Radiative τ-Meson Decay

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The characteristics of the decay process $\tau^{\pm}\rightarrow\pi^{\pm}+\pi^{+}+\pi^{-}+\gamma$ are considered. For a pseudoscalar τ-meson, the probability $R(k)$, relative to the 3π-decay, for emission of a photon of energy k Mev or greater is found to be 1.06×10^{-3} for $k=10$ and 1.1×10^{-4} for $k=30$. Quite similar values are obtained for other spin values compatible with the τ-meson decay data. Branching ratios for other decay processes are briefly discussed and it is suggested that the absence of the decay mode $\pi^{\pm}+\gamma$ or $\pi^{\pm}+e^{+}+e^{-}$ may provide some evidence against a nonzero τ-meson spin. Competing decay processes involving a neutrino are not considered.

AN interesting K-meson decay event has recently been reported by Daniels and Pal,[1] in which the observed decay products were three pions of total kinetic energy 46.1 ($\pm$2.2) Mev and total momentum 32.1 ($\pm$2.7) Mev/c. The most natural interpretation of this event in terms of known particles is that it represents a radiative decay of the τ-meson,

$$\tau^{+}\rightarrow\pi^{+}+\pi^{+}+\pi^{-}+\gamma, \qquad (1)$$

in which [if one assumes a Q-value[2] of 74.3 ($\pm$0.3) Mev for the τ-meson] the energy of the emitted photon was 30.2 ($\pm$1.5) Mev. In this note, the characteristics of this alternative mode of decay will be discussed for several possible τ-meson spin values. Several other τ-meson decay processes which involve the electromagnetic field will also be briefly considered.

The emission of such an accompanying photon in τ-meson decay may arise either (a) as radiation from the pions emitted in the 3π-decay of the τ-meson, by the process of "internal bremsstrahlung," or (b) as a decay mode proceeding through distinct channels, a "direct emission" process, which will generally depend on details of the τ-meson structure. These two mechanisms will be coherent, of course, and their amplitudes will interfere in the total probability. They are represented graphically in Fig. 1.

In process (a), the three pions have been emitted and are penetrating the centrifugal barriers appropriate to the spin j and parity w of the τ-meson when the photon is emitted. For photons of long wavelength, the volume concerned in this emission process may be very considerable and the corresponding matrix element may therefore be very large. This is the dominant process in the emission of low-frequency photons, and it is the classical "internal bremsstrahlung" process, well known in beta-decay[3] and in the $\pi-\mu$ decay.[4] Clearly it is dependent on matrix elements (actually off-diagonal in energy) for the 3π-decay process and its probability may be predicted, to a sufficient approximation, from a knowledge of the matrix elements for the normal 3π-decay.

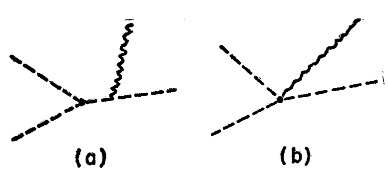

FIG. 1. Diagrams for radiative τ-meson decay.

The amplitude for process (a) may be obtained classically, and for this purpose the following representation of the 3π-decay is adequate. Up to time $t=0$, the τ-meson is at rest at the origin O; suddenly, at $t=0$, three pions, two positive and one negative, begin to move out from O and continue thereafter to move uniformly with momenta $(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$ respectively. The probability amplitude for this pion configuration will be denoted by G. Now it is well known[3] that the amplitude for emission of a photon by a particle of charge e which suddenly begins to move with constant momentum $\mathbf{p}$ [and energy $\omega=(\mu^2+p^2)^{\frac{1}{2}}$], is given by

$$\int d_3r\int_0^\infty dt\, e\frac{\mathbf{p}\cdot\boldsymbol{\varepsilon}}{\omega}\delta\left(\mathbf{r}-\frac{t}{\omega}\mathbf{p}\right)e^{i(kt-\mathbf{k}\cdot\mathbf{r})}=e\frac{\mathbf{p}\cdot\boldsymbol{\varepsilon}}{\omega k-\mathbf{p}\cdot\mathbf{k}}, \qquad (2)$$

since the current density $j(r,t)$ is

$$\mathbf{j}(\mathbf{r},t)=e\frac{\mathbf{p}}{\omega}\delta_3\left(\mathbf{r}-\mathbf{p}\frac{t}{\omega}\right),\quad t>0$$
$$=0. \qquad t<0$$

In Eq. (2) $\mathbf{k}$, $\boldsymbol{\varepsilon}$ are the momentum and the polarization vectors of the photon. For the present, the nonrelativistic approximation will be made, so that the retardation term $\mathbf{p}\cdot\mathbf{k}$ may be neglected in (2) and ω replaced by the pion mass μ. The inclusion of relativistic effects will be discussed in Appendix A. For the τ-meson decay, the amplitude for the internal bremsstrahlung process is then obtained by adding the separate amplitudes for each pion to give[5]

$$-e\frac{(\mathbf{p}_3-\mathbf{p}_1-\mathbf{p}_2)}{\mu k}G. \qquad (3)$$

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[1] R. R. Daniels and Y. Pal, Proc. Inst. Acad. Sci. **40**, 114 (1954).
[2] Amaldi, Baroni, Cortini, Franzinetti, and Manfredini, Nuovo cimento **12**, Supplement 2, 181 (1954).
[3] C. S. W. Chang and D. Falkoff, Phys. Rev. **76**, 365 (1949).
[4] H. Primakoff, Phys. Rev. **84**, 1255 (1951); and W. C. Fry, Phys. Rev. **91**, 130 (1951).
[5] Since it is the pion momenta before photon emission which appear in G, the inclusion of the photon recoil would replace (3).

For the three-pion system, it is convenient to introduce internal coordinates

$$\mathbf{p}=\mathbf{p}_3-\tfrac{1}{2}(\mathbf{p}_1+\mathbf{p}_2),\quad \mathbf{q}=\tfrac{1}{2}\sqrt{3}(\mathbf{p}_1-\mathbf{p}_2). \tag{4}$$

For a τ-meson initially at rest ($\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{k}=0$), (3) then becomes

$$-e\frac{(4\mathbf{p}+\mathbf{k})\cdot\boldsymbol{\varepsilon}}{\mu k}G=-\frac{4e}{3}\frac{\mathbf{p}\cdot\boldsymbol{\varepsilon}}{\mu k}G, \tag{5}$$

since ϵ is perpendicular to the photon direction. The energy equation for the system is now

$$\frac{1}{3\mu}(p^2+q^2)+k+\frac{k^2}{6\mu}=E, \tag{6}$$

where $E=M_\tau-3\mu$ is the energy release in the decay.

For spin (0−), it will be shown below that the amplitude for the direct emission process is unlikely to be appreciable in comparison with (5). For this case then, using the coordinates (4) and summing over photon polarizations, the probability per unit time for the decay mode (1) is given by

$$P(3\pi+\gamma)=\frac{16\alpha}{9}\int G^2\frac{p^2\sin^2\theta}{\mu^2k^2}\frac{2\pi}{k}\frac{d_3kd_3pd_3q}{(2\pi)^6}\times\left(\frac{2}{3\mu\sqrt{3}}\right)^3 2\pi\delta\left(\frac{p^2+q^2}{3\mu}-\Delta\right), \tag{7}$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{k}$, and

$$\Delta=E-k-(k^2/6\mu) \tag{8}$$

is the energy left to the three pions in their barycentric system. For comparison the probability per unit time for the 3π-decay is

$$P(3\pi)=\int G^2\frac{d_3qd_3p}{(2\pi)^3}\left(\frac{2}{3\mu\sqrt{3}}\right)^3 2\pi\delta\left(\frac{p^2+q^2}{3\mu}-E\right), \tag{9}$$

with the value $\pi G^2E^2/3\sqrt{3}$. For given k, the average value of p^2 is $3\mu\Delta/2$, and the relative probability of photon emission is then

$$\frac{P(3\pi+\gamma)}{P(3\pi)}=\frac{16}{9}\frac{\alpha}{\pi}\frac{\Delta^3}{\mu E^2}\frac{dk}{k}, \tag{10}$$

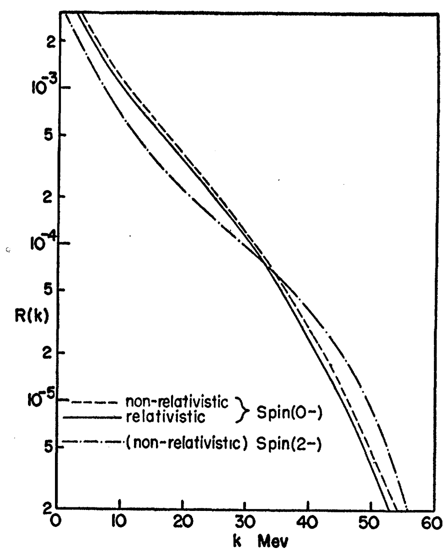

FIG. 2. The probability $R(k)$ for emission of a photon exceeding k Mev.

the maximum photon energy being $(9\mu^2+6\mu E)^{\frac{1}{2}}-3\mu\simeq 69$ Mev. This probability falls off rapidly with increasing k for two reasons, (i) the volume of phase space available to the final pions decreases quadratically with the energy left them by the outgoing photon, and (ii) the photon emission is proportional to the square of the current of the final system and hence to the final energy of the pions. According to (10), the branching ratio for emission of a photon $k>10$ Mev has the value 1.2×10^{-3}, and for a photon $k>30$ Mev, 1.2×10^{-4}, relative to the 3π-decay. The inclusion of the relativistic and retardation terms results in surprisingly little change in these figures, although the pion momenta are always of order μ. For both cases, the probability $R(k)$ for emission of a photon exceeding k Mev is plotted in Fig. 2.

Hence this classical process can account adequately for this one example of Daniels and Pal. The probability that one radiative decay event with photon exceeding 30 Mev should occur among the ~80 τ-meson decays now observed is about 1%, not an unreasonable chance. The probability for less energetic photons is rather greater, but the expected number of events with $k>1$ Mev is still only 0.50. Also, below about 5 Mev, the efficiency of detection of photon emission drops off rapidly, owing to technical difficulties in plate measurements—there has been considerable variation in the reported Q-values and it has often not been possible to establish coplanarity to better than several degrees. It should be added here that the detailed configuration of this event is not in conflict with this classical interpretation. According to expression (7), the photon emission has a $\sin^2\theta$ distribution: the observed angle θ is close to 130°. For given photon energy k, expression (7) indicates that the emission is more probable the greater the magnitude of p, i.e., the greater the pion current: the

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by

$$-e\epsilon\cdot\left[p_3G(p+k,q)-p_1G\left(p-\tfrac{1}{2}k,q+\frac{\sqrt{3}}{2}k\right)-p_2G\left(p-\tfrac{1}{2}k,q-\frac{\sqrt{3}}{2}k\right)\right]\Big/\mu k$$
$$=-e\left[\epsilon\cdot p\left(2G(p+k,q)+G\left(p-\tfrac{1}{2}k,q+\frac{\sqrt{3}}{2}k\right)+G\left(p-\tfrac{1}{2}k,q-\frac{\sqrt{3}}{2}k\right)\right)-\sqrt{3}\epsilon\cdot q\left(G\left(p-\tfrac{1}{2}k,q+\frac{\sqrt{3}}{2}k\right)-G\left(p-\tfrac{1}{2}k,q-\frac{\sqrt{3}}{2}k\right)\right)\right]\Big/3\mu k.$$

However, since the photon momentum is generally somewhat smaller than p or q, this modification will generally be found unimportant.

observed value of p is 95% of the maximum value permissible for the observed k.

The "direct emission" process (b) may be regarded as an electromagnetic transition from the initial τ-meson state (j,w) of the system to final states of various angular momenta J and parities W which then break up into three outgoing pions. Since the photons emitted in this process have wavelength long compared with the τ-meson radius R (which we will assume to be $R\approx h/M_\tau c$), only the lowest multipoles allowed by angular momentum and parity conservation will be effective. The three outgoing pions are then left to penetrate the centrifugal barriers appropriate to the state (J,W) and the amplitude for this may be estimated from considerations of barrier penetrability and angular momentum conservation, as described elsewhere.[6] However it will be found below that, for the cases of interest (even spin j and odd parity w), the structure-dependent multipole terms of the direct process are not generally important, because (a) they contain an additional factor $kR(\lesssim\frac{1}{8})$ and (b) the significant photon momenta are less than the corresponding pion momenta—photons of the highest possible energies are always improbable owing to the rapid decrease of the available phase space and of barrier penetrability with the decreasing energy left to the pions. For spin $j=0$, all direct emission terms are especially unfavoured since the only permissible photon emissions (zero-zero transitions being forbidden) lead to states $J>0$, in which the barriers against 3π-emission greatly exceed those in the initial state $j=0$.

In the limit of low frequencies, only the amplitude for electric dipole emission can be nonvanishing and, in fact, its value in this limit may be obtained from the value of G, the amplitude for the normal 3π-decay process, by gauge invariance arguments. For zero frequency, the external potential is uniform and may be removed by a gauge transformation whose effect is to replace each particle momentum $\mathbf{p}_i$ by $\mathbf{p}_i-e_i\mathbf{A}$. The amplitude $G_A(\mathbf{p},\mathbf{q},\mathbf{X})$ for the 3π-decay in the presence of an external potential is then given by $G(\mathbf{p}+2e\mathbf{A},\ \mathbf{q},\ \mathbf{X}-e\mathbf{A})$, $\mathbf{X}$ being the total momentum of the pion system. The zero-frequency matrix element for the direct emission process is then given by the linear term in the expansion of G_A in powers of $e\mathbf{A}$. This is discussed further in Appendix B. For spin $(0-)$ this leads to an amplitude

$$2e\lambda R^2 G\mathbf{p}\cdot\boldsymbol{\varepsilon}, \tag{11}$$

where λ is a constant (of order unity) which depends on a derivative of G off the energy shell. This term is generally negligible compared with (5)—a value $\lambda\sim1$ changes the above value of R(30 Mev) only by $\sim5\%$. Assuming spin $j=0$, an accurate value of $P(3\pi+\gamma)/P(3\pi)$ would determine λ and thereby give new information on the $(\tau\rightarrow3\pi)$ matrix element. For higher spin values, it may be shown that this gauge-dependent amplitude may be obtained, to a good approximation, simply by the explicit replacement $\mathbf{p}\rightarrow\mathbf{p}+2e\mathbf{A}$ in the expression for G—here this direct-emission amplitude is independent of the τ-meson structure except through the normal 3π-amplitude G.

All other direct emission amplitudes must be gauge-invariant and therefore depend only on the field $\mathbf{E}(k)(=k\boldsymbol{\varepsilon})$ or $\mathbf{H}(k)(=\mathbf{k}\times\boldsymbol{\varepsilon})$ of the photon, according as the transition is electric or magnetic. Following the methods used in the discussion of the τ-meson decay,[6] the form of these amplitudes may readily be written down, particularly for the simplest (and most important) transitions. For 2^l-pole photon emission $(l=j-J)$, from state (j,w) to state (J,W), this amplitude will be a sum of the symmetric, traceless tensors of rank j which are linear in $\mathbf{E}(k)$ or $\mathbf{H}(k)$ (according as the emission is electric or magnetic) and which have $(l-1)$ factors $\mathbf{k}$, the remaining factors consisting of either (i), if $W=(-1)^{J+1}$, an even number ($2n$, say) of factors q, the last $(J-2n)$ factors being $\mathbf{p}$, or (ii), if $W=(-1)^J$, then one factor $\mathbf{p}\times\mathbf{q}$, an odd number $(2n-1)$ of factors $\mathbf{q}$, the remaining $(J-2n)$ factors being $\mathbf{p}$. The state $(J=1,\ W=-1)$ may be omitted, as the barriers against dissociation into three pions are then especially high[7]—in fact, for given J, 3π-emission is more difficult from state $W=(-1)^J$ than from a state $W=(-1)^{J+1}$ and the states (ii) above are not usually important. The coefficients of the various tensors (i) above contain a factor R^{j+1}—their momentum variation may be neglected. A complete expression will not be written out here but the above remarks will be illustrated by discussion of the case $(2-)$. The present statistics[8] on 3π-decay of the τ-meson suggest strongly that the spin j is even and the parity odd, so that this is the simplest case of nonzero spin which need be considered.

For spin $(2-)$, the amplitude G for 3π-decay is given by

$$G=[AT_{ij}(\mathbf{p},\mathbf{p})+BT_{ij}(\mathbf{q},\mathbf{q})]R^2, \tag{12}$$

where the symmetric, traceless, second-rank tensor $(x_iy_j+x_jy_i-\frac{2}{3}x\cdot y\delta_{ij})$ has been denoted by $T_{ij}(x,y)$. The present data on the energy correlations between pions in τ-meson decay requires that $|A|\sim|B|$ and that A and B should differ in phase by $\sim\pi/2$. The

[6] R. H. Dalitz, Phil. Mag. 44, 1068 (1953) and Phys. Rev. 94, 1046 (1954); E. Fabri, Nuovo cimento 11, 479 (1954).

[7] The case of spin $(1-)$ is an exceptional one among low-spin values, A 3π-disintegration is very much unfavored because the relative motion of the like pions and the motion of the unlike pion relative to them must each carry two units of angular momentum, and therefore face large centrifugal barriers. E. Fabri and B. F. Touschek [Nuovo cimento 11, 96 (1954)] have pointed out that these barriers may contribute a factor $\sim10^{-10}$ to the partial lifetime for 3π-decay and that this alone could account for the long observed τ-meson lifetime if all other competing processes were also strongly forbidden. However if the long life arises only from barrier effects, it is not easily understood why other processes for which the barriers are much less should not proceed rapidly—even if the 2π-decay is partially forbidden by some weak selection rule, the decay mode $2\pi+\gamma$ should proceed rapidly. In fact, even $3\pi+\gamma$ decay should proceed more rapidly than 3π-decay, by an $M1$ direct emission to the $(0-)$ state.

[8] R. H. Dalitz, Proceedings of the Fifth Rochester Conference, February, 1955 (to be published). A more detailed report is in preparation.

internal bremsstrahlung amplitude may now be written down, using the expression of reference 5 and the amplitude (12). The gauge-dependent $E1$ matrix element is now

$$4R^2AT_{ij}(\boldsymbol{\varepsilon},\mathbf{p}). \tag{13}$$

The only two gauge-invariant transitions to be considered are an $E1$ emission leading to $J=(1+)$ with amplitude

$$\rho AR^3T_{ij}(\mathbf{E},\mathbf{p})=\rho A(kR)T_{ij}(\boldsymbol{\varepsilon},\mathbf{p})R^2,$$

and an $E2$ emission leading to $J=(0-)$ with amplitude

$$\eta AR^3T_{ij}(\mathbf{E},\mathbf{k})=\eta A(kR)T_{ij}(\boldsymbol{\varepsilon},\mathbf{k})R^2.$$

Unless ρ or η are rather large, neither of these terms will contribute significantly to $P(3\pi+\gamma)$—for example, the η^2 term in the probability will increase $R(30)$ only in the proportion $0.002\eta^2$. Finally, with $B=iA$ and keeping only appreciable terms,

$$\frac{P(3\pi+\gamma)}{P(3\pi)}=\frac{16\alpha}{9\pi}\left\{\left(\frac{\Delta}{k}\right)^2-2\frac{\Delta}{k}+4\right\}\frac{\Delta^3kdk}{\mu E^4}. \tag{14}$$

That expressions (14) and (10) agree for very soft photons reflects the fact that the matrix element G used in each case reproduces the important features of the observed distributions in the 3π-decay. The form of the spectrum (14) is not greatly different from that predicted for zero spin—$R(30)$ is now 0.98×10^{-4} and $R(10)$ is 6.3×10^{-4}—the function $R(k)$ corresponding to (14) is plotted in Fig. 2.

It is also of interest to discuss on similar lines some further decay processes possible for the τ-meson. A model is necessary for the estimation of their branching ratios, and here we shall assume simply that all these processes are determined by a common "weak link" in the sequence of virtual processes between τ-meson and the final state. For order of magnitude estimates, it is unnecessary to specify the nature of this "weak link" in detail but, for example, it may be that the Λ^0-decay provides the basic weak process, so that

$$\tau^+\rightarrow\bar{\Lambda}^0+p\rightarrow\bar{p}+\pi^++p\rightarrow \text{final decay products.} \tag{15}$$

Certainly the decay processes of the K-mesons and the hyperons must be linked together through their joint coupling with nucleons, but the situation may well be more complicated than this. With this fairly general model, the probability of a particular decay process may now be estimated from considerations of angular momentum conservation, barrier penetrability, and the volume of phase space allowable to the final particles, and will depend on the mass and radius of the τ-meson and on the coupling strength of the weak link. Such estimates will clearly be uncertain by numerical factors of order unity.

For spin $j\neq0$, the process

$$\tau^+\rightarrow\pi^++\gamma \tag{16}$$

would be rather rapid, owing to the large energy release. For spin $(2-)$ the matrix element has the form $gRT_{ij}(\mathbf{E},\mathbf{k})$, leading to decay probability $16\pi\alpha g^2(kR)^2k^3/M$ per unit time. Assuming $g\sim A$ [Eq. (12)], the branching ratio is

$$\frac{P(\pi+\gamma)}{P(3\pi)}\sim6\left(\frac{k}{E}\right)^4\frac{k}{M}\frac{1}{(\mu R)^2}\sim24. \tag{17}$$

This ratio is typical of the higher spin values. The absence of evidence for the decay process (16) from multiplate cloud-chamber studies is at present an important difficulty in the assumption of a nonzero τ-meson spin, although this could result if g/A were rather small (<0.1).

For $j=0$, the zero-zero transition (16) is, of course, completely forbidden. However, the internal pair conversion process

$$\tau^+\rightarrow\pi^++e^++e^- \tag{18}$$

is possible for all spin values. In plates, this process would be readily identified, whereas there are known to be a number of K-meson decays which lead to one relativistic charged particle and with which an event (16) may be confused. For spin $j\neq0$, process (18) results from internal conversion due to the transverse field produced in the decay and, as in the π^0-decay, the pair produced would be closely collimated opposite the direction of the pion. The internal conversion coefficient is 0.90×10^{-2}, so that the branching ratio $P(\pi+e^++e^-)/P(3\pi)$ would be ~0.2 for spin $(2-)$. The absence of this process in plates at present requires $g/A<\frac{1}{4}$ if the τ-meson has spin $(2-)$. For $j=(0-)$, the virtual electromagnetic field is spherically symmetric (and therefore longitudinal) and the pairs produced are wide angle pairs with angular separation $\sim(1+\cos\theta)d(\cos\theta)$. The coupling with the electric field will have the form $CR\mathbf{k}\cdot\mathbf{E}(k)$ and the branching ratio is then rather small, being $3\alpha^2(\mu R)^2(C/A)^2(M-\omega)^5/\mu^4\omega\sim5\times10^{-5}(C/A)^2$.

The further decay mode

$$\tau^+\rightarrow\pi^++\pi^0+\gamma \tag{19}$$

may occur by a direct emission process. The phase space available is 5.4 times that for the 3π-decay. For spin $(0-)$, this process will proceed by $M1$ photon emission to a $(1-)$ state which then breaks up to two pions. Owing to the increased barriers in the final state, the matrix element $\sigma R^2\mathbf{q}\cdot\mathbf{H}(k)$ is rather small and the relative probability for this mode is reduced to a value $(\mu R)^4 10^{-2}\sim10^{-4}$ for $\sigma\sim G$. For spin $(2-)$, the most important matrix element is $\eta R^2T_{ij}(\mathbf{H}(k),\mathbf{q})$ and the relative probability to 3π-decay is close to $\sim4\%$.

Other decay processes leading to final states which include neutrinos may also occur but the estimation of their probabilities will require far more detailed assumptions than are needed above—these processes will not be discussed here.

Hence, for a pseudoscalar τ-meson, it would be readily understood for the dominant decay mode (excluding neutrino processes) to be 3π-decay. The next probable decay process is then the radiative 3π-decay and it is not unreasonable to observe one such event at the present stage. For higher spin values compatible with the 3π-decay statistics, it would be natural to expect the $(\pi+\gamma)$ decay to compete favorably with 3π-decay—the absence of evidence for this decay process or for the related internal conversion process (18) may provide some argument against nonzero spin values for the τ-meson as data on the decay processes accumulate.

In conclusion, I am pleased to thank Dr. M. Gell-Mann for a discussion concerning these questions. I am also much indebted to Dr. B. Peters for early information concerning the radiative τ-meson decay event observed by the Bombay group.

APPENDIX A. RELATIVISTIC CALCULATION OF THE INTERNAL BREMSSTRAHLUNG

In the simple discussion of internal bremsstrahlung given above, retardation terms and other relativistic effects were neglected. However the retardation terms are not generally small compared with the terms retained. In discussing these relativistic terms here, it proves most direct to carry the calculation through in a completely relativistic way.

Using the covariant notation for the various graphs of type (a), the probability for emission of a photon k and pions of four-momenta p_1, p_2, p_3 is given by

$$\Sigma_\nu M_\nu^2(2\pi)^4\delta_4(p_1+p_2+p_3+k-P)\times\frac{d_3p_1}{(2\pi)^2\omega_1}\frac{d_3p_2}{(2\pi)^2\omega_2}\frac{d_3p_3}{(2\pi)^2\omega_3}\frac{d_3k}{(2\pi)^2k},\quad\text{(A1)}$$

where P is the four-momentum of the τ-meson and M_μ is given by

$$M_\nu=eG\left(\frac{P_\nu}{P\cdot k}-\frac{p_{1\nu}}{p_1\cdot k}-\frac{p_{2\nu}}{p_2\cdot k}+\frac{p_{3\nu}}{p_3\cdot k}\right).\quad\text{(A2)}$$

In this expression the scalar products are four-dimensional and G denotes the amplitude for emission of the three pions, assumed independent of the momenta.

Observations on the photon would naturally be made for τ-mesons at rest, whereas the integration over the pion configurations is most readily carried through in the c.m. system of the three pions. Hence, for fixed photon momentum k, we transfer to the pion c.m. system. In this system the photon has four-momentum $(K_0,\mathbf{K})$ where $\mathbf{K}=\mathbf{k}[M/(M-2k)]^{\frac{1}{2}}$ and $K_0=K$. The τ-meson four-momentum $(P_0,\mathbf{P})$ is given by $P_0=(M^2+K^2)^{\frac{1}{2}}$, $\mathbf{P}=\mathbf{K}$, where M denotes the τ-meson mass, and the pion momenta will be denoted by $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$. Since the volume elements of (A1) are invariant, the probability expression becomes

$$\frac{1}{(2\pi)^4}\Sigma_\nu M_\nu^2\delta_3(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3)\delta(\omega_1+\omega_2+\omega_3-E_k)\times\frac{d_3q_1}{\omega_1}\frac{d_3q_2}{\omega_2}\frac{d_3q_3}{\omega_3}\frac{d_3k}{k},\quad\text{(A3)}$$

where $E_k=(M^2-2Mk)^{\frac{1}{2}}$ and, now,

$$\Sigma_\nu M_\nu^2=\frac{M-2k}{M}\left[\frac{M^2}{(P\cdot k)^2}+\frac{\mu^2}{(q_1\cdot k)^2}+\frac{\mu^2}{(q_2\cdot k)^2}+\frac{\mu^2}{(q_3\cdot k)^2}-\frac{2P\cdot q_1}{(P\cdot k)(q_1\cdot k)}-\frac{2P\cdot q_2}{(P\cdot k)(q_2\cdot k)}+\frac{2P\cdot q_3}{(P\cdot k)(q_3\cdot k)}+\frac{2q_1\cdot q_2}{(q_1\cdot k)(q_2\cdot k)}-\frac{2q_2\cdot q_3}{(q_2\cdot k)(q_3\cdot k)}-\frac{2q_3\cdot q_1}{(q_3\cdot k)(q_1\cdot k)}\right].\quad\text{(A4)}$$

Now in the integrations to be carried out there is complete symmetry between q_1, q_2 and q_3, so we may replace (A4) by

$$\frac{M-2k}{M}\left[\frac{M^2}{k^2(M-2k)}+\frac{3\mu^2}{(q_3\cdot k)^2}-\frac{2P\cdot q_3}{(P\cdot k)(q_3\cdot k)}-\frac{2q_1\cdot q_2}{(q_1\cdot k)(q_2\cdot k)}\right].\quad\text{(A5)}$$

To carry through the integrations over q_1, q_2, a four-dimensional notation is very convenient. The q_1, q_2 phase-space volumes may be written

$$\frac{d_3q_1}{\omega_1}\frac{d_3q_2}{\omega_2}=4\delta(q_1^2-\mu^2)\delta(q_2^2-\mu^2)d_4q_1d_4q_2=\delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)d_4Qd_4R,\quad\text{(A6)}$$

where $Q=q_1+q_2$, $R=q_1-q_2$. The δ-functions of (A3) imply that Q is $(E_k-\omega_3, -\mathbf{q}_3)$ and only the R integration is left. With the following two integrals,

$$\int\delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)d_4R=2\pi\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{\frac{1}{2}},$$

$$\int\frac{Q^2-R^2}{(Q\cdot k)^2-(R\cdot k)^2}\delta(Q\cdot R)\delta(Q^2+R^2-4\mu^2)d_4R=\frac{4\pi(Q^2-2\mu^2)}{(Q\cdot k)^2}\arctan\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{\frac{1}{2}}.$$

Integration over directions of $\mathbf{q}_3$ and $\mathbf{k}$ may then be carried through to give the final result

$$P(3\pi+\gamma)=2G^2\frac{2\alpha}{\pi}\left(1-\frac{2k}{M}\right)\Phi(k)\frac{dk}{k},\quad\text{(A7)}$$

where

$$\Phi(k)=\int\left\{q[1+x^2(k)]\operatorname{arctanh}x(k)+x(k)\omega\operatorname{arctanh}\frac{q}{\omega}-2qx(k)\right\}d\omega,\quad\text{(A8)}$$

and $x(k)=\{(E_k^2-2\omega E_k-3\mu^2)/(E_k^2-2\omega E_k+\mu^2)\}^{\frac{1}{2}}$.

Similarly $P(3\pi)$ is given by

$$P(3\pi)=2G^2\int qx(0)d\omega, \tag{A9}$$

with the value $0.153G^2\mu^2$, assuming an energy release of 74.3 Mev in τ-meson decay. The integral

$$R(k)=\int_k^{k_{\max}} P(3\pi+\gamma)/P(3\pi)$$

is plotted in Fig. 2 as function of k (Mev)—the value $R(30)$ is now 1.1×10^{-4}, $R(10)$ being 1.06×10^{-3}.

APPENDIX B. MATRIX ELEMENTS FOR $E1$ TRANSITIONS

In covariant notation, the (off-diagonal) matrix element for 3π-decay of a spin j τ-meson may be denoted by $N_{\alpha\beta\cdots}(P,p,q)$, where $N_{\alpha\beta\cdots}$ has j suffices and P, p, q are four-momenta defined by

$$\begin{aligned}P&=p_1+p_2+p_3,\\ p&=p_3-\tfrac{1}{2}(p_1+p_2),\\ q&=\tfrac{1}{2}\sqrt{3}(p_1-p_2),\end{aligned} \tag{B1}$$

where p_3 and p_1, p_2 refer to the unlike and like pions respectively. The matrix element $N_{\alpha\beta\cdots}$ is a symmetrical, traceless tensor of rank j, which satisfies also

$$P_\alpha N_{\alpha\beta\cdots}=0. \tag{B2}$$

In the c.m. system, $P=(M_\tau,0,0,0)$ so that only completely space-like components of $N_{\alpha\beta\cdots}$ are then nonzero. $N_{\alpha\beta\cdots}$ therefore has just $(2j+1)$ independent components.

If this decay process proceeds in the presence of a weak field $A_\nu(k)$, there will be a first-order addition to the above matrix element, which we denote by $N_{\alpha\beta\cdots,\nu}(P,p,q;k)A_\nu(k)$. In particular, the matrix element for emission of a photon of polarization vector ϵ_ν together with the three pions is then given by $\epsilon_\nu N_{\alpha\beta\cdots,\nu}$. For frequencies $kR\ll1$, this matrix element will vary only little with k if it approaches a nonzero limit as $k\rightarrow0$. This zero-frequency limit gives the matrix element for the electric dipole transition, with errors of order (kR) for nonzero k. When $k=0$, the external field $A_\nu(k)$ is a constant field a_ν, say, which can be removed by a gauge transformation. This transformation replaces $N_{\alpha\beta\cdots}(P,p,q)$ by

$$N_{\alpha\beta\cdots}(P-ea,\, p+2ea,\, q), \tag{B3}$$

so that the required $E1$ matrix element (at zero frequency) is

$$\epsilon_\nu N_{\alpha\beta\cdots,\nu}=e\epsilon_\nu\left(2\frac{\partial N_{\alpha\beta\cdots}}{\partial p_\nu}-\frac{\partial N_{\alpha\beta\cdots}}{\partial P_\nu}\right). \tag{B4}$$

After these differentiations, relationships between p, q and P which are effective on the energy shell may then be substituted. Two cases will be considered: (a) *spin 0.* Expanding N in powers of pR and qR gives

$$N=A\left(\frac{P^2}{M^2}\right)+\frac{P\cdot p}{M}RB\left(\frac{P^2}{M^2}\right)+p^2R^2C\left(\frac{P^2}{M^2}\right)+q^2R^2D\left(\frac{P^2}{M^2}\right)+\cdots, \tag{B5}$$

where terms of order $(\mu R)^3$ are omitted, and then

$$N_\nu=e\left[-\frac{2P_\nu}{M^2}\left(A'+B'\frac{P\cdot p}{M}R+\cdots\right)-\frac{B}{M}p_\nu R+2B\frac{P_\nu}{M}R+2Cp_\nu R^2+\cdots\right].$$

In c.m. system, the only part of this appropriate for transverse polarizations is

$$\epsilon_\nu N_\nu=ep\cdot\epsilon R^2\left(2C-\frac{B}{MR}\right), \tag{B6}$$

where the coefficient is evaluated on the energy shell in the τ-meson rest system. (b) *spin 1.* The form of N_α is given by

$$N_\alpha=(P^2p_\alpha-P\cdot pP_\alpha)(R/M^2)\phi(P^2,P\cdot p,\cdots). \tag{B7}$$

It will be clear that, in calculating (B4), the terms coming from will be of order $(pR)^2$ relative to the main terms, and we will therefore neglect them to find

$$N_{\alpha,\nu}=(-2P_\nu p_\alpha+p_\nu P_\alpha-2P_\nu P_\alpha+\delta_{\alpha\nu}(2P^2+P\cdot p))\frac{R}{M^2}\phi+\cdots. \tag{B8}$$

Only space-like components are relevant in c.m. system and then

$$\epsilon_\nu N_{\alpha,\nu}=2\epsilon_\alpha\left(1-\frac{\omega}{2M}\right)R\phi(P^2=M^2,0,0\cdots). \tag{B9}$$

For consistency the term $\omega/2M$ should also be neglected, to give the result $2\phi R\epsilon$ for this matrix element.

This result (B9) is typical of nonzero spin values and it will readily be seen that generally the dominant term of the zero-frequency $E1$ matrix element is obtained simply by the explicit replacement $p\rightarrow p+2eA$ in the matrix element for 3π-decay.