low"-energy case). Secondly, the maximum around 1 Bev was believed to be much broader[2] than that around 200 Mev. Still it was thought worthwhile carrying out an investigation similar to Dyson's for this case as well.

The interesting feature of this "second" maximum is that it occurs only in the π^--p cross section, unlike the previous case where both the π^+-p and π^--p cross sections showed maxima at the same energy (~200 Mev); no maximum in (π^+p) scattering has been observed in the region of 1 Bev.[3] Thus a possible explanation of the "second" maximum would require an isotopic spin state $T=\frac{1}{2}$, rather than $T=\frac{3}{2}$, since the latter would contribute mostly to π^+-p scattering. Now the most important feature of the T.D. method (as has been clearly shown by Dyson's work) is that it considerably enhances the Born approximation phase shift for attractive states and reduces it for repulsive states. Hence, in order to interpret the second maximum as a one-meson resonance, one has to look for some state of $T=\frac{1}{2}$ which has an attractive interaction. Now it can be shown generally[1,4] that for $T=\frac{1}{2}$, the states of $j=2n+\frac{1}{2}$ (where 'n' is an integer) have an attractive interaction, while for $T=\frac{3}{2}$ it is the states $j=2n-\frac{1}{2}$ which are attractive.

Among the states $j=2n+\frac{1}{2}$, the states $j=\frac{1}{2}$ occupy a special position because only in these states can the meson be absorbed by the nucleon *before* the latter emits one. As a result, these states exhibit self-energy effects due to the possibility of successive emission and reabsorption of a meson by the nucleon, in the framework of the T.D. formalism. Consequently these states require a special renormalization which has been treated by Dalitz and Dyson.[5] On account of these special features these states are not expected to be strongly attractive. (Moreover, their j- value is much too small to account for the magnitude of the cross section at the observed maximum.) This leaves as the first strongly attractive states those of $j=5/2$. At an energy as high as 1 Bev, it is reasonable to consider states of such high j; indeed the large cross section observed around 1 Bev (~50 mb) requires a j at least as high as this.

Now for a given j, the state with $l=j-\frac{1}{2}$ has always the stronger interaction; for $l=j+\frac{1}{2}$ the interaction begins to be important only in the relativistic region (but cannot exceed the former). So one may expect the $D_{\frac{3}{2}}$ state to be the more important one, rather than $F_{\frac{5}{2}}$. For the same reason one would not expect important effects from the $D_{\frac{3}{2}}$ state of $T=\frac{3}{2}$ (which would,

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[1] Dyson, Ross, Salpeter, Schweber, Sundaresan, Visscher, and Bethe, Phys. Rev. **95**, 1644 (1954); referred to as A.

[2] This was the situation in January, 1954 when this calculation was undertaken. Recently, more accurate data have appeared which indicate that the curve drops fairly sharply on the high-energy side.

[3] The more recent data indicate that there is a *minimum* in the (π^+p) cross section somewhat below 1 Bev.

[4] Bethe, DeHoffman, and Schweber, *Mesons and Fields* (Row, Peterson and Company, Evanston, 1955), Vol. 2.

[5] R. H. Dalitz and F. J. Dyson, Phys. Rev. **99**, 301 (1955).

moreover, contribute predominantly to π^+-p scattering).

Having thus decided on this choice of the state to be considered, we shall take the paper of Dyson *et al.*[1] as basic and freely use the various equations and formulas of this paper with the same notation as far as possible.

2. NECESSARY FORMALISM

From Eqs. (23), (27), and (27a) of A, we find that the wave function $f(p)$ of the $D_{\frac{3}{2},\frac{1}{2}}$ state satisfies the equation:

$$f(p)=2\lambda\int_0^\infty \frac{s^2}{(E_p\omega_p E_s\omega_s)^{\frac{1}{2}}}L_1(p,s) \times\left[\delta(E-E_s-\omega_s)+P\frac{f(s)}{E-E_s-\omega_s}\right], \quad (1)$$

where

$$\lambda=G^2/(16\pi^2), \quad (2)$$

and

$$L_1(p,s)=-\tfrac{1}{2}[(E_p+M)\{(A-2M)K_2(C) +(E-M)K_2(B)\}+sp(E_s+M)^{-1} \times\{AK_3(C)+(E+M)K_3(B)\}]. \quad (3)$$

The other symbols are as defined in A. $f(p)$ is related to the phase shift according to the equation:

$$\tan\delta=-\pi f(k). \quad (4)$$

It is somewhat more convenient to work in terms of the quantities $f_1(p)$ and x defined as follows:

$$f_1(p)=(E_p\omega_p)^{\frac{1}{2}}f(p), \quad (5)$$

$$x=E_s+\omega_s-E. \quad (6)$$

The transformations (5) and (6) bring (1) into the form:

$$f_1(p)=f_B(p)+\lambda\int L(p,s)f_1(s)(dx/x), \quad (7)$$

where

$$L(p,s)=-2s(E_s+\omega_s)^{-1}L_1(p,s), \quad (8)$$

$$f_B(p)=-\lambda(E_k\omega_k)^{\frac{1}{2}}L(p,k), \quad (9)$$

$$\tan\delta=-\pi(E_k\omega_k)^{-\frac{1}{2}}f_1(k). \quad (10)$$

The quantities $f_1(s)$ and $L(p,s)$ should now be regarded as functions of x rather than s. This has the advantage that the singularity at $x=0$ can be evaluated very simply in a semianalytical manner, without introducing any complexities elsewhere.

So far, the Eqs. (1) or (7) do not take account of the possibility of meson production. However, if the total energy of the system is sufficiently large, the latter process becomes energetically possible (threshold energy in the c.m. system being only $E=M+2\mu$), and its importance increases with energy. Since the energies to be considered are rather large ($\gtrsim 1$ Bev), it is expected that these reactive processes will play an important part, and their effects should therefore be taken into account.

The Eq. (15) of A (before angular integration) suggests the modification necessary for including meson production effects. We find that one of the terms in the kernel (corresponding to the (1,2) state) involves an energy denominator $E-E_{p+s}-\omega_p-\omega_s$ which can vanish only for $E\geq M+2\mu$. Now in Eq. (A15), the integral is evaluated as a principal value, representing standing waves. To include the effects of meson production, we should now add an *outgoing* two-meson wave to the one-meson equation (A15) through the modification:

$$P\frac{1}{E_1}\longrightarrow\lim_{\epsilon\to 0}\frac{1}{E_1+i\epsilon}=P\frac{1}{E_1}+i\pi\delta(E_1), \quad (11)$$

where

$$E_1=E-\omega_s-\omega_p-E_{p+s}, \quad (11a)$$

and $\epsilon>0$. The kernel therefore becomes a complex function of p and s, and in place of (1), we now have a *complex* integral equation of the form:

$$f(p)+i\phi(p)=2\lambda\int_0^\infty s^2ds(E_s\omega_sE_p\omega_p)^{-\frac{1}{2}}\{L_1(p,s) +iL_2(p,s)\}\times\left[\delta(E-\omega_s-E_s) +P\frac{1}{E-E_s-\omega_s}\{f(s)+i\phi(s)\}\right]. \quad (12)$$

The evaluation of $L_2(p,s)$ may be carried out by noting that, according to (11), the quantities $K_n(C)$ in (3) [see Eq. (A22)] are changed into

$$K_n(C)+i\pi(2ps)^{-1}P_n[(c^2-1-r^2)/2r], \quad (13)$$

where the P_n's are the usual Legendre polynomials. It may be noted that the above modification does not change the B-terms in (3). Thus we have:

$$L_2(p,s)=-\pi(ps)^{-1}[(E_p+M)(A-2M)P_2(y) +spA(E_s+M)^{-1}P_3(y)], \quad (14)$$

where

$$y=(c^2-1-r^2)/2r. \quad (15)$$

The important property of $L_2(p,s)$ is that it is zero unless $E=\omega_p+\omega_s+E_{p+s}$ as may be seen from the appearance of the δ function in (11). This condition cannot be satisfied for real values of θ (angle between **p** and **s**) unless *both* p and s are *less* than k in magnitude. In particular, $L_2(p,s)$ vanishes on and above the energy the energy shell (*i.e.*, when $p\geq k$ or $s\geq k$ or both).

Transformation of (12) according to (5) and (6) leads finally to the equation:

$$f_1(p)+i\phi_1(p)=f_B(p)+\lambda\int\{L(p,s) +iL'(p,s)\}[f_1(s)+i\phi_1(s)](dx/x), \quad (16)$$

where

$$L+iL'=-2s(E_s+\omega_s)^{-1}(L_1+iL_2). \quad (17)$$

The quantity $f_B(p)$ in (16) is purely real and is given as before by (9), since $L'(p,k)=0$ (from what has been said in the last paragraph).

The complex phase shift $\delta+i\eta$ is now given by:

$$\tan(\delta+i\eta)=-\pi(E_k\omega_k)^{-\frac{1}{2}}[f_1(k)+i\phi_1(k)]=A+iB. \quad (18)$$

The contributions of a particular state (j,T) to the total cross sections (elastic and inelastic) are[6]

$$\sigma_{\rm el}(j,T)=\pi\lambda\!\!\!^{-2}(j+\tfrac{1}{2})\,|\exp(2i\delta-2\eta)-1|^2, \quad (19a)$$

$$\sigma_{\rm in}(j,T)=\pi\lambda\!\!\!^{-2}(j+\tfrac{1}{2})[1-\exp(-4\eta)]. \quad (19b)$$

In terms of the quantities A and B of (18), the cross sections are easily seen to be:

$$\sigma_{\rm el}(j,T)=2\pi\lambda\!\!\!^{-2}(2j+1)(A^2+B^2)[(1+B)^2+A^2]^{-1}, \quad (20a)$$

$$\sigma_{\rm in}(j,T)=2\pi\lambda\!\!\!^{-2}(2j+1)B[(1+B)^2+A^2]^{-1}. \quad (20b)$$

For the state $T=\frac{1}{2}$, the contributions to (π^-p) scattering are finally given by:

$$\sigma_{\rm el}(\pi^-p)=\tfrac{2}{3}\sigma_{\rm el}(j,\tfrac{1}{2}),$$
$$\sigma_{\rm in}(\pi^-p)=\tfrac{2}{3}\sigma_{\rm in}(j,\tfrac{1}{2}). \quad (21)$$

3. KERNEL OF THE INTEGRAL EQUATION

The solution of the complex integral equation (16) is an extremely complicated process, since it appears in a coupled form when the real and imaginary parts are separated out. It is more feasible first to solve the real equation (7) (which does not take account of meson production) and then to use this solution as a basis for the more general case of (16). Even for the simpler case of (7) a completely analytical solution is virtually impossible, and a semi-analytical approach seems to be the most that can be done. For this purpose, it is useful first to discuss some of the important features of the kernel $L(p,s)$. $L(p,s)$ contains the various functions $K_n(b)$ and $K_n(C)$ which have been defined in A. It is useful to express these functions in terms of the parameters u and v defined by:

$$u=r/(1+b), \quad v=r/(1+c); \quad (22)$$

where

$$c=C/\bar{E}, \quad b=B/\bar{E}; \quad (22a)$$

$$\bar{E}=\tfrac{1}{2}[E(p+s)+E(p-s)]; \quad (22b)$$

$$C=\omega_p+\omega_s-E, \quad r=sp/\bar{E}^2. \quad (22c)$$

Thus, explicitly,

$$v=sp/\bar{E}(\bar{E}+C)=\{E(p+s)-E(p-s)\}/\{E(p+s)+E(p-s)-2C\}. \quad (22d)$$

A similar expression holds for u. We now have the following expressions for the K_n-functions:

$$K_0(C)=\tfrac{1}{2}(sp)^{-1}\ln[(1+v)/(1-v)]; \quad (23a)$$

$$K_1(C)=yK_0(C)+\tfrac{1}{2}(1-c)/sp; \quad (23b)$$

$$K_2(C)=\tfrac{3}{4}yK_1(C)+\tfrac{1}{4}c/\bar{E}^2-\tfrac{1}{2}K_0(C); \quad (23c)$$

$$K_3(C)=(5/3)yK_2(C)-\tfrac{2}{3}K_1(C)-\tfrac{1}{6}cr/\bar{E}^2. \quad (23d)$$

Exactly similar expressions hold for $K_n(B)$.

For small values of u and v, the logarithmic terms $\ln[(1+v)/(1-v)]$ and $\ln[(1+u)/(1-u)]$ can be expanded in ascending powers of these parameters and a certain number of terms may be retained, depending on the accuracy needed. On the other hand, for $|v|$ or $|u|\geqslant\frac{1}{3}$, the closed forms are more suitable for numerical work.

The asymptotic behavior of the kernel can be obtained by expanding in powers of $(p_</p_>)$ (where $p_<$ and $p_>$ are respectively the smaller and larger of p and s). Following the procedure of Bethe and others,[7] one has the following asymptotic form of $L(p,s)$, for the case of $j=l+\frac{1}{2}$:

$$L(p,s)\approx\tfrac{1}{2}ps^{-1}[H_{j-\frac{1}{2}}(p/s)+H_{j+\frac{1}{2}}(p/s)], \quad (s>p); \quad (24)$$

where

$$H_n(x)=\ln(1+x)-x+\tfrac{1}{2}x^2-\cdots+(-1)^nx^n/n. \quad (24a)$$

Retaining the most important terms in the expansion of $H_n(x)$ for $|x|<1$, one finds that (24) reduces to the following expression $(l=2)$:

$$L(p,s)\approx 38/405(p/s)^3+O(p/s)^4. \quad (24b)$$

Using (24), one can derive the asymptotic behavior of $f_1(p)$, assuming $f_1(p)\to p^{-n}$, $(n>0)$, and substituting in (7) to determine n in terms of λ. This relation is given by[4]:

$$-\lambda/Q'=\frac{1}{n(1-n)}\left[\frac{\pi}{\sin\pi n}-\frac{1}{n(1-n)}+2\right]+2\sum_{k=1}^{j-\frac{1}{2}}(-1)^k\left[\frac{1}{k^2-n^2}+\frac{1}{k^2-(1-n)^2}\right]+(-1)^{j+\frac{1}{2}}\left[\frac{1}{(j+\frac{1}{2})^2-n^2}+\frac{1}{(j+\frac{1}{2})^2-(1-n)^2}\right]. \quad (25)$$

Using the argument of Bethe *et al.*,[4] one can derive from (25) an upper limit λ_0 on the value of λ, from the requirement of square integrability of the wave function which demands that $n>\frac{1}{2}$. This "critical" coupling constant $(4\pi\lambda_0)$ is obtained by putting $n=\frac{1}{2}$ in (25) (limit of normalizability) and using the values of

[6] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., 1952), Chap. 8.

[7] Dalitz, Sundaresan, and Bethe, Proc. Cambridge Phil. Soc. (to be published).

$j=5/2$ and $Q'=-1$. This gives finally:

$$4\pi\lambda_0=G_0^2/(4\pi)=176.8. \tag{25a}$$

It may be noted from the expressions (23) that the functions $K_n(C)$ are real as long as $|v|<1$, diverge logarithmically when $v=\pm1$, and become complex when $|v|>1$.[8] Thus the imaginary part of the kernel vanishes except when $|v|>1$. This gives a simple criterion for the existence of the imaginary part. For the energies we shall consider, $|u|$ is always less than unity, so that $K_n(B)$ is always real. Moreover, for the real equation (7), $\ln x_1$ is replaced by $\ln|x_1|$, where $x_1=(1+v)/(1-v)$.

Since $|v|>1$ makes the kernel complex this condition must be equivalent to the possibility of simultaneous production of two mesons, of momenta p and s, with conservation of energy. Such production is possible if, for *some* relative direction of p and s, the energy conservation equation [see (11)],

$$E(\mathbf{p}+\mathbf{s})+\omega_s+\omega_p=E, \tag{26}$$

can be fulfilled. This is obviously the case if, for the given p and s,

$$E(p-s)+C\leqslant 0\leqslant E(p+s)+C, \tag{26a}$$

where C is defined in (22c). But according to the definition (22d) of v, $v=\pm1$ means that

$$E(p\mp s)+C=E(p\mp s)+\omega_p+\omega_s-E=0. \tag{27}$$

This equation is clearly equivalent to the two limiting cases of Eq. (26a). It can then easily be shown that the energy equation (26a) can indeed be fulfilled if and only if $|v|\geqslant 1$.

The region in which the kernel is complex is then the one between the two curves in the $p-s$ plane defined by Eq. (27), with the $-$ and $+$ signs in $E(p\mp s)$ respectively; they are plotted in Fig. 1 for the case $k=1.0M$.

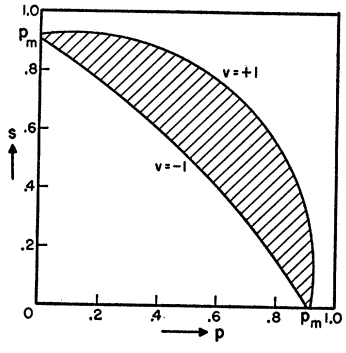

FIG. 1. The curves $v=\pm1$; the shaded portion between the curves is the region of meson production.

[8] This follows from the relation:
$\ln[(1+v)/(1-v)]=\ln|(v+1)/(v-1)|+i\pi$, for $|v|>1$.

If p is kept fixed and s varied, then for $s=0$ we have

$$\bar{E}+C=E(p)+\omega(p)+\mu-E. \tag{28}$$

Production of an additional meson s together with p is only possible if the expression (28) is negative which will only be the case if $p<p_{\max}$, where $p_{\max}$ is defined by:

$$E(p_{\max})+\omega(p_{\max})+\mu=E, \tag{28a}$$

and is always somewhat smaller than k.

The nature of the kernel is quite different according as $p\lessgtr p_{\max}$. For $p\leqslant p_{\max}$, v can vary from $-\infty$ to $+\infty$, so that the $K_n(c)$ functions have logarithmic singularities. Thus for such values of p the kernel is expected to show rather large variations as a function of s. On the other hand, for $p>p_{\max}$, $|v|$ is less than unity for all s, so that the kernel is expected to be a smooth function.

The quantity $L(p,s)$ was first tabulated numerically by programming it for the Card-Programmed-Computer

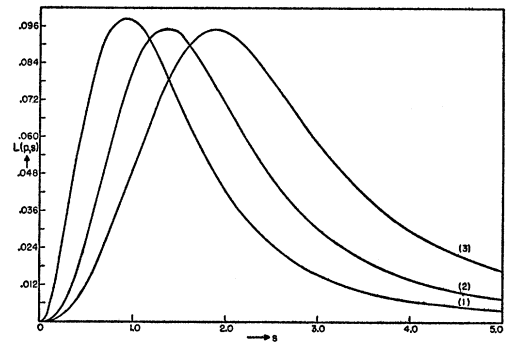

FIG. 2. The function $L(p,s)$ as a function of s, when $p>p_{\max}$ and $k=1.0M$. Curves (1), (2), and (3) correspond respectively to $p=1.0M$, $p=1.4M$ and $p=2.0M$.

(C.P.C.) situated at the Cornell Computing Center. Two different initial C.M. momenta, $k=0.7M$ and $k=1.0M$ were considered. These correspond respectively to 1.135-Bev and 2.128-Bev meson kinetic energies in the laboratory system. For each of these two energies, the kernel was programmed as a function of the "variable" s and the "parameter" p. Use of two different "instruction fields" greatly facilitated the programming for $|v|>\frac{1}{3}$ and $|v|\leqslant\frac{1}{3}$ respectively. The regions where the kernel was expected to show rather marked variations (according to the previous paragraph) were given due weights by taking smaller intervals of s and p for them. For the asymptotic region (large s) the behavior of the kernel, as given by (24a) was taken as a rough indication of the maximum value of s needed for a given p, with the specification that the least tabulated value of $L(p,s)$ (for given p) should be about 1% of its maximum value.

Some typical curves are shown in Figs. 2 and 3. For $p>p_{\max}$, Fig. 2 shows that the kernels are indeed smooth functions of s, with fairly well-defined maxima

at roughly $s=p$. There is a tendency for these peaks to be somewhat broader as p is increased.

Figure 3 shows that for $p<p_{\max}$, the kernel has logarithmic singularities. There is a fairly deep minimum in between these singularities. The signularities shift towards the left as p is increased, and finally disappear at $p=p_{\max}$.

4. SOLUTION OF THE REAL EQUATION

For a solution of the integral equation (7), it was necessary first to transform it into a set of linear simultaneous equations in suitable parameters representing the wave function. Because of the presence of various singularities it was thought somewhat more convenient (and probably safer) to use a semianalytical approach by fitting simple functions of the appropriate type to the numerical data representing the kernel, instead of taking only those discrete points at which the kernel had been tabulated. This procedure permitted the integrations over singular regions to be carried out in a relatively simple way. It is particularly useful for $p<p_{\max}$, though not so much for $p>p_{\max}$. However, since the procedure of curve fitting turned out to be very much simpler for $p>p_{\max}$, it was decided to adopt the same procedure throughout.

As already noted, it was found convenient to work in terms of $x(=E_s+\omega_s-E)$ rather than s. This makes the integration over the singularity at $x=0$ a fairly trivial matter.

Since it was not possible to represent $L(p,s)$ by a single function of x over its entire range (without sacrificing a considerable amount of accuracy), the latter was broken up into several suitable intervals in each of which the kernel could be represented accurately by simple functions of 'x'. It was expected that the number of intervals should increase with k in order to achieve the same degree of accuracy; therefore it was decided to take 4 intervals for $k=0.7M$, and 6 for $k=1.0M$.

The wave function $f_1(p)$ could be conveniently represented in the *corresponding* intervals by means of simple analytic functions of a similar type. As regards the *shape* of $f_1(p)$, the function $L(p,k)$ (which is just the Born approximation $f_B(p)$, apart from the factor $-\lambda(E_k\omega_k)^{\frac{1}{2}}$, was taken as a rough indication. The assumption of similarity between $L(p,k)$ and $f_1(p)$ is probably not unreasonable at moderate energies, but may not be valid for larger ones. However, as long as the number of parameters used to represent $f_1(p)$ is reasonably large, the assumption of a particular shape (like the one mentioned above) should not make any important difference, especially since $f_1(p)$ cannot have any unphysical singularity (being a wave function). For $p>p_{\max}$ it was found that the functions $L(p,s)$ as well as $L(s,k)$ could be represented very accuractely

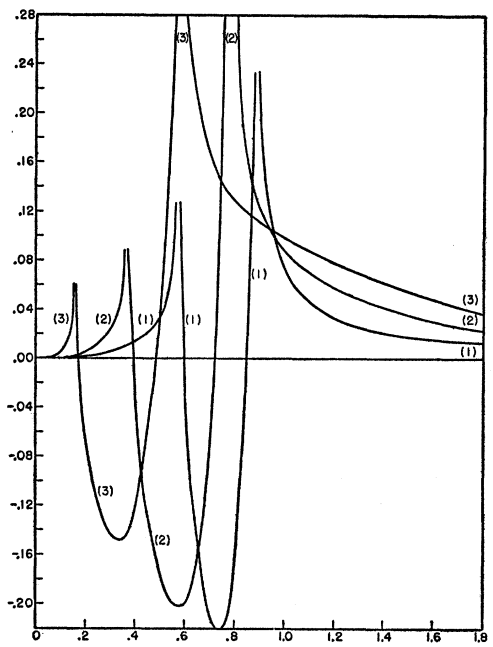

FIG. 3. $L(p,s)$ as a function of s when $p<p_{\max}$ and $k=1.0M$. Curves (1), (2), and (3) corresponding respectively to $p=0.4M$, $p=0.6M$ and $p=0.8M$.

by simple quadratic functions of x for moderate values of s and by functions of the type

$$f_1(s)=\alpha/x+\beta/x^2+\gamma/x^3 \tag{29}$$

in the asymptotic region. This conclusion was thoroughly checked for all the values of $L(p,s)$ $(p>p_{\max})$ which had been tabulated. This gave a basis for the choice of functions needed to represent $f_1(s)$.

As for $p<p_{\max}$, there were some additional complications on account of the appearance of logarithmic singularities in $L(p,s)$ at the values of s corresponding to $v=\pm 1$. To take these features into account, it was necessary to use functions which would have the correct singular behavior of $L(p,s)$ at the appropriate places. With some trial and error it was found that $L(p,s)$ and $(p<p_{\max})$ could be represented in the neighborhoods of $v=1$ and $v=-1$ in the form:

$$L(p,s)=a+bx+cx^2+(Bx+Cx^2)\ln|1+\alpha x|, \tag{30}$$

α being so chosen that the function would diverge logarithmically at the value of x corresponding to $v=+1$ or $v=-1$ as the case might be. A term like $ln|1+\alpha x|$ was not considered in (30) in order to avoid integrals involving Spence functions. The function (30) was thoroughly checked to give the correct shape of $L(p,s)$ near the singularities and the addition of a term like $A\ln|1+\alpha x|$ was not found to improve the fit in any significant way.

TABLE I. Representations of the wave function $f_1(s)$ as a function of x.

$k=0.7M$ (1.135 Bev) Interval	$f_1(s)$	$k=1.0M$ (2.128 Bev) Interval	$f_1(s)$
(1) $0\leqslant s\leqslant 0.5$	$a_1+b_1x+c_1x^2$	(1) $0\leqslant s\leqslant 0.4$	$a_2+b_2x+c_2x^2$
(2) $0.5\leqslant s\leqslant 0.9$	$a+bx+cx^2$	(2) $0.4\leqslant s\leqslant 0.8$	$a_1+b_1x+c_1x^2$
(3) $0.9\leqslant s\leqslant 1.5$	$\alpha+\beta x+\gamma x^2$	(3) $0.8\leqslant s\leqslant 1.2$	$a+bx+cx^2$
(4) $1.5\leqslant s\leqslant \infty$	$\alpha_1/x+\beta_1/x^2+\gamma_1/x^3$.	(4) $1.2\leqslant s\leqslant 1.7$	$\alpha+\beta x+\gamma x^2$
		(5) $1.7\leqslant s\leqslant 2.5$	$\alpha_1/x+\beta_1/x^2+\gamma_1/x^3$
		(6) $2.5\leqslant s\leqslant \infty$	$\alpha_2/x+\beta_2/x^2+\gamma_2/x^3$.

$L(p,s)$ was tabulated for the following values of p:

$$k=0.7M: p=0.2,\ 0.5,\ 0.7,\ 0.9,\ 1.2,\ 1.5,\ 2.0,\ 3.0;$$
$$k=1.0M: p=0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0,\ 1.2,\ 1.4,\ 1.7,\ 2.0,\ 2.5,\ 3.5,\ 5.0. \quad (31)$$

The various functions were then represented in accordance with the general scheme described above for the two cases $p<p_{\max}$ and $p>p_{\max}$.

The wave function $f_1(s)$ was represented as in Table I. The continuity conditions on $f_1(s)$ are as follows:

(1) $f_1(s)$ should be continuous at the junction of two successive intervals;

(2) $f_1(s)$ should vanish at $s=0$ (since both $f_B(p)$ and $L(p,s)$ vanish at $p=0$).

These conditions reduce the number of independent parameters in Table I to 8 and 12 respectively and these are just the number of "data" represented by (31).

The integral equation (7) was converted into a set of linear equations in the parameters representing $f_1(s)$ by carrying out the integrations with the help of (29), (30) and Table I, in a straightforward way. These equations were then solved for two different values of λ, corresponding to $G^2/(4\pi)=15$ and 10π. The results for the phase shifts so obtained are summed up in Table II below, which also includes $\tan\delta_B$.

Table II shows that the percentage increase in the phase shift over the Born approximation is small even for $G^2/(4\pi)$ as large as 10π. The quantity $\tan\delta_B$ was calculated for intermediate energies $k=0.8M$ and $k=0.9M$ as well.

Figure 4, which exhibits $(\pi\lambda)^{-1}\tan\delta_B=L(k,k)$ (independent of λ) as a function of k, shows that the quantity is strictly monotonic in k. This behavior of $L(k,k)$, together with the fact that the percentage increase of the phase shift over the Born approximation is not large, suggests strongly that the exact phase shift also increases monotomically in the entire range from $k=0.7M$ to $k=1.0M$, rather than, for example, passing through a maximum at some intermediate energy. In view of this result it was not thought necessary to calculate the exact $\tan\delta$ for the intermediate energies, but its value was obtained by a simple interpolation formula assuming that $\tan\delta/\tan\delta_B$ is linear[9] in k. The curves for $(\pi\lambda)^{-1}\tan\delta$ which are drawn in Fig. 4 are thus based on the equation $\tan\delta=\tan\delta_B(a+bk)$.

TABLE II. Calculated phase shifts.

$k=0.7M$ (1.135 Bev) $G^2/4\pi$	$\tan\delta$	$\tan\delta_B$	$k=1.0M$ (2.128 Bev) $\tan\delta$	$\tan\delta_B$	$(\tan\delta)_{\text{mod}}$
15	0.1961	0.1777	0.3762	0.3722	0.4075
10π	0.5032	0.3676	1.1164	0.7700	1.5380

The wave functions obtained from a solution of the simultaneous equations are shown in Figs. 5 and 6 for the two cases $k=0.7M$ and $k=1.0M$ respectively. The quantity actually plotted is $g(s)=-\lambda^{-1}(E_k\omega_k)^{-\frac{1}{2}}f_1(s)$. The corresponding quantity in the Born approximation (also shown) is $g_B(s)=L(s,k)$, independent of λ. This allows a direct comparison of the percentage change in the wave function for different values of λ.

Figure 5 shows that for the lower energy ($k=0.7M$) the effect of the integral equation is to increase the peak and to push the entire wave function somewhat towards higher momenta. This only means that the region of s above the energy shell plays a more important part than the Born approximation form indicates. Apart from this feature, $f_1(s)$ seems to be similar in shape to $f_B(s)$. The behavior is qualitatively the same for both the coupling constants; only the effect is more pronounced for the larger one.

The situation seems to be qualitatively different, however, for the case $k=1.0M$, as one sees from Fig. 6. Here the exact function $g(s)$ has sharp oscillations below the energy shell which are not present in $g_B(s)$. This seems to contrast sharply with the situation encountered for $k=0.7M$, where no such oscillations are perceptible. The behavior is again qualitatively similar for the two coupling constants.

It appears that the oscillations in $g(s)$ in Fig. 6 may be due to the peculiar shape of $L(p,s)$ for $p<p_{\max}$, as may be seen from Fig. 3. The importance of this region apparently increases with energy (see Figs. 5 and 6).

An explanation of these oscillations was attempted by looking for the second Born approximation which is given by:

$$f_B^{(2)}(p)=\lambda\int L(p,s)f_B(s)(ds/s). \quad (32)$$

It was expected that if the Born approximation were not too inaccurate, this quantity should approximately represent the difference between $f_1(s)$ and $f_B(s)$, and hence show large oscillations in the appropriate region. The quantitative evaluation of the integral (32) did indicate oscillations in this quantity below the energy shell, but they were found to be too small to account for the actual difference between $f_1(s)$ and $f_B(s)$. Apparently, Born approximation is too inadequate. The solution of the integral equation apparently amplifies the oscillations to a large extent (and also shifts the wave function generally to the right).

[9] This approximation is justified since the percentage increase over $\tan\delta_B$ is small, i.e., the phase shifts are far from resonance.

As regards the lower energy ($k=0.7M$), such oscillations should exist in principle, but since the region of $p<p_{\max}$ is much less important in this case, than for $k=1.0M$, they are probably too small to be detected.

5. THE COMPLEX INTEGRAL EQUATION

So far, we have considered the solution of the integral equation by neglecting the reactive effects arising from the possibility of meson production. To take these effects into account, we have to turn to the more general equation (16) which may be written as:

$$f_1(p)=f_B(p)+\lambda\int\{L(p,s)f_1(s) -L'(p,s)\phi_1(s)\}(dx/x), \quad (33)$$

$$\phi_1(p)=\lambda\int L'(p,s)f_1(s)(dx/x) +\lambda\int L(p,s)\phi_1(s)(dx/x). \quad (34)$$

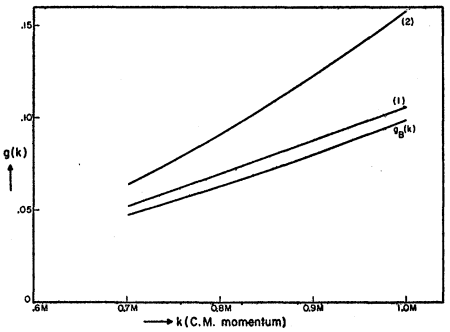

FIG. 4. The quantity $g(k)=(\pi\lambda)^{-1}\tan\delta$ as a function of k. The curves (1) and (2) correspond respectively to the values 15 and 10π for the coupling constant. $g_B(k)=L(k,k)$ is the corresponding quantity in the Born approximztion.

The coupling of these equations makes their general solution a very laborious task. If, however, we neglect the last term on the right-hand side of (33), the simplification will be considerable, since the equations will be decoupled. There are two main arguments in support of this approximation. Firstly, the quantity $L'(p,s)$ vanishes identically for $p>p_{\max}$, so that for such values of p, the Eq. (33) reduces exactly to (7). Now a considerable contribution to the solution of the integral equation comes from the region of high momenta. In fact, Figs. 5 and 6 indicate that the wave function is considerably enhanced for $p\gtrsim k$ and reduced for $p<k$, as compared with the Born approximation. Our approximation is, in fact, exact in the region from which we expect most of the contribution to the integral equation.

Secondly, it has been noted that even in the region of $p<p_{\max}$, $L'(p,s)$ can be nonzero only in a narrow strip of the $p-s$ plane, between the singularities of any of the curves of Fig. 3. Thus if $L(p,s)$ and $L'(p,s)$ are

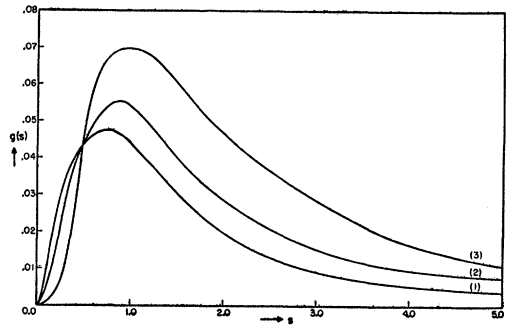

FIG. 5. The function $g(s)=-\lambda(E_k\omega_k)^{-\frac{1}{2}}f_1(s)$, for the case $k=0.7M$. Curve (1) corresponds to the Born approximation $g_B(s)=L(s,k)$. Curves (2) and (3) are for the two coupling constants of 15 and 10π respectively.

of the same orders of magnitude (in the region where L' exists) and if the same is true for $f_1(s)$ and $\phi_1(s)$, the contribution from the second term $\int L'\phi_1(dx/x)$ in the integral of (33) is expected to be small, compared with that from the first term $\int Lf_1(dx/x)$.[10]

Both the arguments tend to emphasize the importance of the term $\int Lf_1(dx/x)$ compared with $\int L'\phi_1 \times(dx/x)$ in the entire two-dimensional domain of p and s. We may therefore feel confident that our approximation will not affect the solution seriously.

With the approximation, (33) is identical with (7) and its solution can be taken over from Sec. 4. Thus in Eq. (34), the first term on the right is a known quantity, and may serve as the "inhomogeneous" term of the integral equation for $\phi_1(s)$. Moreover, the "homogeneous" part of (34) is identical with that of (7), except that $\phi_1(s)$ replaces $f_1(s)$. Thus all the results of Sec. 4 can be used quantitatively if similar assumptions are made about $\phi_1(s)$ as for $f_1(s)$.[11] All that is needed is

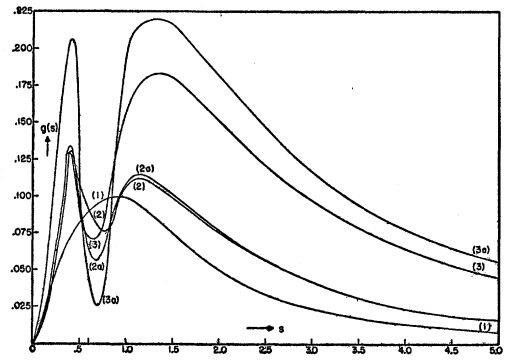

FIG. 6. The function $g(s)=-\lambda(E_k\omega_k)^{-\frac{1}{2}}f_1(s)$ for the case $k=1.0M$. Curve (1) corresponds to the Born approximation $g_B(s)=L(s,k)$. Curves (2) and (3) correspond respectively to the solutions for the two coupling constants 15 and 10π. Curves (2a) and (3a) represent modifications of (2) and (3) when the effects of meson production are taken into account.

[10] These considerations need modification near "resonance" where the relative magnitudes of $\int Lf_1(dx/x)$ and $\int L'\phi_1(dx/x)$ play a more important part than implied here. [For details, see M. Nelkin, Cornell thesis, 1955 (unpublished).]

[11] Mathematically, this simplification stems from the equality of the Green function for both the equations.

TABLE III. Values of $A+iB=-\pi(E_k\omega_k)^{-\frac{1}{2}}[f_1(k)+i\phi_1(k)]$ $=\tan(\delta+i\eta)$.

$G^2/4\pi$	$k=0.7M$ A	B	$k=1.0M$ A	B
15	0.19613	0.00425	0.39638	0.05566
10π	0.50324	0.06810	1.23990	0.60517

to evaluate the quantity:

$$\phi_B(p)=\lambda\int L'(p,s)f_1(s)(dx/x), \tag{35}$$

which plays the role of $f_B(p)$ in (7). This can be obtained by simple quadrature with $f_1(s)$ and $L'(p,s)$ (which is defined by (14) and (17)). From (26), we can deduce immediately that the parameter $y=(c^2-1-r^2)/(2r)$ is simply equal to $\cos\theta$, where θ is the angle between **p** and **s**. Thus y should lie between -1 and $+1$ for real values of θ.

Solution of Eq. (34) in a straightforward manner gave results for the complex phase shift (defined by (18)), which are listed in Table III. A comparison of the relative magnitudes of A and B shows that the latter increases fairly rapidly with energy and coupling constant. These figures serve to give a general idea of the relative importance of the effects of meson production at these energies, as predicted by the T.D. method.

For a discussion of the validity of our "decoupling" approximation, it is enough to confine our attention to $k=1.0M$, since Table III shows already that the approximation is quite good for the lower energy.

An iterative procedure was employed to check the approximation. The last term of (33) was evaluated on the basis of the solution for $\phi_1(s)$ as obtained from the simplified equations. This term was then included in $f_B(p)$ and the equation was solved for $f_1(s)$ for both the coupling constants 15 and 10π. The "modified" phase shifts so obtained are listed in Table II which shows that these quantities are larger than the previous ones. The change is small for $G^2/4\pi=15$, but quite significant ($\sim 20\%$) for $G^2/4\pi=10\pi$. However, even for the latter, the increase is not so large as to invalidate the interation procedure itself; (one might need a few more interations).

The "modified" wave functions are plotted in Fig. 6 alongside the unmodified ones. As might have been expected, the modification mostly affects the region *below* the energy shell, at least for the lower coupling constant. The effect is of course much more pronounced for the larger coupling constant, where even the region above the energy shell is significantly affected. The general tendency is to increase the importance of the region above the energy shell, indicating a somewhat stronger interaction than the real integral equation alone implies.

6. DISCUSSION OF THE RESULTS

The significant feature of the results derived in the previous sections is the total absence of any resonance over a wide range of energies and coupling constants. If the maximum in the π^--p cross section near 1 Bev were indeed due to a one-meson resonance, the value $\delta\approx 90°$ should have occurred at an energy somewhat higher than 1 Bev,[12] and the range of energies considered here would have been sufficient to locate such a point. On the contrary, the results obtained in the previous sections strongly suggest that the phase shift for the $D_{\frac{5}{2},\frac{1}{2}}$ state increases monotonically over a wide range of energies and coupling constants, without ever reaching 90°.

As shown in the introduction, the state chosen for the purpose of the present investigation seemed the most plausible one theoretically as a one-meson resonant state. It is unlikely that any other *single* meson state of the π-p system will give any better results of the type desired. Thus we are forced to conclude that it is apparently not possible to explain the second maximum in terms of single meson resonant states like $P_{\frac{3}{2},\frac{3}{2}}$.

If the $D_{\frac{5}{2},\frac{1}{2}}$ state were considered as an *isolated* problem, $G^2/4\pi$ could have been taken arbitrarily large, subject only to the condition that $G^2/4\pi<G_0^2/4\pi$ [Eq. (25a)], and it is quite conceivable that a resonance would have been obtained for a fairly large value of $G^2/4\pi$ (but less than 176.8). The more important consideration, on the other hand, is how far the coupling constant so obtained is compatible with other conditions.

Increasing evidence has been accumulating in favor of the value $G^2/4\pi\approx 15$, ever since it was determined from the $P_{\frac{3}{2},\frac{3}{2}}$ resonance.[1] A considerable theoretical basis for the interpretation of experimental results in terms of the coupling constant has been provided by a theorem of Kroll and Ruderman[13] and the recent theory of Low.[14] These questions are discussed in detail in Bethe's forthcoming book. It now appears that even a value of 20 for the coupling constant is too high, and we are certainly not justified in considering any value greater than 10π.

For a direct comparison of the experimental results with the theory, we have evaluated the cross sections [using the formulas (20) and (21)] which are described in Table IV.

TABLE IV. Theoretical cross sections in mb.

	$k=0.7M$ (lab energy = 1.135 Bev) $G^2/4\pi=15$	$G^2/4\pi=10\pi$	$k=1.0M$ (lab energy = 2.128 Bev) $G^2/4\pi=15$	$G^2/4\pi=10\pi$
$\sigma_{el}(\pi^-p)$	0.834	4.196	1.400	5.166
$\sigma_{in}(\pi^-p)$	0.092	1.108	0.4866	1.635

[12] This is because of the factor χ^2 in the cross section.
[13] N. M. Kroll and M. Ruderman, Phys. Rev. **93**, 233 (1954).
[14] F. Low, Rochester Conference Proceedings, 1955 (unpublished).

TABLE V. Experimental cross sections.

Process ($\pi^-+p\rightarrow$)	Cross sections (mb) 1.0 Bev	1.5 Bev
1. Diffraction scattering	14	7
2. Elastic (large angle)	5	2
3. Inelastic processes	28–33	26

The experimental curves for the total cross sections[14,15] show a maximum of about 50 mb at 1 Bev in π^--p scattering. The π^+-p scattering at this energy is about 30 mb. From this one estimates the $T=\frac{3}{2}$ contribution to π^--p scattering to be 10 mb. This leaves about 40 mb to be explained by the state $T=\frac{1}{2}$. Unfortunately the values given in Table IV are much too small to account for this magnitude.

It may be worthwhile analyzing this large discrepancy in terms of elastic and inelastic contributions. According to Table IV, the reactive cross section is rather small compared with the elastic contribution, though the former increases with energy. However, even at $k=1.0M$, the ratio $\sigma_{\text{in}}/\sigma_{\text{el}}$ is only $\sim\frac{1}{3}$. The various contributions to the cross sections estimated from the experiments of Walker,[16] Piccioni,[17] and others are given in Table V.

Comparison of these figures with those in Table IV shows that the discrepancy is very much larger for the inelastic processes than for elastic ones. The calculated ratio $\sigma_{\text{in}}/\sigma_{\text{el}}$ is too small to account for the observed ratio of ~1 at 1 Bev and ~2 at 1.5 Bev.

It may be noted from Table V that most of the contribution to the elastic events comes from diffraction scattering (confined to small angles). The large angle scatterings are mostly inelastic. Now a large number of angular momentum states make comparable contributions to diffraction scattering, so that even large l-values are important for this process. On the other hand, we have assumed only one l value ($l=2$), so that this process has not been adequately treated in our theory.

It has been suggested by the Brookhaven group that a possible explanation may be based on a two-meson interaction. As the experiments indicate, the cross section for the production of a meson increases very rapidly with energy. If the total available energy is shared in comparable amounts by the various particles concerned, the conditions may be such that around 1 Bev, one or both the mesons are in a resonant $P_{\frac{3}{2},\frac{3}{2}}$ state with the nucleon. Walker's[16] measurements of the momentum distributions of the scattered mesons at 1 Bev and 1.5 Bev, respectively, throw some valuable light on this point. His analysis shows only one peak (at $p\sim350$ Mev/c) for the 1-Bev mesons and two distinct peaks (at $p\sim300$ Mev/c, $p\sim600$ Mev/c) for the 1.5-Bev mesons. This gives support to the argument that at 1 Bev, both the mesons are at the same resonant state with the nucleon. Assuming therefore a $P_{\frac{3}{2},\frac{3}{2}}$ resonance, it follows that at 1 Bev, one essentially gets the *square* of the enhancement factor which comes from the $P_{\frac{3}{2},\frac{3}{2}}$ state, and this is probably enough to explain the large magnitude of the cross section at 1 Bev.

One difficulty about this explanation is that there is no simple reason why this enhancement will affect only the $T=\frac{1}{2}$ state and not $T=\frac{3}{2}$. According to Ross,[18] the $P_{\frac{3}{2},\frac{3}{2}}$ state might still account for the peak at 1 Bev, but only if the subsequent scatterings of *both* the mesons by the nucleon are important.

An explanation which avoids the enhancement in the $T=\frac{3}{2}$ state has been recently proposed by Dyson.[19] According to him, a "particle" with a strong interaction with the nucleon is probably formed when the π-meson strikes the target. This particle which is assumed to have a mass of about $1000m_e$ and an isotopic spin $T=0$ decays into two π mesons with an unobservably short lifetime. This picture can explain why the $T=\frac{1}{2}$ scattering can be very large in several angular momentum states simultaneously, without affecting the $T=\frac{3}{2}$ state. It may be noted in this connection that a calculation performed some time ago[20] indicated an attractive interaction between two mesons in a state $T=0$. It is therefore suggested that a more detailed study of this interaction may prove useful in this connection.

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[16] W. Walker, Proceedings of the Fifth Annual Rochester Conference on High Energy Physics, 1955 (to be published).
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[18] M. Ross (private communication).
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[20] A. N. Mitra and F. J. Dyson, Phys. Rev. **90**, 372(A) (1953).