described by a normalizable state vector. An example of a calculation using non-normalizable state vectors which violates the above theorem is the work of Sachs and Foldy.[7] These authors calculate the cross section for the scattering of gamma rays from nucleons using nonrelativistic, no-recoil, weak coupling, pseudoscalar meson theory. The state vector representing the nucleon ground state in their calculation is not normalizable in the meson variables. Because of this lack of normalizability, their calculated transition amplitude contains a term [the $\boldsymbol{\sigma}\cdot\mathbf{u}\times\mathbf{u}'$ term in Eq. (20) of reference 4], which results from scattering through excited states of the nucleon, and which is independent of photon energy in the low-energy limit. Therefore, this term violates the above stated theorem. In the following paper, the procedure of Sachs and Foldy is modified in a manner which insures that the state vectors are normalizable, and the cross section is recalculated. The transition amplitude calculated by this procedure no longer contains this energy-independent term.

[7] R. G. Sachs and L. L. Foldy, Phys. Rev. 80, 824 (1950).

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Electromagnetic Properties of the Nucleon in a Finite Source Theory*

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Some electromagnetic properties of nucleons are investigated under the assumption that the nucleon consists of a spin one-half core particle in interaction with the pion field through a gauge-invariant, nonrelativistic, pseudoscalar, finite source interaction. Recoil of the nucleon core is neglected and the weak coupling approximation is used. A method is presented for making the finite source interaction between two particle fields gauge-invariant by introducing filaments of current in the region of the source function. The theoretical contributions of interaction currents to the magnetic moments of the proton and neutron are calculated and found to be small compared to the observed anomalous moments and of the opposite sign. The model is used to calculate, in the limit of zero source size, the total and differential cross sections for scattering of gamma rays from nucleons. The results, which are presented graphically, are found to be in accord with the conditions imposed on a finite theory in the preceding paper. They are compared to the results obtained by Sachs and Foldy on the basis of a point-source theory.

I. INTRODUCTION

The recent discoveries[1] of heavy mesons and hyperons indicate that it is entirely possible that the nucleon core which emits and absorbs pions, instead of being simple, may be structured, i.e., composed of two or more of the "new" particles, one or more of which may carry the source of the pion field. A complete theory of the nucleon would have to incorporate the details of the structured core and would then depend on the nature of the particles which compose this structure.

However, in many processes involving low-energy pions and photons, the complicated details of the source may not be important and therefore it may be possible to account for the important effects of the core structure by introducing a smooth source function into the interaction between the pion field and its source. This means that the annihilation and creation of pions are not limited to a mathematical point in space, but can occur over a small but finite region, the size of which is determined by the dimensions of the source function.

The notion of the extended source has already been introduced,[2] not for the physical reasons just stated, but for the formal advantage that the extended source possesses in regard to circumventing the well-known divergence difficulties that are encountered in the point source theory. An important result of the introduction of the extended source is that high-momentum pions are quenched, so that the state vector of the pion-nucleon system, which was not normalizable in the point-source theory, can now be normalized. The advantage of working with normalizable state vectors has been emphasized in the preceding paper, in which many of the results depend on this property.

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[1] For up-to-date reviews of these matters see: L. Leprince-Ringuet, Ann. Rev. Nuc. Sci. 3, 39 (1953); Many authors, Proc. Roy. Soc. (London) A221, 277–420, (1954); Proceedings of the Fourth Rochester Conference (University of Rochester Press, Rochester, 1954).

[2] W. Pauli and S. M. Dancoff, Phys. Rev. 62, 85 (1942). W. Pauli, *Meson Theory of Nuclear Forces* (Interscience Publishers, Inc., New York, 1946).

It is clear, then, that certain advantages accrue with the advent of the extended source, but on the other hand, the extended source theory raises new problems of its own. As generally applied, this theory is not Lorentz-covariant. But this fact causes no great concern here since the point of view is taken that the extended source theory is not supposed to be a fundamentally rigorous theory but a way to avoid the complicated details of the nucleon core, a procedure which, it is hoped, has validity at least in low-energy processes.

Furthermore, there is the question of making the theory gauge-invariant when the interaction of the pion-nucleon system with photons is considered. As pointed out by Blair *et al.*,[3] the essential difficulty lies in the fact that in the present form of the extended source theory, the pion is created or absorbed at some particular point inside the source, whereas the entire spread-out nucleon changes from a neutron to a proton or *vice versa*, a process which violates the continuity equation of charge and current density. One of the important features of the present paper is the delineation of the manner in which an extended source, pion-nucleon interaction may be made gauge-invariant. The procedure involves the introduction of filaments of current within the source so that the continuity equation is satisfied everywhere. These filaments of current are not to be construed as accurately representing the physical situation inside the source, but merely as some sort of approximation to the actual currents that presumably exist there. Nevertheless, some physical consequences of these "pseudocurrents" are presented. Consideration is given to a peculiar effect that arises from the virtual photons generated by these currents. The contribution of these currents to the nucleon magnetic moments is deduced and found to be zero for the simplest possible kind of filaments, namely, straight lines of current emanating radially from the origin. However, a contribution to the magnetic moment[4] remains from the interaction current that arises in the gradient-type coupling. This effect is found to be small and in the opposite direction to the observed anomalies.

Attention is then given to the implications of the static, gradient-type, extended-source interaction for the differential and total cross sections for the scattering of γ rays by protons and neutrons. The point source in the pion-nucleon interaction employed in the calculation of this process by Sachs and Foldy[5] is replaced by a finite source and the cross sections recalculated. The SF result for the low-energy expression of the transition amplitude contains a spin-dependent term which is independent of the photon frequency. This violates the theorem of Sec. V of the preceding paper, a result which depends on the fact that the point-source theory of SF leads to non-normalizable state vectors. The violation is found not to occur in the normalizable, extended-source theory, even in the limit as the source size is allowed to approach zero. The numerical results given here are obtained in this limiting case, so that the finite source is used here only as a convergence procedure. It does not represent a physical smearing out of the core particle.

Two other modifications of the procedure used in SF are made. First, the absorptive part of the scattering, omitted by SF, is included. Second, the pion-nucleon coupling constant is obtained from experiments performed since the publication of SF. The discussion of the γ-ray scattering problem in the present paper is an amplification of some remarks published previously.[6]

II. GAUGE INVARIANCE OF THE INTERACTION

The Hamiltonian of a static pion field interacting with an infinitely heavy source can be separated into two parts:

$$H=H_f+H',$$

where H_f is the total energy of the free pion field,[7] and H' is the interaction energy between the pions and the source. For pseudoscalar pions interacting in a charge symmetric way with an extended, spin $\frac{1}{2}$, isotopic spin $\frac{1}{2}$ source, a rather commonly used[2] expression for H' is

$$H'=(4\pi)^{\frac{1}{2}}(gc/\mu)\int d^3r f(r)\tau_+\boldsymbol{\sigma}\cdot\nabla\varphi(\mathbf{r})+\text{h.c.},\qquad(1)$$

where g is the coupling constant in units of electric charge, μ the reciprocal pion Compton wavelength, $f(r)$ the spherically symmetric source function normalized so that $\int f(r)d^3r=1$, $\varphi(\mathbf{r})$ a component of the charged pion field which creates a negative or absorbs a positive pion, and $\tau_+=\frac{1}{2}(\tau_1+i\tau_2)$ is an isotopic spin operator which converts a neutral into a positively charged source. Only charged pions have been included in H', since the major concern here is the behavior of the system in the presence of an electromagnetic field, with which neutral pions presumably have no direct interaction.

The importance of introducing an electromagnetic field into the Hamiltonian in such a way as to satisfy the condition of gauge invariance has been emphasized in the preceding paper. If $H(\mathbf{A})$ designates the matter Hamiltonian modified to take into account the interaction of matter with photons, the gauge condition of the preceding paper becomes, for the particular Hamiltonian under consideration,

$$\begin{aligned}e^{iD}H(\mathbf{A})e^{-iD}&=e^{iD}[H_f(\mathbf{A})+H'(\mathbf{A})]e^{-iD}\\&=H_f(\mathbf{A}+\text{grad }G)+H'(\mathbf{A}+\text{grad }G),\end{aligned}\qquad(2)$$

[3] Blair, Chew, Friedman, and Salzman (privately circulated report, 1952).

[4] No consideration is given in this paper to the magnetic moments that derive from the conventional pion convection currents.

[5] R. G. Sachs and L. L. Foldy, Phys. Rev. **80**, 824 (1950). Hereinafter, this paper will be referred to by the symbol SF. Numbered equations in SF will be referred to as Eq. (SF-1), etc.

[6] R. H. Capps and R. G. Sachs, Phys. Rev. **96**, 540 (1954).

[7] For the form of H_f, see Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949), p. 49.

in which $G(\mathbf{r})$ is a guage function which may be any electromagnetic field operator that commutes with $\mathbf{A}$. The expression for D is

$$D=(1/\hbar c)\int[\rho_\pi(\mathbf{r})+\tfrac{1}{2}(1+\tau_3)e\delta(\mathbf{r})]G(\mathbf{r})d^3r, \quad (3)$$

where $\rho_\pi(\mathbf{r})$ is the charge density operator for the pion field. The second term in Eq. (3) is concerned with the charge on the source.

It is of interest to determine the form of $H(\mathbf{A})$ satisfying Eq. (2). The H_f part can be made gauge-invariant by the well-known prescription,[8] of replacing each grad φ and grad φ^* by $[\text{grad}-(ie/\hbar c)\mathbf{A}]\varphi$ and $[\text{grad}+(ie/\hbar c)\mathbf{A}]\varphi^*$, respectively. However, because of the presence of the extended source, H' modified only in this way does not satisfy the gauge condition, but requires an additional factor which[9] can assume the form

$$\exp\left[\pm(ie/\hbar c)\int_0^{\mathbf{r}} A_s ds\right], \quad (4)$$

the argument of the exponential being a line integral of the vector potential from the origin to the point in the source where the pions are created or absorbed. The proper modification of H' in the presence of an electromagnetic field then is

$$H'(\mathbf{A})=(4\pi)^{\frac{1}{2}}(gc/\mu)\int d^3r\biggl\{f(r)\tau_+ \times\exp\left[(-ie/\hbar c)\int_0^{\mathbf{r}} A_s ds\right]\boldsymbol{\sigma} \cdot[\nabla-(ie/\hbar c)\mathbf{A}]\varphi\biggr\}+\text{h.c.} \quad (5)$$

The fact that this expression for $H'(\mathbf{A})$ satisfies Eq. (2) with D given by Eq. (3) follows from the statements

$$e^{iD}\tau_\pm e^{-iD}=\tau_\pm \exp[(\pm ie/\hbar c)G(0)], \quad (6)$$

$$e^{iD}\{\boldsymbol{\sigma}\cdot[\nabla-(ie/\hbar c)\mathbf{A}]\varphi\}e^{-iD}=\exp[(-ie/\hbar c)G(\mathbf{r})] \times\boldsymbol{\sigma}\cdot[\nabla-(ie/\hbar c)(\mathbf{A}+\text{grad } G)]\varphi(\mathbf{r}). \quad (7)$$

Just as the extended source is a phenomenological device to make a complex system susceptible to calculations yielding finite results, so the exponential of the vector potential in $H'(\mathbf{A})$ is a convenient means of preserving the gauge invariance of the theory, i.e., the effect of the exponential is to replace the presumably complicated current distribution inside the source by a more simple current distribution, which nevertheless keeps the continuity equation of charge and current density satisfied. Indeed, the physical effect of this added factor is best seen through an examination of the current distribution that it produces.

Since the small currents produced by external fields are of no interest, the part of the current independent of the electromagnetic field is desired. When $H_1'(\mathbf{A})$, the term in $H'(\mathbf{A})$ linear in $\mathbf{A}$, is cast into the form

$$H_1'(\mathbf{A})=-c^{-1}\int \mathbf{j}_{\text{int}}\cdot\mathbf{A}d^3r, \quad (8)$$

the relevant current density can be identified as the quantity which is denoted by $\mathbf{j}_{\text{int}}$ in the expression.

The current defined in this way satisfies Maxwell's equations, and is referred to as the interaction current, since it arises from the special form of the interaction H'. To obtain the part of $H'(\mathbf{A})$ linear in $\mathbf{A}$, the exponential in Eq. (5) is expanded and the linear terms are found to be

$$H_1'(\mathbf{A})=-(4\pi)^{\frac{1}{2}}(ige/\mu\hbar)\int d^3rf(r) \times\left[\left(\int_0^{\mathbf{r}} A_s ds\right)(\tau_+\boldsymbol{\sigma}\cdot\nabla\varphi-\tau_-\boldsymbol{\sigma}\cdot\nabla\varphi^*) +(\tau_+\boldsymbol{\sigma}\cdot\mathbf{A}\varphi-\tau_-\boldsymbol{\sigma}\cdot\mathbf{A}\varphi^*)\right]. \quad (9)$$

Now the arbitrary line integral from the origin to the point $\mathbf{r}$ may be rewritten in the form,

$$\int_0^{\mathbf{r}} A_s ds=\int_0^{\mathbf{r}}\left[d\mathbf{s}\cdot\int d^3\mathbf{r}'\mathbf{A}(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{s})\right], \quad (10)$$

where $\mathbf{s}$ represents a vector to the point on the line and $d\mathbf{s}$ is an infinitesimal vector tangent to the line at the endpoint of the vector $\mathbf{s}$.

Then

$$H_1'(\mathbf{A})=-(4\pi)^{\frac{1}{2}}(ige/\mu\hbar)\int d^3r'\biggl(\mathbf{A}(\mathbf{r}')\cdot\int d^3rf(r) \times\biggl\{\left[\int_0^{\mathbf{r}} d\mathbf{s}\delta(\mathbf{r}'-\mathbf{s})\right][\tau_+\boldsymbol{\sigma}\cdot\nabla\varphi(\mathbf{r})-\tau_-\boldsymbol{\sigma}\cdot\nabla\varphi^*(\mathbf{r})] +\boldsymbol{\sigma}\delta(\mathbf{r}'-\mathbf{r})[\tau_+\varphi(\mathbf{r})-\tau_-\varphi^*(\mathbf{r})]\biggr\}\biggr). \quad (11)$$

Comparison with Eq. (8) shows that the current is

$$\mathbf{j}_{\text{int}}(\mathbf{r}')=(4\pi)^{\frac{1}{2}}(igec/\mu\hbar)\int d^3rf(r)\biggl\{\left[\int_0^{\mathbf{r}} d\mathbf{s}\delta(\mathbf{r}'-\mathbf{s})\right] \times[\tau_+\boldsymbol{\sigma}\cdot\nabla\varphi(\mathbf{r})-\tau_-\boldsymbol{\sigma}\cdot\nabla\varphi^*(\mathbf{r})] +\boldsymbol{\sigma}\delta(\mathbf{r}'-\mathbf{r})[\tau_+\varphi(\mathbf{r})-\tau_-\varphi^*(\mathbf{r})]\biggr\}. \quad (12)$$

[8] See reference 7, p. 66.

[9] This form was suggested to the authors by R. G. Sachs. Compare R. G. Sachs, Phys. Rev. **74**, 433 (1948). Factors of this kind were also considered by Peierls and Chretien with regard to nonlocal interactions in field theory, R. Peierls and M. Chretien, Proc. Roy. Soc. (London) **A233**, 468 (1954). See also C. Bloch, Kgl. Danske Videnskab Selskab, Mat.-fys Medd. **27**, 8 (1952).

This is just the current that comes from $H'(\mathbf{A})$, no consideration having been given to current that arises from $H_f(\mathbf{A})$. The first part of the expression, Eq. (12), derives from the exponential term in Eq. (5) and can be interpreted as a line of current[10] flowing between the origin and the point in the source where the pions are created or destroyed. Once again the phenomenological nature of this current should be pointed out. It doubtless does not correctly represent the current distribution within the source and has been inserted for the sake of keeping inviolate the law of conservation of electric charge within the source. The second term in Eq. (12) is the well-known current given previously by Pauli and Dancoff.[2]

III. UNIQUENESS OF THE ELECTROMAGNETIC INTERACTION

In the absence of an electromagnetic field, the form of a linear, charge-symmetric, static interaction of pseudoscalar pions with an infinitely heavy, extended, spin $\frac{1}{2}$, isotopic spin $\frac{1}{2}$ source is given uniquely by Eq. (1). An alternative expression[3] is

$$H'_{\mathrm{alt}}=(4\pi)^{\frac{1}{2}}(gc/\mu)\int d^3rF(r)\tau_+\varphi(\mathbf{r})\boldsymbol{\sigma}\cdot\mathbf{r}+\mathrm{h.c.}, \quad (13)$$

but H'_{alt} can be obtained from H' by an integration by parts of the latter expression, which shows that $H'=H'_{\mathrm{alt}}$ if

$$F(r)=(1/r)(df/dr).$$

In the presence of an electromagnetic field, however, $H'_{\mathrm{alt}}(A)$, which has the gauge invariant form,

$$H'_{\mathrm{alt}}(A)=(4\pi)^{\frac{1}{2}}(gc/\mu)\int d^3r\bigg\{F(r)\tau_+\varphi(\mathbf{r})\boldsymbol{\sigma}\cdot\mathbf{r}\times\exp\bigg[(-ie/\hbar c)\int_0^{\mathbf{r}}A_sds\bigg]\bigg\}+\mathrm{h.c.}, \quad (14)$$

is not equivalent to $H'(\mathbf{A})$. This can be seen by an integration by parts of $H'(\mathbf{A})$, which gives

$$H'(\mathbf{A})=-(4\pi)^{\frac{1}{2}}(gc/\mu)\int d^3r\bigg\{\tau_+\varphi(\mathbf{r})\boldsymbol{\sigma}\cdot\bigg[(\mathbf{r}/r)(df/dr)+(ie/\hbar c)f(r)\bigg(\mathbf{A}-\mathrm{grad}\int_0^{\mathbf{r}}A_sds\bigg)\bigg]\times\exp\bigg[(-ie/\hbar c)\int_0^{\mathbf{r}}A_sds\bigg]\bigg\}+\mathrm{h.c.} \quad (15)$$

For purposes of simplification, the line integral is taken along a straight line. Then

$$\boldsymbol{\sigma}\cdot\nabla\int_0^{\mathbf{r}}A_sds=\boldsymbol{\sigma}\cdot\nabla\int_0^1 d\alpha[\mathbf{r}\cdot\mathbf{A}(\alpha\mathbf{r})]=\boldsymbol{\sigma}\cdot\bigg\{\mathbf{A}(\mathbf{r})+\int_0^1 d\alpha[\mathbf{r}\times\mathrm{curl}\mathbf{A}(\alpha\mathbf{r})]\bigg\}, \quad (16)$$

whereupon

$$H'(\mathbf{A})=-(4\pi)^{\frac{1}{2}}(gc/\mu)\int d^3r\bigg\{\boldsymbol{\sigma}\cdot\bigg[(\mathbf{r}/r)(df/dr)-(ie/\hbar c)f(r)\int_0^1\mathbf{r}\times\mathrm{curl}\mathbf{A}(\alpha\mathbf{r})d\alpha\bigg]\exp\bigg[(-ie/\hbar c)\times\int_0^1\mathbf{A}(\alpha\mathbf{r})\cdot\mathbf{r}d\alpha\bigg]\tau_+\varphi(\mathbf{r})\bigg\}+\mathrm{h.c.} \quad (17)$$

The nonequivalence of $H'(\mathbf{A})$ and $H'_{\mathrm{alt}}(\mathbf{A})$ then resides in the presence of the term in $H'(\mathbf{A})$ containing curl$\mathbf{A}$. Whether or not such a term exists must be decided by experiment.

Any linear combination of $H'(\mathbf{A})$ and $H'_{\mathrm{alt}}(\mathbf{A})$ will satisfy the conditions of gauge invariance. This is a manifestation of the fact that any term of the form $H(\mathrm{curl}\mathbf{A})$ that satisfies the condition $[\rho,H(\mathrm{curl}\mathbf{A})]=0$, where ρ is the charge density operator for the matter system, may be added to a gauge-invariant Hamiltonian without destroying gauge invariance.

IV. VIRTUAL PHOTONS

Assuming for the moment that $H'(\mathbf{A})$ does give a correct account of the current distribution within the source, let us investigate some of its electromagnetic consequences. First we consider the effects of virtual photons produced by the vector potential in the exponential factor of $H'(\mathbf{A})$. Usually such photons, arising from the interaction of charged particles with the electromagnetic field, cause self-energy divergences such as that encountered, for instance, in the $A^2\varphi^2$ term of the gauge invariant form of H_f.

In contrast to this relatively simple A^2 dependence, arbitrarily high powers of $\mathbf{A}$ are present in the exponential under study, so that the divergence arises here in a quite complicated manner. If all virtual photons are included in a calculation, every matrix element of $H'(\mathbf{A})$ will include a factor of $e^{-\infty}$, which will cause the matrix elements to vanish.[11] This can be seen in the following way. Suppose, for the sake of simplicity, that the line integral is taken along a straight line; then an expansion of the vector potential into plane waves

[10] A filament of current of a very similar nature has been introduced by Adams in connection with nuclear exchange currents. E. N. Adams II, Phys. Rev. **81**, 1 (1951). See also R. K. Osborn and L. L. Foldy, Phys. Rev. **79**, 795 (1950).

[11] This point has been observed by Heitler and communicated to us by W. Thirring. See also R. Peierls and M. Chretien, Proc. Roy. Soc. (London) **A223**, 468 (1954).

yields

$$\exp\left[(\pm ie/\hbar c)\mathbf{r}\cdot\int_0^1 A(\alpha\mathbf{r})d\alpha\right]$$

$$=\exp\left\{(\pm ie/\hbar c)(\hbar c2\pi/V)^{\frac{1}{2}}\sum_{u,\omega}\left[(A_{\omega,u}+A^*_{-\omega,u})\right.\right.$$

$$\left.\left.\times\omega^{-\frac{1}{2}}\mathbf{u}\cdot\mathbf{r}\int_0^1\exp(i\boldsymbol{\omega}\cdot\alpha\mathbf{r})d\alpha\right]\right\}, \quad (18)$$

in which $\mathbf{u}$ is the photon polarization, $\boldsymbol{\omega}$ is its wave vector, A and A^* are photon creation and annihilation operators, and V is the volume of an enclosure used for normalization of the photon waves. The effect of the virtual photons can be isolated by arranging the terms in Eq. (18) in such a way that the annihilation operators act first. To accomplish this, use is made of the theorem[12] for two operators, a and b, whose commutator is a c-number,

$$\exp(a+b)=(\exp a)(\exp\tfrac{1}{2}[b,a])(\exp b). \quad (19)$$

If a is identified with the A^* part and b with the A part of the expression, Eq. (18), this expression may be written in the form of the right side of Eq. (19). If the sum over the two directions of polarization is carried out, the terms corresponding to $\exp(\frac{1}{2}[b,a])$ may be written in the following form,

$$\exp\left((-e^2/4\pi\hbar c)\int_{-1}^1 dx\left\{(1-x^2)\int_0^\infty \rho(2/\rho x)^2\right.\right.$$

$$\left.\left.\times[\sin^2(\rho x/2)]d\rho\right\}\right), \quad (20)$$

where $\rho=\omega r$. The integral over ρ in this expression diverges logarithmically at infinity causing $H'(\mathbf{A})$ to vanish.

Of course, one way to circumvent this predicament in calculations involving real photon processes is to ignore the effect of virtual photons altogether, i.e., replace the expression, Eq. (20) by one. Perhaps a more physically meaningful approach to the removal of the divergence under study is to note that, for extremely short wavelengths, the details of the structure of the source would become important. The consideration of such details would not be consistent with the notion that the complexities of the source could be replaced by the smooth source function $f(r)$. Therefore, the photon wavelengths should be smaller that the dimensions of the source and thus should be limited by the condition

$$\omega r\leqq 1. \quad (21)$$

[12] Harold T. Davis, *The Theory of Liner Operators* (Principia Press, Inc., Bloomington, 1936), p. 198.

Application of this condition limits the integration over ρ in the expression, Eq. (20) to the range $0\leqq\rho\leqq 1$, and Eq. (20) now becomes $\exp(-e^2/6\pi\hbar c)$. This expression is very nearly unity and the vanishing of $H'(\mathbf{A})$ no longer occurs. It is concluded that the formally gauge invariant expression Eq. (5) for $H'(\mathbf{A})$ can be employed in a perturbation calculation of electromagnetic processes for which the condition Eq. (21) is satisfied.

V. MAGNETIC MOMENTS

Having given consideration to the effect of virtual photons, we now investigate some of the implications of $H'(\mathbf{A})$ with regard to real photon processes. Attention will first be given to the contribution of $H'(\mathbf{A})$ to the static nucleon magnetic moments. The moment will be calculated to lowest order in g, i.e., to order g^2. The magnetic moment operator is defined by

$$M_z(\text{int})=\tfrac{1}{2}c^{-1}\int d^3r'[\mathbf{r}'\times\mathbf{j}_{\text{int}}(\mathbf{r}')]_z, \quad (22)$$

where $\mathbf{j}_{\text{int}}(\mathbf{r}')$ is the interaction current density. The insertion of Eq. (12) for the current density yields

$$M_z(\text{int})=(i\pi^{\frac{1}{2}}ge/\mu\hbar)\int\int d^3r'd^3rf(r)\mathbf{i}_z$$

$$\cdot\left(\mathbf{r}'\times\left\{\left[\int_0^{\mathbf{r}} d\mathbf{s}\delta(\mathbf{r}'-\mathbf{s})\right]\right.\right.$$

$$\times[\tau_+\boldsymbol{\sigma}\cdot\nabla\varphi(\mathbf{r})-\tau_-\boldsymbol{\sigma}\cdot\nabla\varphi^*(\mathbf{r})]$$

$$\left.\left.+\boldsymbol{\sigma}\delta(\mathbf{r}'-\mathbf{r})[\tau_+\varphi(\mathbf{r})-\tau_-\varphi^*(\mathbf{r})]\right\}\right), \quad (23)$$

where $\mathbf{i}_z$ denotes a unit vector in the positive z direction.

Note that $M_z(\text{int})$ changes sign under the mirror operation in charge space, the operation which reflects the i-spin space through a plane containing the z axis, i.e., $M_z(\text{int})\leftrightarrow -M_z(\text{int})$ when $\tau_+\leftrightarrow\tau_-$, $\varphi\leftrightarrow\varphi^*$ and $\tau_z\leftrightarrow -\tau_z$. But under the assumption that the interaction between mesons and nucleons is charge symmetric, this operation changes a neutron state to a proton state and *vice versa*. Consequently, the expectation value $\langle M_z(\text{int})\rangle$ is equal in magnitude but opposite in sign for the proton as compared to the neutron. Thus, the mirror theorem of Sachs,[13] concerning the sum of neutron and proton moments, is not affected by any contribution from these interaction currents.

To obtain an idea of the size of $\langle M_z(\text{int})\rangle$, we consider the simplest form of $M_z(\text{int})$, which corresponds to choosing the path of the line integral to be a straight

[13] R. G. Sachs, Phys. Rev. **87**, 1100 (1952).

line. Then,

$$M_z(\mathrm{int}) = (i\pi^{\frac{1}{2}} ge/\mu\hbar) \int\int d^3r' d^3r f(r) \mathbf{i}_z \cdot \left(\mathbf{r}' \times \left\{ \left[\mathbf{r} \int_0^1 \delta(\mathbf{r}' - \alpha\mathbf{r}) d\alpha \right] \times [\tau_+ \boldsymbol{\sigma} \cdot \nabla\varphi(\mathbf{r}) - \tau_- \boldsymbol{\sigma} \cdot \nabla\varphi^*(\mathbf{r})] + \boldsymbol{\sigma}\delta(\mathbf{r}' - \mathbf{r})[\tau_+\varphi(\mathbf{r}) - \tau_-\varphi^*(\mathbf{r})] \right\} \right). \quad (24)$$

Because of the factor $\delta(\mathbf{r}' - \alpha\mathbf{r})$, $\mathbf{r}'$ and $\mathbf{r}$ are always parallel and $\mathbf{r}' \times \mathbf{r}$ vanishes. This result is simply a manifestation of our particular choice of the path of integration, which constrains the line current to flow radially, whereas a circulating current is required to produce a magnetic moment. Thus, the contribution to the magnetic moment of the phenomenological line current has been made zero, so that

$$M_z(\mathrm{int}) = (i\pi^{\frac{1}{2}} ge/\mu\hbar) \times \int d^3r [f(r)(\mathbf{r} \times \boldsymbol{\sigma})_z (\tau_+\varphi - \tau_-\varphi^*)], \quad (25)$$

which is the result that would be obtained[2] even if the line currents or the exponential factor had not been introduced.

To find $\langle M_z \rangle$ to order g^2, the ground-state eigenfunction of H to order g is needed. This eigenfunction for a nucleon with spin up is, to the required order

$$\Phi = \Phi\{0\} + \Phi\{1\} = \Phi\{0, \sigma_z = 1, \tau_z = \pm 1\} + \sum_{\substack{k,\ \mathrm{spin,} \\ \mathrm{isotopic\ spin}}} \frac{\langle \Phi\{\mathbf{k}, \sigma_z, \tau_z\} | H' | \Phi\{0, \sigma_z = 1, \tau_z = 1\} \rangle \times \Phi\{\mathbf{k}, \sigma_z \tau_z\}}{-\hbar c(\mu^2 + k^2)^{\frac{1}{2}}}, \quad (26)$$

where $\Phi\{0, \sigma_z = 1, \tau_z = \pm 1\}$ is the functional describing the state with no pions present, namely, just the core of the nucleon, with $\tau_z = \pm 1$ referring to the positive and neutral cores respectively. Similarly, $\Phi(\mathbf{k}, \sigma_z \tau_z)$ represents the core surrounded by one pion in a state of momentum $\hbar\mathbf{k}$. Since the calculation is carried out in momentum space, we record the operators H' and $M_z(\mathrm{int})$ in that representation

$$H' = i(gc/\mu)(2\pi\hbar/cV)^{\frac{1}{2}} \sum_{\mathbf{k}} \{ f(k) \boldsymbol{\sigma} \cdot \mathbf{k} (\mu^2 + k^2)^{-\frac{1}{4}} \times [(a_{\mathbf{k}} + b_{-\mathbf{k}}^*)\tau_+ + (a_{-\mathbf{k}}^* + b_{\mathbf{k}})\tau_-] \}, \quad (27)$$

$$M_z(\mathrm{int}) = -(ge/\mu)(\pi/2\hbar cV)^{\frac{1}{2}} \sum_{\mathbf{k}} \{ (k^{-1} df/dk) \times (\mu^2 + k^2)^{-\frac{1}{4}} (\mathbf{i}_z \cdot \boldsymbol{\sigma} \times \mathbf{k}) \times [(a_{\mathbf{k}} + b_{-\mathbf{k}}^*)\tau_+ - (a_{-\mathbf{k}}^* + b_{\mathbf{k}})\tau_-] \}, \quad (28)$$

where $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^*$ are the annihilation and creation operator's for positive mesons and the b's refer similarly to negative mesons. The function $f(k)$ is the Fourier transform of the spherically symmetric source function $f(r)$. Then,

$$\langle M_z(\mathrm{int}) \rangle = \langle \Phi | M_z | \Phi \rangle = \mp \frac{g^2 e}{3\pi\hbar c\mu^2} \int_0^\infty dk f^2(k) \frac{d}{dk} \left(\frac{k^3}{\mu^2 + k^2} \right), \quad (29)$$

the negative sign referring to the proton and the positive to the neutron. The fact that the integral is always positive for any real function $f(k)$ leads to the conclusion that $\langle M_z(\mathrm{int}) \rangle$ gives a magnetic moment anomaly opposite to that observed.[14]

The magnitude of the anomaly depends on the particular form of $f(k)$. For the choice

$$f(k) = 1, \quad k \leqq K, \qquad = 0, \quad k > K, \quad (30)$$

then

$$\langle M_z(\mathrm{int}) \rangle, = \mp (g^2 e/3\pi\hbar c\mu^2) K^3 (\mu^2 + K^2)^{-1}. \quad (31)$$

If the cut-off momentum is large, i.e.,

$$K^2 \gg \mu^2, \quad (32)$$

then

$$\langle M_z(\mathrm{int}) \rangle = \mp (2g^2/3\pi\hbar c)(MK/\mu^2)(e/2M), \quad (33)$$

where M is the reciprocal nucleon Compton wavelength. Now $\langle M_z(\mathrm{int}) \rangle$ may be expressed in terms of the probability $P_1{}^c$ that a charged pion is present in the nucleon. This quantity is given by

$$P_1{}^c = \langle \Phi(1) | \Phi(1) \rangle,$$

where $\Phi(1)$ is given by Eq. (26). Then,

$$P_1{}^c = (g^2/\pi\hbar c\mu^2) \int_0^\infty dk [f^2(k) k^4 (\mu^2 + k^2)^{-\frac{3}{2}}], \quad (35)$$

and if the conditions, Eqs. (30) and (32), are used,

$$P_1{}^c = (g^2/\hbar c)(1/2\pi)(K^2/\mu^2), \quad (36)$$

whence

$$\langle M_z \rangle = \mp P_1{}^c (4M/3K), \quad (37)$$

in units of the nuclear magneton. In order to obtain specific numbers, suppose $K = M$ and[13] $P_1{}^c = 10\%$, then $\langle M_z \rangle \approx \mp 0.13$ nuclear magnetons, where again the $\mp$ refers to the proton or neutron, respectively. This number is rather small compared with the observed anomalies of 1.78 nuclear magnetons for the proton and -1.91 for the neutron.

Under the same assumptions that have characterized this calculation based on $H'(\mathbf{A})$, the interaction $H'_{\mathrm{alt}}(\mathbf{A})$ would give no contribution to the magnetic moment. Furthermore, in view of the smallness of the effect derived from $H'(\mathbf{A})$ and in view of the fact that the sign is given incorrectly, the magnetic moment data provide little evidence which can be used to make a choice between different linear combinations of $H'(\mathbf{A})$ and $H'_{\mathrm{alt}}(\mathbf{A})$.

[14] This conclusion is drawn by Blair *et al.*, in reference 3.

VI. SCATTERING OF LOW-ENERGY PHOTONS BY NUCLEONS

Sachs and Foldy[5] have given an expression for the cross section for the scattering of gamma rays from nucleons, calculated on the basis of nonrelativistic, no-recoil, point-source theory. The spin-dependent term in their final expression for the transition amplitude is independent of frequency at low frequency. This is a direct contradiction of the theorem in Sec. V of the preceding paper, namely that the transition amplitude should contain no constant term arising from excitation of the nucleon.

The reason for this contradiction can best be seen by examining the method of approximate stationary states used in SF and described in Appendix I of that paper. The approximate stationary states of the nucleon were derived from weak-coupling pseudoscalar meson theory, with neglect of recoil, by use of the following interaction between the nucleon core particle and the pion field.

$$H'=(4\pi)^{\frac{1}{2}}(gc/\mu)\int d^3r[\delta(\mathbf{r})\tau_+\boldsymbol{\sigma}\cdot\nabla\varphi(\mathbf{r})]+\text{h.c.} \quad (38)$$

In the presence of an electromagnetic field, H' was modified by replacing the gradient operator by the operator $[\nabla-(ie/\hbar c)\mathbf{A}]$. Therefore, $H'(\mathbf{A})$ satisfied the condition of gauge invariance, Eq. (2). However, the state vectors derived from Eq. (38) were not normalizable; hence, it is not surprising that the theorem of the preceding paper does not apply to the cross section derived in this manner.

In the present paper, the form of the interaction between the core particle and pions has been modified in such a way as to insure that the state vectors are normalizable. The delta function in the interaction, Eq. (38), has been replaced by a source function with a finite range. The source function $f(r)$ is required to be spherically symmetric and to satisfy the following two additional requirements:

$$\int f(r)d^3r=1, \quad (39a)$$

and

$$rf(r) \text{ nonsingular.} \quad (39b)$$

The method of approximate stationary states is then used to study the low-energy behavior of the spin-dependent term in the transition amplitude. The approximate stationary states calculated from the modified interaction are normalizable.

In the presence of an electromagnetic field, the interaction term in the Hamiltonian is assumed to have the form of Eq. (5). In this calculation, as in the calculation of SF, the only contribution to the spin-dependent term in the transition amplitude arises in the second order of the perturbation theory, through the intermediate states of the type denoted in SF by β and γ. A β state is a state whose zero-order term corresponds to one free meson in addition to the core particle, while the zero-order term of a γ state represents two free mesons and the core.

There are now three terms in the Hamiltonian which are linear in the vector potential, and hence can make a second-order contribution to the transition amplitude:

$$H_e=(iec/\hbar)\int d^3r[\tau_+\mathbf{A}(\mathbf{r})\cdot\varphi^*(\mathbf{r})\nabla\varphi(\mathbf{r})]+\text{h.c.}, \quad (40a)$$

$$H_{eg}=-i(4\pi)^{\frac{1}{2}}(eg/\hbar\mu)\int d^3r[\tau_+f(r)\boldsymbol{\sigma}\cdot\mathbf{A}\varphi]+\text{h.c.}, \quad (40b)$$

$$H'_{eg}=-i(4\pi)^{\frac{1}{2}}(eg/\hbar\mu)\int d^3r\left[\tau_+f(r)\left(\int_0^{\mathbf{r}}A_s ds\right)\times\boldsymbol{\sigma}\cdot\nabla\varphi\right]+\text{h.c.} \quad (40c)$$

The interactions H_e and H_{eg} are analogous to the interactions H_e and H_{eg} of SF. The interaction H'_{eg} arises from the exponential term $\exp[-(ie/\hbar c)\int_0^{\mathbf{r}}A_s ds]$ in Eq. (5). Although H'_{eg} is zero if $f(r)$ is replaced by a delta function, its influence on the transition amplitude will not vanish in the limit as $f(r)$ collapses to a delta function.

The only spin dependent terms in the transition amplitude which are independent of energy at low energy arise in the long-wavelength electric-dipole approximation, so we shall confine our attention to these terms here. In the electric dipole approximation, the transition amplitude does not depend on the choice of the path of integration in the line integral, $\int_0^{\mathbf{r}}A_s ds$. This follows from the facts that in this approximation, the function $\mathbf{u}\exp(i\boldsymbol{\omega}\cdot\mathbf{r})$ is replaced by $\mathbf{u}$, and the integral $\int_0^{\mathbf{r}}u_s ds=\mathbf{u}\cdot\mathbf{r}$ is independent of the path of integration.

In the electric-dipole approximation, the spin dependent terms in the transition amplitude may be written as a series of integrals:

$$\frac{4ie^2g^2}{\mu^2V\hbar c\omega}(\boldsymbol{\sigma}\cdot\mathbf{u}'\times\mathbf{u})\left\{\int_0^\infty dk\left[\frac{\omega}{k_0^2-\omega^2}\left(\frac{4}{3}\frac{k^4}{k_0^3}f^2(k)-\frac{k^2}{k_0}f^2(k)-\frac{2}{3}\frac{k^3}{k_0}f(k)\frac{df}{dk}\right)\right]-\int_0^\infty dk\left(\frac{1}{3}\frac{\omega}{k_0^2-\omega^2}\frac{k^4}{k_0^3}f^2(k)\right)\right\}, \quad (41)$$

where V is the volume of an enclosure introduced for purposes of normalization, the unit vectors $\mathbf{u}$ and $\mathbf{u}'$ denote the polarization directions of the initial and final photon, the symbol ω represents the magnitude of the wave vector of the incident or final photon, and k_0 is defined by the relation $k_0^2=\mu^2+k^2$. The function $f(k)$ is the Fourier transform of the source function $f(r)$, which is spherically symmetric in both r space and k space, i.e.,

$$f(k)=(2/\pi)^{\frac{1}{2}}k^{-1}\int_0^\infty rf(r)\sin krdr. \qquad (42)$$

The first integral in Eq. (41) results from a sum over intermediate states of type β, while the second integral results from a sum over γ states. The term containing $f(k)df/dk$ involves the two interactions, H_{eg} and H'_{eg}. No other term of Eq. (41) involves H'_{eg}.

When the source function satisfies the requirement of Eq. (39b), its Fourier transform $f(k)$ must decrease at least as fast as k^{-2} as k approaches infinity. Thus, both integrals of Eq. (41) are convergent.

Since we are interested in the energy independent portion of these integrals, we expand in powers of (ω^2/k_0^2) and retain only the constant term. Then the frequency independent term in the transition amplitude can be written in the form

$$\frac{4ie^2g^2}{\mu^2V\hbar c}(\boldsymbol{\sigma}\cdot\mathbf{u}'\times\mathbf{u})\int_0^\infty dk\frac{d}{dk}\left[-\frac{1}{3}\frac{k^3}{k_0^3}f^2(k)\right]. \qquad (43)$$

The function $(k^3/k_0^3)f^2$ vanishes at both limits, hence it is clear that this term, Eq. (43), vanishes. Thus the use of the finite source has removed the term of SF which contradicted the theorem of the previous paper.

It is interesting to compare Eq. (41) with the corresponding equation which occurs in SF in order to see how the disparity arises. The SF equation can be deduced easily from Eq. (41) by setting $f^2(k)$ equal to one. The result is[15]

$$\frac{4ie^2g^2}{\mu^2V\hbar c}(\boldsymbol{\sigma}\cdot\mathbf{u}'\times\mathbf{u})\left[\int_0^\infty \frac{1}{k_0^2-\omega^2}\left(\frac{4}{3}\frac{k^4}{k_0^3}-\frac{k^2}{k_0}\right)dk -\frac{1}{3}\int_0^\infty \frac{k^4}{(k_0^2-\omega^2)k_0^3}dk\right]. \qquad (44)$$

Each of these integrals is divergent, but in order that a finite result be obtained, the sum of the integrals was replaced by the integral of the sum of the integrands. The resulting integral is convergent, but leads to a result which does not vanish at zero energy.

One might have expected that the finite source procedure, in the limit as the range of the source function becomes zero, would lead to the same results as the point source procedure. But this is not the case. In this limit, the terms in Eq. (41) involving $f^2(k)$ do lead to the point source results obtained in SF, but the term involving $f(df/dk)$, which resulted from the inclusion of the exponential $\exp[-(ie/\hbar c)\int_0^r A_s ds]$ in $H'(\mathbf{A})$, leads to a finite result of proper magnitude and sign to cancel the result of SF at zero energy. Thus the condition of gauge invariance in the finite source plays a direct role in the cancellation, even in the limit as the source range approaches zero.

[15] This result can also be deduced from Eq. (SF-8), by setting $\boldsymbol{\omega}=0$ in such combinations as $\mathbf{k}-\boldsymbol{\omega}$ and carrying out the angular part of the integrals in the spin dependent term.

VII. CROSS SECTION AS A FUNCTION OF ENERGY

If the meson-nucleon interaction used in SF is replaced by an interaction of the form given by Eq. (5), the transition amplitude for the scattering of gamma rays from nucleons may be recalculated. If the incident and scattered gamma rays have wavelengths long compared with the range of the source function in the meson-nucleon interaction, the calculated transition amplitude should not be sensitive to the form of the source function. In the present calculation, the transition amplitude and the cross section will be evaluated in the limit as the range of the source function approaches zero; hence they will be completely independent of the form of the source function. Thus, the finite source is used here only as a device for insuring that the wave function be normalizable, and that the integrals occurring in the transition amplitude converge. As in SF, the transition amplitude is calculated here only to lowest order in $e^2/\hbar c$. The integrals in the expression for the transition amplitude have been evaluated by expanding the angular dependent denominators in the integrands in powers of the parameter $\eta=2\omega k/(k^2+\mu^2+\omega^2)$. The results are computed to order η^2 in this expansion.

It has been found that at all energies, the transition amplitude calculated using the finite source procedure differs from the corresponding quantity in SF by only one additive, energy-independent term. The proof of this statement and the explicit expression for the transition amplitude are given in Appendix A.

In addition to including this constant term, a further correction to the transition amplitude obtained by SF must be made for energies above pion production threshold. Because of the possibility of pion production, the transition amplitude at these energies contains an absorptive part, in addition to the dispersive part which is present at all energies. In SF, the absorptive part was omitted, but it is included in the present calculation. The correction involves only those integrals of the transition amplitude whose integrands contain poles, and it is to be made by integrating each such integral along a path beneath the pole, rather than by replacing the integral by its principal part, as was done in SF.

The correction has a profound effect on the energy dependence of the cross section, as can be seen by comparing the calculated cross section for scattering from neutrons, Fig. 1, with the corresponding cross section shown in Figs. SF-2. The neutron cross section of SF rises to a peak at meson production threshold, then decreases again while the corrected cross section continues to increase.

The proton cross section consists of three terms, an energy independent term resulting from Thomson scattering from the proton as a whole, a term proportional to g^4 resulting from Rayleigh scattering from the meson cloud and a g^2 term from the interference between Thomson and mesonic scattering. The neutron cross section is equal to the g^4 term in the proton cross section.

In order to obtain numerical results, one may assign a value to g^2 by comparing some experimental result to the corresponding prediction of weak coupling, pseudoscalar meson theory. In SF, the experimental result used for this purpose was the nearly zero binding energy to the ^1S state of the deuteron. Since the publication of SF, the low-energy cross section for the photoproduction of positive pions from protons has been measured.[16,17] Since photomeson production is a phenomenon closely related to the scattering of gamma rays, the positive pion production cross section will be used here to fix g^2. The value of $(g^2/\hbar c)$ is taken to be 0.116.[18]

With this choice of g^2 the mesonic scattering term and the interference term in the proton cross sections are of the order of magnitude of the Thomson cross section, at energies in the range 100–200 Mev. Because of this fact, the total cross section and angular distribution for the proton are sensitive to a change in the coupling constant. For example, at a center-of-mass energy of 139 Mev the differential cross section is greater at 180° than at 0° by a factor of 1.8. However, if the coupling constant were multiplied by a factor of 1.7, the cross section would be nearly the same as 0° as at 180°, and at still higher values of the coupling constant, forward scattering would predominate.

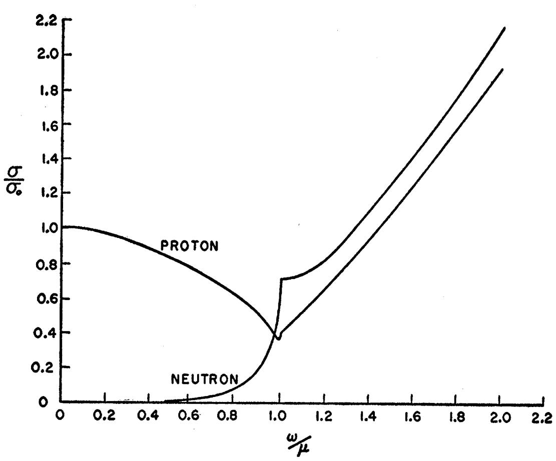

FIG. 1. Calculated total cross sections, in units of the Thomson cross section σ_0, for scattering of gamma rays from neutrons and protons. The quantity ω/μ is the photon energy in units of the pion mass.

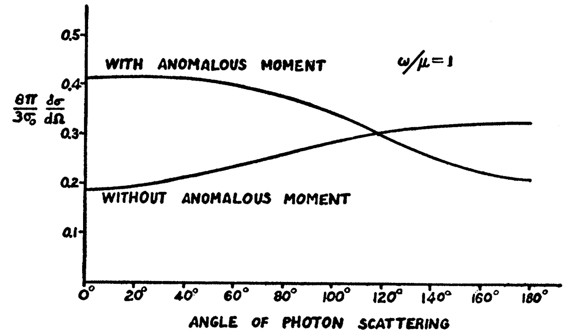

FIG. 2. Calculated differential cross section at 140 Mev center-of-mass energy for the scattering of gamma rays from protons, showing the effect of scattering from the proton anomalous moment.

At present there are no experimental data available concerning the cross section for the scattering of gamma rays from hydrogen. It seems likely that when experiments are performed in the energy range 100–150 Mev, the results will reveal deviations from the Thomson cross section which will serve to check the present model. A reasonable procedure would be to fit the coupling constant g^2 to the observed total cross section, and use the observed angular distribution and energy dependence to check the model. If the measured cross section were significantly smaller than the Thomson cross section, the results presented here would not be meaningful, since recoil terms and terms of higher order in $(g^2/\hbar c)$ would then be important.

Up to this point, it has been assumed that the meson currents are responsible for the anomalous magnetic moments. If one assumes that the anomalous moment is to be associated with the core, the effect of the anomalous moment on the proton cross section can be approximated by introducing into the Hamiltonian an interaction of the form

$$-(\lambda e\hbar/2\mu c)(\boldsymbol{\sigma}\cdot\mathrm{curl}\mathbf{A}), \tag{45}$$

where λ is the anomalous moment measured in nuclear magnetons. This leads to an additional term in the transition amplitude which has the form

$$(\lambda^2\pi e^2\hbar^2/M^2c^2V)[i\boldsymbol{\sigma}\cdot(\mathbf{v}'\times\mathbf{u}')\times(\mathbf{v}\times\mathbf{u})], \tag{46}$$

where $\mathbf{v}=(\boldsymbol{\omega}/\omega)$ and $\mathbf{v}'=(\boldsymbol{\omega}'/\omega')$. Although this is a term of the same order as recoil effects, it makes an appreciable contribution to the cross section because of the large value of the anomalous moment. The contribution to the angular distribution is shown, at the energy $(\omega/\mu)=1$, in Fig. 2.

The results, when the anomalous moment contribution is not included, are presented in Figs. 1, 3, and 4. Figure 1 shows the total cross section for the proton and neutron as a function of energy. The cusps which appear in these curves at the threshold energy for

[16] G. Bernardini and E. L. Goldwasser, Phys. Rev. 95, 857 (1954).

[17] G. S. Janes and W. L. Kraushaar, Phys. Rev. 93, 900 (1954); Jenkins, Luckey, Palfrey, and Wilson, Phys. Rev. 95, 179 (1954).

[18] This value was taken from an analysis of low-energy photomeson production data by G. Bernardini and E. L. Goldwasser (private communication). The constant f^2 in the work is related to g^2 by $(g^2/\hbar c)=2f^2$. After the numerical work presented here had been completed, Bernardini and Goldwasser, refernce 7, revised their estimate of f^2 to the value 0.066.

photopion production are discussed in Appendix B. The angular distribution of the scattering from protons is shown for each of five values of the energy in Fig. 3, and the corresponding curves for scattering from neutrons are given in Fig. 4.

VIII. CONCLUSION

The principal purpose of this work has been to investigate the consequences of using an extended meson source as a convergence procedure for the treatment of the electromagnetic interactions of nucleons. To establish the gauge invariance of the extended source theory, it has been necessary to introduce filaments of current within the source. Some physical consequences resulting from the presence of these currents have been deduced.

The nucleon magnetic moments are not affected when a simple choice is made for the form of the current filaments. However, the effect of the currents on the γ-ray scattering by nucleons is more profound. The anomalous term (which violates the gauge-invariance theorem of the preceding paper) found by Sachs and Foldy in their point-source theory does not occur in this calculation. In fact, the only effect of using the finite-source procedure of Sec. VI, where the source size is taken to be arbitrarily small, is the cancellation of this one anomalous term, a very interesting point in view of the many divergent integrals that appeared in the SF calculation. Furthermore, the results of Sec. VI do not depend on the choice of the path of integration in the line integral $\int_0^{\mathbf{r}} A_s ds$. Thus, this particular method of obtaining gauge invariant results introduces no spurious effects.

Although the cross section for the scattering of γ rays by nucleons has been considered here only in the weak-coupling approximation, the usefulness of the concept of a pion-nucleon interaction of finite range is not limited to weak coupling calculations. In any calculation where a finite source interaction is used, the exponential term provides a convenient method of preserving gauge invariance. Furthermore, if the results of any calculation are finite in the limit as the source size approaches zero, they should not differ appreciably from the results corresponding to a fixed source size, provided this fixed distance is small compared to the wavelength of the radiation under consideration.

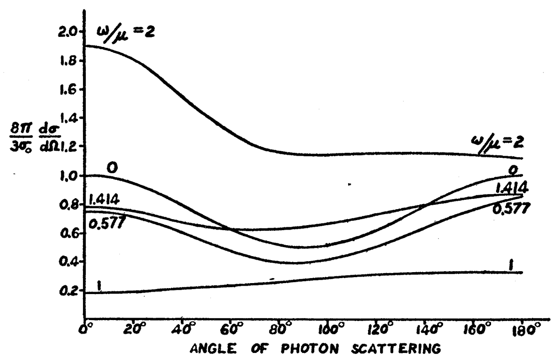

FIG. 3. Calculated differential cross sections for scattering of gamma rays from protons. The quantity ω/μ is the photon energy in units of the pion mass.

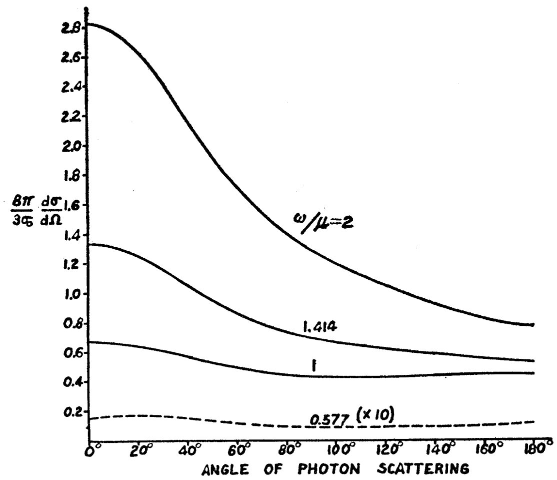

FIG. 4. Calculated differential cross sections for scattering of gamma rays from neutrons.

The calculation of the gamma-ray scattering was performed by direct use of the detailed tables of their work kindly provided by Professor Sachs and Professor Foldy. Many valuable discussions were had with Professor Sachs concerning the material of this article.

APPENDIX A. CALCULATION OF THE TRANSITION AMPLITUDE

The form of the transition amplitude for the scattering of gamma rays by nucleons has been calculated, assuming that the interaction between the core particle and the meson field is given by Eq. (5). If this interaction is used there are three terms in the internal Hamiltonian of the nucleon which are linear in the vector potential, and three terms quadratic in the vector potential. The linear terms are enumerated in Eq. (40). In addition to the term H_{ee} of Eq. (SF-4), there are two quadratic terms involving the integral $\int_0^{\mathbf{r}} A_s ds$. They are:

$$H_{eeg} = -(4\pi)^{\frac{1}{2}}(e^2g/\hbar\mu)\int d^3r\left[\tau_+ f(r)\boldsymbol{\sigma}\cdot\mathbf{A}(\mathbf{r})\varphi(\mathbf{r}) \times\left(\int_0^{\mathbf{r}} A_s ds\right)\right]+\text{h.c.}, \quad \text{(A1)}$$

$$H'_{eeg} = -(4\pi)^{\frac{1}{2}}(e^2g/2\hbar\mu)\int d^3r\left[\tau_+ f(r)\boldsymbol{\sigma}\cdot\nabla\varphi(\mathbf{r}) \times\left(\int_0^{\mathbf{r}} A_s ds\right)^2\right]+\text{h.c.} \quad \text{(A2)}$$

The transition amplitude T for the scattering of electromagnetic radiation from protons may then be written in the following form, correct to order $(g^2/\hbar c)$

and (μ/M), but not including terms of order $(g^2/\hbar c)\times(\mu/M)$.

$$T=T_a+T_b, \tag{A3}$$

$$\begin{aligned}
T_a=&(2\pi\hbar e^2/Mc\omega V)(\mathbf{u}\cdot\mathbf{u}')+(e^2g^2/\pi\hbar c\omega V\mu^2)\int d^3k\Bigg(\Bigg\{\frac{(\mathbf{u}\cdot\mathbf{u}')(\mathbf{k}-\boldsymbol{\omega}')\cdot(\mathbf{k}-\boldsymbol{\omega})}{(\mathbf{k}-\boldsymbol{\omega})_0{}^2(\mathbf{k}-\boldsymbol{\omega}')_0{}^2}\\
&-\frac{1}{k_0{}^2-\omega^2}\left[4\frac{(\mathbf{u}\cdot\mathbf{k})(\mathbf{u}'\cdot\mathbf{k})(\mathbf{k}-\boldsymbol{\omega})\cdot(\mathbf{k}-\boldsymbol{\omega}')}{(\mathbf{k}-\boldsymbol{\omega})_0{}^2(\mathbf{k}-\boldsymbol{\omega}')_0{}^2}-2\frac{(\mathbf{u}'\cdot\mathbf{k})\mathbf{u}\cdot(\mathbf{k}-\boldsymbol{\omega}')}{(\mathbf{k}-\boldsymbol{\omega}')_0{}^2}-2\frac{(\mathbf{u}\cdot\mathbf{k})\mathbf{u}'\cdot(\mathbf{k}-\boldsymbol{\omega})}{(\mathbf{k}-\boldsymbol{\omega})_0{}^2}+(\mathbf{u}\cdot\mathbf{u}')\right]\\
&-\frac{i\omega}{k_0(k_0{}^2-\omega^2)}\left[4\frac{(\mathbf{u}\cdot\mathbf{k})(\mathbf{u}'\cdot\mathbf{k})\boldsymbol{\sigma}\cdot(\mathbf{k}-\boldsymbol{\omega}')\times(\mathbf{k}-\boldsymbol{\omega})}{(\mathbf{k}-\boldsymbol{\omega})_0{}^2(\mathbf{k}-\boldsymbol{\omega}')_0{}^2}-2\frac{(\mathbf{u}'\cdot\mathbf{k})\boldsymbol{\sigma}\cdot(\mathbf{k}-\boldsymbol{\omega}')\times\mathbf{u}}{(\mathbf{k}-\boldsymbol{\omega}')_0{}^2}+2\frac{(\mathbf{u}\cdot\mathbf{k})\boldsymbol{\sigma}\cdot(\mathbf{k}-\boldsymbol{\omega})\times\mathbf{u}'}{(\mathbf{k}-\boldsymbol{\omega})_0{}^2}+\boldsymbol{\sigma}\cdot\mathbf{u}'\times\mathbf{u}\right]\Bigg\}\\
&\times f(|\mathbf{k}-\boldsymbol{\omega}|)f(|\mathbf{k}-\boldsymbol{\omega}'|)+\frac{2i\omega}{k_0{}^2}\left[\frac{1}{(\mathbf{k}-\boldsymbol{\omega}')_0}\frac{(\mathbf{u}'\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\mathbf{u}\times\mathbf{k})}{[k_0+(\mathbf{k}-\boldsymbol{\omega}')_0]^2-\omega^2}-\frac{1}{(\mathbf{k}-\boldsymbol{\omega})_0}\frac{(\mathbf{u}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\mathbf{u}'\times\mathbf{k})}{[k_0+(\mathbf{k}-\boldsymbol{\omega})_0]^2-\omega^2}\right]\\
&\times f(k)f(|\mathbf{k}-\boldsymbol{\omega}-\boldsymbol{\omega}'|)+4i\omega\frac{(\mathbf{u}'\cdot\mathbf{k})(\mathbf{u}\cdot\mathbf{k})\boldsymbol{\sigma}\cdot(\mathbf{k}-\boldsymbol{\omega}')\times(\mathbf{k}-\boldsymbol{\omega})}{k_0[(\mathbf{k}-\boldsymbol{\omega})_0{}^2-(\mathbf{k}-\boldsymbol{\omega}')_0{}^2]}\left[\frac{1}{(\mathbf{k}-\boldsymbol{\omega})_0{}^2}\frac{1}{\omega^2-[k_0+(\mathbf{k}-\boldsymbol{\omega}')_0]^2}\right.\\
&\left.-\frac{1}{(\mathbf{k}-\boldsymbol{\omega}')_0{}^2}\frac{1}{\omega^2-[k_0+(\mathbf{k}-\boldsymbol{\omega}')_0]^2}\right]f(|\mathbf{k}-\boldsymbol{\omega}|)f(|\mathbf{k}-\boldsymbol{\omega}'|)\Bigg),
\end{aligned} \tag{A4}$$

$$\begin{aligned}
T_b=&(e^2g^2/\pi\hbar c\omega V\mu^2)\int d^3k\Bigg\{\frac{(\mathbf{k}\cdot\mathbf{u}')f(k)\eta_{\omega u}(\mathbf{k}-\boldsymbol{\omega}')}{k_0{}^2}+\frac{(\mathbf{k}\cdot\mathbf{u})f(k)\eta_{\omega'u'}(\mathbf{k}-\boldsymbol{\omega})}{k_0{}^2}-\frac{k^2f(k)\eta'(\mathbf{k})}{k_0{}^2}\\
&-\frac{1}{k_0{}^2-\omega^2}\Bigg[2\frac{(\mathbf{u}'\cdot\mathbf{k})(k^2-\mathbf{k}\cdot\boldsymbol{\omega}')f(|\mathbf{k}-\boldsymbol{\omega}'|)\eta_{-\omega u}(\mathbf{k})}{(\mathbf{k}-\boldsymbol{\omega}')_0{}^2}+2\frac{(\mathbf{u}\cdot\mathbf{k})(k^2-\mathbf{k}\cdot\boldsymbol{\omega})f(|\mathbf{k}-\boldsymbol{\omega}|)\eta_{-\omega'u'}(\mathbf{k})}{(\mathbf{k}-\boldsymbol{\omega})_0{}^2}\\
&-(\mathbf{u}'\cdot\mathbf{k})f(|\mathbf{k}-\boldsymbol{\omega}'|)\eta_{-\omega u}(\mathbf{k})-(\mathbf{u}\cdot\mathbf{k})f(|\mathbf{k}-\boldsymbol{\omega}|)\eta_{-\omega'u'}(\mathbf{k})+k^2\eta_{-\omega u}(\mathbf{k})\eta_{-\omega'u'}(\mathbf{k})\Bigg]\\
&-\frac{i\omega}{k_0(k_0{}^2-\omega^2)}\Bigg[-2\frac{(\mathbf{u}'\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\boldsymbol{\omega}'\times\mathbf{k})f(|\mathbf{k}-\boldsymbol{\omega}'|)\eta_{-\omega u}(\mathbf{k})}{(\mathbf{k}-\boldsymbol{\omega}')_0{}^2}+2\frac{(\mathbf{u}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\boldsymbol{\omega}\times\mathbf{k})f(|\mathbf{k}-\boldsymbol{\omega}|)\eta_{-\omega'u'}(\mathbf{k})}{(\mathbf{k}-\boldsymbol{\omega})_0{}^2}\\
&-(\boldsymbol{\sigma}\cdot\mathbf{u}'\times\mathbf{k})f(|\mathbf{k}-\boldsymbol{\omega}'|)\eta_{-\omega u}(\mathbf{k})-(\boldsymbol{\sigma}\cdot\mathbf{k}\times\mathbf{u})f(|\mathbf{k}-\boldsymbol{\omega}|)\eta_{-\omega'u'}(\mathbf{k})\Bigg]+\frac{2i\omega}{k_0{}^2}\Bigg[-\frac{(\mathbf{u}'\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\boldsymbol{\omega}'\times\mathbf{k})f(k)\eta_{\omega u}(\boldsymbol{\omega}'-\mathbf{k})}{(\mathbf{k}-\boldsymbol{\omega}')_0\{[k_0+(\mathbf{k}-\boldsymbol{\omega}')_0]^2-\omega^2\}}\\
&+\frac{(\mathbf{u}\cdot\mathbf{k})(\boldsymbol{\sigma}\cdot\boldsymbol{\omega}\times\mathbf{k})f(k)\eta_{\omega'u'}(\boldsymbol{\omega}-\mathbf{k})}{(\mathbf{k}\cdot\boldsymbol{\omega})_0\{[k_0+(\mathbf{k}-\boldsymbol{\omega})_0]^2-\omega^2\}}\Bigg]\Bigg\}.
\end{aligned} \tag{A5}$$

The transition amplitude for the scattering from neutrons differs from the above expression only in that the first term of T_a is absent.

The corresponding result obtained by SF is given in Eq. (SF-8). The symbol T_a denotes that part of T which does not involve the line integral $\int_0^{\mathbf{r}}A_s ds$, while T_b denotes the part of T which does involve this integral. The functions $\eta_{\omega u}(\mathbf{k})$ and $\eta'(\mathbf{k})$ are related to $\int_0^{\mathbf{r}}A_s ds$; if the path of integration in this integral is taken to be a straight line, η and η' are the Fourier transforms of the functions

$$\eta_{\omega u}(\mathbf{r})=-if(r)(\mathbf{u}\cdot\mathbf{r})\int_0^1\exp(i\boldsymbol{\omega}\cdot\alpha\mathbf{r})d\alpha,$$

$$\eta'(\mathbf{r})=-f(r)(\mathbf{u}\cdot\mathbf{r})(\mathbf{u}'\cdot\mathbf{r})\left[\int_0^1\exp(i\boldsymbol{\omega}\cdot\alpha\mathbf{r})d\alpha\right]\times\left[\int_0^1\exp(i\boldsymbol{\omega}'\cdot\alpha'\mathbf{r})d\alpha'\right].$$

It can be shown that $f(k)$, $\eta_{\omega u}(\mathbf{k})$, and $\eta'(\mathbf{k})$ are all real functions. The symbol $(\mathbf{k}-\boldsymbol{\omega})_0$ in Eqs. (A4) and (A5) denotes $[\mu^2+(\mathbf{k}-\boldsymbol{\omega})^2]^{\frac{1}{2}}$.

If the integration over $\mathbf{k}$ space in Eq. (A4) is carried out, and then the range of the source function is allowed to approach zero, the resulting expression for T_a will become equal to the transition amplitude of SF, Eq. (SF-8). It will now be shown that if this procedure is carried out for Eq. (A-5), the resulting expression will reduce to one energy-independent term, the term mentioned in Sec. VI.

In the following argument, use is made of several properties of the source function. To satisfy the condition Eq. (39a), we write $f(r)$ in the form $f(r)=R^{-3}g(r/R)$, where R represents the range of the source function. The Fourier transform of $f(r)$ is then a function of the product (kR), i.e., $f(k)=\varphi(kR)$. Since $f(r)$ has the property, Eq. (39b), $\varphi(kR)$ must vanish at least as fast as $(kR)^{-2}$ as (kR) approaches infinity. Hence, all derivatives of $\varphi(kR)$ with respect to (kR) must also vanish at least as fast as $(kR)^{-2}$.

It is convenient to expand the exponential in the functions $\eta_{\omega u}(\mathbf{r})$ and $\eta'(\mathbf{r})$ in power series, i.e.,

$$(\mathbf{u}\cdot\mathbf{r})\int_0^1 \exp(i\boldsymbol{\omega}\cdot\alpha\mathbf{r})d\alpha=(\mathbf{u}\cdot\mathbf{r})+i(\mathbf{u}\cdot\mathbf{r})(\boldsymbol{\omega}\cdot\mathbf{r})+\cdots. \tag{A6}$$

By making use of this expansion, $\eta'_{\omega u}(\mathbf{r})$ and $\eta'(\mathbf{r})$ may be written as sums of terms, each term depending on r to the first or higher power. Hence, $\eta_{\omega u}(\mathbf{k})$ may be written as a sum of terms in the form

$$G_n(\mathbf{u},\mathbf{k})R(R\omega)^{n-1}\varphi^{(n)}(kR), \tag{A7}$$

where G_n is some function independent of R and ω; the exponent n is a positive integer and $\varphi^{(n)}$ represents the nth derivative of the source function $\varphi(kR)$ with respect to (kR). Similarly, $\eta'(\mathbf{k})$ may be written as a sum of terms of the form,

$$G_n(\mathbf{u},\mathbf{u}',\mathbf{k})R^2(R\omega)^{n-2}\varphi^{(n)}(kR). \tag{A8}$$

Thus, T_b may be written as a sum over n and n' of terms of the following form

$$R^\nu\int d^3k[F_{nn'}(k_0,\mathbf{k},\boldsymbol{\omega},\boldsymbol{\sigma},\mathbf{u},\mathbf{u}')\varphi^{(n)}(k'R)\varphi^{(n')}(k''R), \tag{A9}$$

where the new symbols have the following meanings: The function F_{nn} is bounded in $\mathbf{k}$ and independent of R; the symbols n and n' represent non-negative integers, and the exponent $\nu=n+n'$ is a positive integer. The variable k', or the variable k'', represents some one of the following five quantities, in each of the various terms to be considered: k, $|\mathbf{k}\pm\boldsymbol{\omega}|$, $|\mathbf{k}\pm\boldsymbol{\omega}'|$.

It is convenient to express the integral over all k-space in Eq. (A9) as a sum of integrals over two regions, i.e.,

$$R^\nu\left(\int_0^K d^3kF_{nn'}\varphi^{(n)}\varphi^{(n')}+\int_K^\infty d^3kF_{nn'}\varphi^{(n)}\varphi^{(n')}\right). \tag{A10}$$

The first includes the region in k-space inside a sphere of arbitrary radius K; the second region includes the space outside this sphere.

Since ν is greater than zero, it is clear that the first integral in Eq. (A10) does not contribute to the transition amplitude, as R approaches zero. Thus, we need consider only the second integral. If K is chosen very large compared with μ and ω, k_0 may be set equal to k, and all terms may be expanded in powers of (ω/k). The angular integrations may then be carried out, and thus T_b may be written as a sum (over n, n', and m) of terms of the following nature:

$$R^\nu\omega^m\int_K^\infty dk[F_{nn'm}(k,\boldsymbol{\sigma},\mathbf{u},\mathbf{u}')\varphi^{(n)}(kR)\varphi^{(n')}(kR)]. \tag{A11}$$

Examination of Eq. (A5) leads to the conclusion that $m\geqq -1$. The theorem in Sec. V of the preceding paper tells us that the terms of T_b with $m=0$ and $m=-1$ must have the proper magnitude and angular dependence to cancel the ω^0 and ω^{-1} terms in T_a. In Sec. VI, this was demonstrated for the ω^0 terms. The ω^{-1} terms have been calculated for an arbitrary value of R and the result is in agreement with the theorem, namely the ω^{-1} terms of T_b cancel against the ω^{-1} terms of T_a. In the limit as R approaches zero, the ω^{-1} terms of T_a and T_b separately vanish.

A simple dimensional argument may now be used to demonstrate that all terms in Eq. (A11) corresponding to $m\geqq 1$ will vanish as R approaches zero. The combination $\omega^mR^\nu\int_K^\infty dkF_{nn'm}(k,\boldsymbol{\sigma},\mathbf{u},\mathbf{u}')$ must be dimensionless. Hence, $F_{nn'm}$ must be of the form $F_{nn'm}=k^{\nu-m-1}Q_{nn'm}(\boldsymbol{\sigma},\mathbf{u},\mathbf{u}')$, since k is the only dimensional variable in F. Thus Eq. (A11) may be written in the form

$$\omega^mR^\nu Q_{nn'm}(\boldsymbol{\sigma},\mathbf{u},\mathbf{u}')\int_K^\infty dk[k^{\nu-m-1}\varphi^{(n)}(kR)\varphi^{(n')}(kR)]. \tag{A12}$$

Examination of Eqs. (A5), (A7), and (A8) reveals that $m\geqq\nu-3$ or $\nu-m-1\leqq 2$. Since $\varphi^{(n)}$ vanishes at least as fast as k^{-2} as k approaches infinity, the integral in Eq. (A12) is therefore always convergent for finite R. If the integral is written in the form of an integral over the variable (kR), then it can be seen that the resulting combination,

$$R^m\int_{KR}^\infty d(kR)[\varphi^{(n)}(kR)\varphi^{(n')}(kR)(kR)^{\nu-m-1}], \tag{A13}$$

vanishes in the limit as R approaches zero if the integer m is positive. This completes the proof that T differs from the T of SF by only one term. This term was discussed in Section VI, and found to be equal to

$$-(8ie^2g^2/3\mu^2V\hbar c)(\boldsymbol{\sigma}\cdot\mathbf{u}'\times\mathbf{u})\times\int_0^\infty dk[(k^3/k_0^3)f(k)df/dk]=(4ie^2g^2/3\mu^2V\hbar c)(\boldsymbol{\sigma}\cdot\mathbf{u}'\times\mathbf{u}). \tag{A14}$$

This result does not depend on the choice of the path of integration in the line integral $\int_0^{\mathbf{r}}A_sds$ because in the limit as the source size R approaches zero, the only nonvanishing terms of T_b are terms which would arise in the electric dipole approximation and, as discussed in Sec. VI, in the electric dipole approximation the integral is independent of the path of integration for any value of R.

APPENDIX B. CUSPS IN THE TOTAL CROSS SECTIONS

The discontinuity in the energy derivative of the total scattering cross section at photopion production threshold is a very general phenomenon; similar discontinuities occur in many cross sections. The appearance of such cusps in nuclear cross sections has been discussed by Wigner.[19]

[19] E. P. Wigner, Phys. Rev. 73, 1002 (1948). The fact that cusps probably would appear in the present work was first suggested to the authors by Professor Wigner.

The existence of discontinuities of this type will be demonstrated for a general class of scattering problems. We consider the collision of two particles denoted by A and B, at such an energy that only two reaction channels are open, corresponding to elastic scattering, and to the production of two other particles, C and D, which have a greater total rest mass than the particles A and B. One of the incident particles may be a photon. It is assumed that the particles $C+D$ may be produced in an S state, so that the cross section near threshold for this process has the $(E-E_t)^{\frac{1}{2}}$ energy dependence characteristic of an S state. For simplicity it is also assumed that the particles are all spinless, though this assumption is not necessary for the validity of the conclusions.

The elastic scattering cross section is related to the cross section for production of the particles $C+D$ by the condition that the scattering matrix U be unitary. That part of the U matrix corresponding to zero angular momentum is a two-by-two matrix; the two elements of the first row may be written as

$$U_{AB,AB}=e^{i\delta}(1-r^2)^{\frac{1}{2}};\quad U_{AB,CD}=ire^{i\rho},\qquad \text{(B1)}$$

where r, δ, and ρ are real functions of energy. The sum of the absolute squares of the two elements has been set equal to one to satisfy the requirement of unitarity. If the matrix $\mathcal{T}$ is defined to be the matrix equation $U-1=-i\mathcal{T}$, the two S-wave cross sections are related to the corresponding matrix elements of $\mathcal{T}$ by the equation $\sigma_{if}=(\pi/k_i^2)|\mathcal{T}_{if}|^2$ where i and f refer to the initial and final states, respectively, and k_i is the relative momentum in the initial state. If the quantity r^2 of Eq. (B1) is small, $\mathcal{T}_{AB,AB}$ may be expanded in powers of r^2 and written in the form,

$$\mathcal{T}_{AB,AB}=-(\sin\delta)(1-\tfrac{1}{2}r^2+\cdots)+i[(\cos\delta)(1-\tfrac{1}{2}r^2+\cdots)-1].\qquad \text{(B2a)}$$

At an energy slightly above the threshold for the production of the state $C+D$, $\sigma_{AB,CD}=(\pi/k_{AB}^2)r^2$ is proportional to $\omega^{\frac{1}{2}}$, where $\omega=E-E_t$, E_t being the threshold energy. Thus, r^2 will be small near threshold and will be given by $r^2=a^2\omega^{\frac{1}{2}}$ where a^2 is a positive constant. Then it can be seen from Eq. (B2a) that, in general, both the real and imaginary parts of $\mathcal{T}_{AB,AB}$ contain terms of order $\omega^{\frac{1}{2}}$.

It has been shown by Eden[20] that, for a very general class of scattering problems, the elastic scattering U-matrix element is an analytic function of energy which has a branch point at $E=E_t$, and that the assumption that the particles interact through their retarded fields may be used to specify the proper path of analytic continuation around the branch point. In the case considered here the continuation must be in the region corresponding to a positive imaginary part of E. Thus from the form of $\mathcal{T}_{AB,AB}$ above threshold, Eq. (B2a), we may deduce that its form just below threshold must be given by[21]

$$\mathcal{T}_{AB,AB}=-\sin\delta+\tfrac{1}{2}(\cos\delta)a^2|\omega|^{\frac{1}{2}}+i[\cos\delta-1+\tfrac{1}{2}(\sin\delta)a^2|\omega|^{\frac{1}{2}}].\qquad \text{(B2b)}$$

Hence, to order $r^2=a^2\omega^{\frac{1}{2}}$, the zero angular momentum part of the total scattering cross section may be written, in the neighborhood of threshold,

$\omega>0$;
$$\sigma_{AB,AB}=C^2[2-2\cos\delta-a^2\omega^{\frac{1}{2}}+(\cos\delta)a^2\omega^{\frac{1}{2}}],\qquad \text{(B3a)}$$
$\omega<0$;
$$\sigma_{AB,AB}=C^2[2-2\cos\delta-(\sin\delta)a^2|\omega|^{\frac{1}{2}}].\qquad \text{(B3b)}$$

where C^2 is a positive constant. Therefore, the energy derivative of the elastic scattering cross section will, in general, become infinite as the energy approaches threshold from both above and below.

The discussion of the preceding paragraphs may be applied to the cross sections calculated in Sec. VII if particle A is identified with a gamma ray, particles B and C with nucleons, and D with a charged pion. Although this case is slightly complicated by the spins of the particles, the argument is fundamentally the same. Since pions may be photoproduced into S states, the unitarity of the U matrix implies that part of the elastic scattering cross section (that part which results from the scattering of a photon wave of odd parity and angular momentum $\frac{1}{2}$) may be written in the form of Eqs. (B3a) and (B3b) at energies close to photopion threshold.

In Sec. VII only the term of lowest order in the electric charge was calculated. If the quantities a^2 and δ, corresponding to the gamma-ray nucleon-pion problem are expanded in powers of e^2, the lowest order term of each will be proportional to e^2, i.e., $a^2\propto e^2$; $\delta\propto e^2$. To order e^2 Eqs. (B2a) and (B2b) may then be written

$$\omega>0;\quad \mathcal{T}_{\gamma N,\gamma N}=-\delta-\tfrac{1}{2}ia^2\omega^{\frac{1}{2}},\qquad \text{(B4a)}$$

$$\omega<0;\quad \mathcal{T}_{\gamma N,\gamma N}=-\delta+\tfrac{1}{2}a^2|\omega|^{\frac{1}{2}}.\qquad \text{(B4b)}$$

Hence, Eqs. (B3a) and (B3b) become, to order e^4

$$\omega>0;\quad \sigma_{\gamma N,\gamma N}=C^2\delta^2,\qquad \text{(B5a)}$$

$$\omega<0;\quad \sigma_{\gamma N,\gamma N}=C^2(\delta^2-\delta a^2|\omega|^{\frac{1}{2}}).\qquad \text{(B5b)}$$

Thus, in this case, there is no term in $\sigma_{\gamma N,\gamma N}$ proportional to $|\omega|^{\frac{1}{2}}$ above threshold, but there is such a term below threshold. By making use of the tables prepared by Sachs and Foldy, the cross sections calculated in Sec. VII have been expanded in powers of $|\omega|^{\frac{1}{2}}$, for energies just below pion production threshold. The results to order $|\omega|$ are, for the proton case $\sigma/\sigma_0=0.40-0.89|\omega/\mu|^{\frac{1}{2}}+5.4|\omega/\mu|$, and for the neutron case, $\sigma/\sigma_0=0.71-3.1|\omega/\mu|^{\frac{1}{2}}+6.9|\omega/\mu|$, where σ_0 is the Thomson cross section, and μ is the rest energy of the pion.

[20] R. J. Eden, Proc. Roy. Soc. (London) **A210**, 388 (1952).

[21] The function δ cannot contain a part proportional to a half-odd-integral power of $|\omega|$. This follows from the requirement that the U matrix be unitary below threshold, together with fact that $U_{AB,AB}$ is analytic with a branch point at threshold.