Multiple Scattering Corrections in $\pi^{\pm}$ Deuteron Scattering*

S. D. DRELL AND L. VERLET
Department of Physics and Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts
(Received March 9, 1955)

Multiple-scattering corrections to the impulse approximation are calculated for the scattering of $\pi^{\pm}$ mesons by deuterons. These corrections are model-dependent since the scattered wave propagates off the energy shell between the first and last scatterings whereas experimental scattering phase shifts are available only for elastic scattering. Various assumptions are made concerning the off-the-energy-shell behavior of the scattering amplitudes and the double-scattering approximation is analyzed for meson energies of 45 Mev and 169 Mev. The results are tabulated and are found to be sensitive to the different scattering models. The cross sections at 169 Mev were calculated with both the Fermi and Bethe solutions for the meson-nucleon scattering phase shifts and are in close agreement. It thus does not seem to be feasible to use $\pi^{\pm}-$D scattering as a means of distinguishing between these two solutions.

I. INTRODUCTION

THE impulse approximation provides a straightforward phenomenological approach to the analysis of the scattering of π mesons by deuterons or heavier nuclei. In this approximation the nucleons are considered to scatter as free particles which have the same momentum distribution as the initially bound nucleons. It is thus possible to characterize the scattering purely in terms of the experimentally observed phase shifts for π-nucleon scattering and in terms of the nuclear momentum distribution. Corrections to the impulse approximation arise due to the effect of the binding force during the scattering, the diminution in amplitude of the incident wave in crossing the nucleus, and the multiple-scattering effects, as discussed by Chew, Wick, Ashkin, and Goldberger.[1–3]

This note considers the multiple-scattering corrections to the impulse approximation in the calculation of the scattering cross sections of $\pi^{\pm}$ mesons by deuterons. In consequence of the small deuteron binding energy and of the fact that the deuteron consists of but two nucleons, it is expected that multiple-scattering corrections will be the only ones of importance for incident π mesons of kinetic energies greater than ≈ 50 Mev.

The motivation of this work derives from the experimental feasibility of measuring $\pi^{\pm}$ deuteron scattering cross sections, and from the theoretical interest in testing the accuracy of the impulse approximation and in finding the dominant features of the multiple-scattering correction.

To a certain extent this correction is model-dependent since the scattered wave propagates off the energy shell between the first and last scatterings, whereas experimental scattering phase shifts are available only for elastic scattering (on the energy shell). Various assumptions concerning the off-energy-shell behavior of the scattering amplitude are made in order to test the sensitivity of the results to specific models. In particular, a point-scattering model has been discussed earlier by Chew and Wick[1] and by Brueckner[4] and will be considered within the framework of our results.

II. OUTLINE AND RESULTS

We describe here the approximation and results of these calculations.

Firstly, we neglect the binding and motions of the sources during the scattering, and use the Hulthén wave function to describe the deuteron. *Secondly*, we neglect the possibility of absorption of the meson by the deuteron. *Thirdly*, we treat only the double-scattering corrections. In principle it is possible to retain the higher-order scattering corrections with more complicated algebra to take into account the different scattering phase shifts for various isotopic spin and angular momentum states. However, we expect that the main multiple-scattering corrections arise from the double scattering so that this procedure should yield the qualitative features of these corrections. *Fourthly*, we consider scattering of the meson by a nucleon to occur only in the six states ($S_{\frac{1}{2}}$, $P_{\frac{1}{2}}$, and $P_{\frac{3}{2}}$; isotopic spin $=\frac{1}{2}$ and $\frac{3}{2}$) which have been used in the analysis of π-hydrogen cross sections.

Now the experiments on π-nucleon scattering furnish information on the elastic scattering in the form of a scattering amplitude,

$$f(\mathbf{k},\mathbf{k}_0)=k^{-1}(\alpha+\beta\mathbf{k}\cdot\mathbf{k}_0-i\gamma\boldsymbol{\sigma}\cdot\mathbf{k}_0\times\mathbf{k}), \qquad (1)$$

where $\mathbf{k}_0$ and $\mathbf{k}$ are respectively the momenta of the ingoing and outgoing particles, with $k=k_0$; $\boldsymbol{\sigma}$ is the 2×2 Pauli spin matrix for the nucleon; and we write

$$\alpha=\eta, \quad \beta k^2=\eta^-+2\eta^+, \quad \gamma k^2=\eta^--\eta^+, \qquad (2)$$

where η, η^+, and η^-, refer respectively to the angular momentum states S, $P_{\frac{3}{2}}$, and $P_{\frac{1}{2}}$. Further, the η's are operators in isotopic space expressing scattering in states of isotopic spin $\frac{3}{2}$ and $\frac{1}{2}$. They appear in the general form for the α, βk^2, γk^2, of the type

$$\eta=\eta'+\mathbf{t}\cdot\boldsymbol{\tau}\eta'', \qquad (3)$$

* This work was supported in part by the Office of Naval Research and the U. S. Atomic Energy Commission.
[1] G. Chew and G. Wick, Phys. Rev. **85**, 636 (1952).
[2] J. Ashkin and G. Wick, Phys. Rev. **86**, 686 (1952).
[3] G. Chew and M. Goldberger, Phys. Rev. **87**, 778 (1952).
[4] K. Brueckner, Phys. Rev. **89**, 834 (1953); Phys. Rev. **90**, 715 (1953).

where $\boldsymbol{\tau}$ is the isotopic spin of the nucleon and $\mathbf{t}$ that of the meson.

In treating the multiple-scattering problem we must go beyond the experimentally available η's given previously and construct specific models to describe the scattering of the meson which propagates (virtually) off the energy shell after scattering by the first nucleon and prior to scattering by the second one. That is, we must assume a form for the scattering amplitude for a meson which is scattered by one nucleon off the energy shell from an initial state $\mathbf{k}_0$ to an intermediate state of momentum $\mathbf{q}$, with $q \neq k_0$, before the second nucleon scatters it back to a final state $\mathbf{k}$, with $k = k_0$.

We may assume that the form of Eq. (1), with α, β, and γ dependent upon the momenta of the incident pion only, is valid even for scattering off the energy shell. This case, studied by Brueckner[4] for point sources, corresponds to the use of a singular propagator for the intermediate meson wave of the form $g_0 = \exp(ikR)/kR$, with $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$ the distance between the two scattering sources. It predicts large scattering amplitudes for scatterings far off the energy shell which lead to virtual intermediate mesons of high momentum. Thereby it attaches great importance to the region of quite small separation between the nucleons where the above approximations may be poor.

To get around this difficulty we can suppose that the scattering amplitude smoothes out for large momenta. This would result physically for scattering by sources of finite extent. It is possible to achieve this effect by considering scattering from a separable potential such as studied by Wentzel,[5] Blatt,[6] Goldberger,[7] and Yamaguchi.[8] In this method we approximate the Schrödinger equation describing the scattering of a particle by a potential, by replacing the wave function which appears in the interaction term by its average over the potential. With this approximation the scattering problem reduces to quadratures. For each nucleon we choose the potential to be a square well of fixed radius, with the depths for S and P states so adjusted as to reproduce the experimental results for the pion-nucleon scattering. With these assumptions we find that when the sources don't overlap, the intermediate wave propagates the same way as for point sources, i.e., $g_0 = \exp(ikR)/kR$. When the sources overlap, however, the propagation factor depends on the well parameters, but remains finite. Its detailed variation has but a very slight effect on the cross sections averaged over the Hulthén function.

A third possible model of the momentum variation of the scattering amplitude considers only scattering on the energy shell in the intermediate state. In this approximation the intermediate wave propagates as $g_0 = i\sin(kR)/kR$. Since this form of the propagator is finite and smooth for $kR < 1$, it doesn't matter whether the sources are assumed to be of zero or of finite extent.

The aforementioned three models give vastly different momentum dependences for the off-the-energy-shell scattering amplitudes and provide a comparison basis for an understanding of the significance of our results. We discuss the predictions of these models for S-wave scattering from two spinless sources in the following paragraph. Our aims are to exhibit the method of obtaining the cross sections with a minimum of algebra, to compare the predictions of the different models, and to analyze the double-scattering approximation. Numerical calculations are presented in paragraph IV, for the actual case of $\pi^{\pm} - D$ scattering at energies of 45 and 169 Mev.

To summarize in brief our results: at 45 Mev the double-scattering correction to the elastic cross section $\pi^{\pm} + D \to D + \pi^{\pm}$, is of the order of 10 percent or less and is model-dependent. At 169 Mev the same correction for forward scattering depends on the model, but for backward scattering, it is fairly independent of the assumptions made and leads to a 30 percent reduction of the cross sections. For the total cross sections (elastic, inelastic, and absorption) as deduced from the imaginary part of the scattering amplitude in the forward direction, the correction is of the order of 10 percent or less and is quite model-dependent.

Finally, we have considered the possibility of using the $\pi^{\pm} + D \to D + \pi^{\pm}$ reaction as a means of selecting between the various sets of phase shifts for the π-nucleon cross sections as proposed by Fermi and by Bethe. In view of the uncertainties involved in this calculation, no such distinction can be drawn.

III. S-WAVE SCATTERING BY TWO IDENTICAL SOURCES

Before discussing the physical pion-deuteron scattering problem, we analyze in this section the S-wave scattering by two spinless identical sources. We aim thus to illustrate the methods of this work with a minimum of algebraic complications, to compare the predictions of the various models discussed in the previous paragraph, and to analyze the double-scattering approximation.

Chew and Goldberger[3] have explicitly written the double-scattering term in Eq. (29a) of their paper. It is simply rederived in Appendix A. For two identical sources[9] separated by a distance R, we have the following expression for the elastic (single plus double) scattering of a meson wave with initial wave number $\mathbf{k}_0$ and final wave number $\mathbf{k}$:

$$f(\mathbf{k},\mathbf{k}_0) = f_1(\mathbf{k},\mathbf{k}_0) + f_2(\mathbf{k},\mathbf{k}_0) + \frac{4\pi}{(2\pi)^3}\int \frac{d^3\mathbf{k}}{q^2 - k^2 - i\epsilon} \times \{f_1(\mathbf{k},\mathbf{q}) f_2(\mathbf{q},\mathbf{k}_0) + f_2(\mathbf{k},\mathbf{q}) f_1(\mathbf{q},\mathbf{k}_0)\}, \quad (4)$$

[5] A. Wentzel, Helv. Phys. Acta **15**, 111 (1942).
[6] J. M. Blatt, Phys. Rev. **72**, 466 (1947).
[7] M. Goldberger, Phys. Rev. **84**, 929 (1951).
[8] S. Yamaguchi, Phys. Rev. **95**, 1628 (1954).

[9] The limitation to identical sources is not necessary.

where $f_i(\mathbf{k},\mathbf{q})$ represents the scattering amplitude for scattering by the ith source from $\mathbf{q}$ to $\mathbf{k}$ and

$$f_2(\mathbf{k},\mathbf{q})=e^{i(\mathbf{q}-\mathbf{k})\cdot\mathbf{R}}f_1(\mathbf{k},\mathbf{q}). \quad (5)$$

Equation (4) exhibits explicitly the sum over the intermediate "off-the-energy-shell" momenta in the double-scattering sum. If we assume for S scattering that $f_1(\mathbf{k},\mathbf{q})$ is a function of the incident energy only and is independent of $\mathbf{q}$, i.e., $f_1=\alpha(k)/k$, then, using Eq. (5), we have directly:

$$f_{\mathrm{I}}(\mathbf{k},\mathbf{k}_0)=\frac{\alpha}{k}\left(e^{-i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{R}/2}+e^{i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{R}/2}+\alpha\frac{e^{ikR}}{kR}(e^{i(\mathbf{k}_0+\mathbf{k})\cdot\mathbf{R}/2}+e^{-i(\mathbf{k}_0+\mathbf{k})\cdot\mathbf{R}/2})\right). \quad (6)$$

With this model of the scattering amplitude and with the approximation of only single and double scattering, the region of close coincidence of the two scattering sources is falsified badly, since the scattering amplitude diverges as $R\rightarrow 0$. Actually, we have for $R=0$ scattering from a source of double the strength of the individual scattering sources. The effect on the scattering amplitude of doubling the source strength is, of course, model-dependent except in the limit where the Born approximation is valid, and the scattering amplitude is also doubled. We can avoid this divergence as $R\rightarrow 0$ by summing up directly all of the triple and higher-order scattering effects. As shown, in Appendix A, it is possible to effect this sum for constant $f_1(\mathbf{q},\mathbf{k})$, with the result that the scattering amplitude becomes

$$f_{M\,\mathrm{I}}(\mathbf{k},\mathbf{k}_0)=[1-\alpha^2(e^{ikR}/kR)^2]^{-1}f_{\mathrm{I}}(\mathbf{k},\mathbf{k}_0), \quad (7)$$

as deduced by Chew and Wick[1] and Brueckner.[4] We see here that the interference among the multiple-scattered waves leads to the prediction that the scattering amplitude decreases linearly with R, vanishing as $R\rightarrow 0$. This again falsifies badly the region of close coincidence of the two sources as discussed above. Equations (6) and (7) indicate the dangers concomitant with the assumption of constant off-the-energy-shell scattering amplitudes as applied to any problem for which the region $kR<1$ is of importance. That this region is of importance in the analysis of the multiple-scattering corrections for π^+-D scattering is seen in Fig. 3 and Table II of reference 4. The main multiple-scattering correction there comes from the region $kR<1$.

We discuss now alternate models which predict widely different behaviors for the scattering amplitude due to two sources near one another. As an extreme assumption we might postulate in Eq. (4) that the individual amplitudes $f_i(\mathbf{k},\mathbf{q})$ vanish for scattering off the energy shell. We then have, since the principal value contribution to the sum in Eq. (4) is omitted according to this assumption,

$$f_{\mathrm{II}}(\mathbf{k},\mathbf{k}_0)=\frac{\alpha}{k}\left(e^{-i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{R}/2}+e^{i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{R}/2}+\alpha\frac{i\sin kR}{kR}(e^{i(\mathbf{k}_0+\mathbf{k})\cdot\mathbf{R}/2}+e^{-i(\mathbf{k}_0+\mathbf{k})\cdot\mathbf{R}/2})\right). \quad (8)$$

The intermediate propagator $g_0=i(\sin kR)/kR$ is nonsingular according to this model and yields a reasonable scattering amplitude in the $R\rightarrow 0$ limit. Thus we see by Eq. (8) that in this limit of doubled source strength, the scattering amplitude becomes $2[\alpha(k)/k][1+i\alpha(k)]$. With this assumption of no scattering off the energy shell, one can also sum up the multiple-scattering effects, obtaining in analogy with Eq. (7),

$$f_{M\,\mathrm{II}}(\mathbf{k},\mathbf{k}_0)=[1+\alpha^2(\sin kR/kR)^2]^{-1}f_{\mathrm{II}}(\mathbf{k},\mathbf{k}_0). \quad (9)$$

A less extreme model based on the assumption of a separable potential also admits exact multiple-scattering solutions. According to this model, the Schrödinger equation for the scattering of a wave by a spherical potential λU is replaced by the inhomogeneous equation

$$(\Delta+k^2)\psi(\mathbf{r})=\lambda U(r)\bar{\psi}, \quad (10)$$

where $\bar{\psi}=\int\psi(\mathbf{r})U(r)d^3\mathbf{r}$, with $U(r)$ normalized to unit volume, denotes the wave function averaged over the potential. The solution of this equation corresponding to outgoing scattered waves reads

$$\psi(\mathbf{r})=e^{i\mathbf{k}_0\cdot\mathbf{r}}-\frac{\lambda}{4\pi}\bar{\psi}\int\frac{\exp ik|\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|}U(r')d^3\mathbf{r}'.$$

Averaging over the source, we obtain:

$$\bar{\psi}=\langle\varphi(\mathbf{k}_0)\rangle_{\mathrm{Av}}-\lambda\bar{\psi}G,$$

with the defining relations

$$\langle\varphi(\mathbf{k}_0)\rangle_{\mathrm{Av}}=\int\exp(i\mathbf{k}_0\cdot\mathbf{r})U(r)d^3\mathbf{r},$$

and

$$G=(4\pi)^{-1}\int\int\frac{\exp ik|\mathbf{r}-\mathbf{r}'|}{k|\mathbf{r}-\mathbf{r}'|}U(r)U(r')d^3\mathbf{r}d^3\mathbf{r}'.$$

The scattering amplitude is, accordingly,

$$f_1(\mathbf{k},\mathbf{k}_0)=-(\lambda/4\pi)\bar{\psi}\langle\varphi(-\mathbf{k})\rangle_{\mathrm{Av}}=-(\lambda/4\pi)(1+\lambda G)^{-1}\times\langle\varphi(\mathbf{k}_0)\rangle_{\mathrm{Av}}\langle\varphi(-\mathbf{k})\rangle_{\mathrm{Av}}\equiv\alpha/k, \quad (11)$$

Extending this development to two sources, we replace Eq. (10) by

$$(\Delta+k^2)\psi(\mathbf{r})=\lambda(U_1\bar{\psi}_{(1)}+U_2\bar{\psi}_{(2)}),$$

with

$$\bar{\psi}_{(i)}\equiv\int\psi(\mathbf{r})U_i(\mathbf{r})d^3\mathbf{r}.$$

The corresponding scattering solution reads

$$\psi(\mathbf{r})=e^{i\mathbf{k}_0\cdot\mathbf{r}}-\frac{\lambda}{4\pi}\int\frac{\exp ik|\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|}(U_1\bar{\psi}_{(1)}+U_2\bar{\psi}_{(2)}).$$

Averaging over the two sources individually, we obtain coupled equations for $\bar{\psi}_{(1)}$ and $\bar{\psi}_{(2)}$ which can be solved directly to give the scattering amplitude due to two identical sources. The result of this calculation is

$$f_{M\ \mathrm{III}}(\mathbf{k},\mathbf{k}_0)=(\alpha/k)(1-\alpha^2W^2)^{-1}(e^{-i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{R}/2}+e^{i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{R}/2}+\alpha W(e^{i(\mathbf{k}_0+\mathbf{k})\cdot\mathbf{R}/2}+e^{-i(\mathbf{k}_0+\mathbf{k})\cdot\mathbf{R}/2})), \tag{12}$$

where

$$W=\frac{\displaystyle\int\int\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{k|\mathbf{r}-\mathbf{r}'|}U_1(\mathbf{r})U_2(\mathbf{r}')d^3\mathbf{r}d^3\mathbf{r}'}{\displaystyle\left(\int\exp(i\mathbf{k}_0\cdot\mathbf{r})U_1(\mathbf{r})d^3\mathbf{r}\right)\left(\int\exp(-i\mathbf{k}\cdot\mathbf{r})U_1(\mathbf{r})d^3\mathbf{r}\right)}, \tag{13}$$

and α is as in Eq. (11). The factor $(1-\alpha^2W^2)$ in the denominator represents all the multiple scattering contributions beyond single and double scattering. For spherical potentials $U_i(\mathbf{r})$ of arbitrary radial profile, Eq. (13) integrates directly to

$$W=e^{ikR}/kR \tag{14}$$

in the case that the separation, R, between the sources is sufficiently large so that they have no common region of overlap. Equation (14) is valid for all values of R in the limit of point sources, in which case Eq. (12) reduces to the previously discussed case of Brueckner. If we assume a square well of radius a for the potentials $U_i(\mathbf{r})$, Eq. (14) is valid for $R\geq 2a$. For $R<2a$, the value of W is dependent on the assumed shape of $U_i(\mathbf{r})$ and can be readily calculated. In the approximation $ka<1$, W reduces simply for $R=0$ to

$$W=6/5ak+i+O(ak). \tag{15}$$

The results of the separable potential model expressed by Eqs. (12)–(15) can be readily applied for various assumed well radii in order to provide a quantitative study of the approach to the point-scattering behavior of Eq. (7).

The physically interesting case corresponds to an average of the above scattering amplitudes over the relative space distribution of the two particles in a bound system. In particular, we may consider the deuteron case. The quantities of interest are thus, for the differential elastic scattering cross section,

$$\sigma(\mathbf{k},\mathbf{k}_0)=\left|\int d^3\mathbf{R}\psi_D^*(R)f_{(R)}(\mathbf{k},\mathbf{k}_0)\psi_D(R)\right|^2, \tag{16}$$

and for the total cross section

$$\sigma_{\mathrm{Tot}}=(4\pi/k)\times\mathrm{Im}\left(\int d^3\mathbf{R}\psi_D^*(R)f_{(R)}(\mathbf{k}_0,\mathbf{k}_0)\psi_D(R)\right). \tag{17}$$

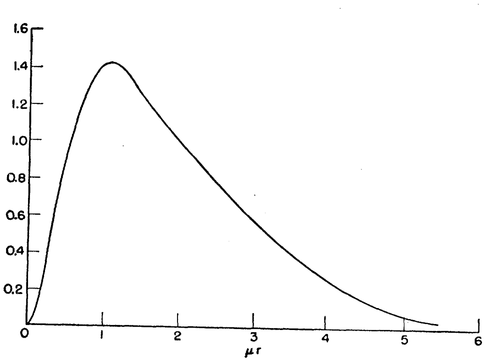

FIG. 1. The deuteron radial distribution, $r^2|\psi_D(r)|^2$, according to the Hulthén function, in arbitrary units.

For the deuteron wave function $\psi_D(R)$ we use the Hulthén function. As illustrated in Fig. 1, the radial density $(R^2|\psi_D(R)|^2)$ cuts down contributions from the $\mu R<\frac{1}{2}$ region. Thus, for[10] $k/\mu\gtrsim 2$ (π-meson kinetic energies >300 Mev, we expect the multiple-scattering corrections to the impulse approximation to be small, and the presence or absence of the singularity in the intermediate-wave propagator to be unimportant in the discussion of the forward elastic scattering or of the total cross section. However, backward elastic scattering will be quite sensitive to the small-distance behavior (or large-momentum components) of both the propagators and of the wave function. This is readily understood since backward scattering results in a large transfer of momentum to the deuteron. This large transfer can be effected only when the two nucleons are in close coincidence in the case that the deuteron remains bound. Thus the probability of finding the two nucleons near one another as well as the behavior of the intermediate-wave propagator for small distances are of decisive importance. In this case, we expect important multiple-scattering corrections to the impulse approximation as well as considerable dependence on the specific form used for the intermediate-wave propagator.

On the other hand, for $k/\mu\lesssim 1$, the entire range of μR values is sensitive to the choice of wave propagator, and we may expect results to be sensitive to the model employed. We also anticipate appreciable multiple-scattering corrections to the impulse approximation in this case for large scattering phase shifts.

[10] $\mu^{-1}=1.4\times10^{-13}$ cm is the meson Compton wavelength in units of $\hbar=c=1$.

The validity of the foregoing remarks has been confirmed in a series of calculations designed primarily to test the reliability of using the double-scattering term alone as a measure of the multiple-scattering corrections. For the three different models discussed previously, we have computed total and differential elastic cross sections according to Eqs. (16) and (17) for two identical S-wave scattering sources with a relative spatial distribution given by the Hulthén wave function for the deuteron:

$$\psi_D(R)=\left(\frac{\alpha+\beta}{\alpha-\beta}\right)\left(\frac{\alpha\beta}{2\pi(\alpha+\beta)}\right)^{\frac{1}{2}}(e^{-\alpha R}-e^{-\beta R})/R,$$

with $\beta/\alpha=6.35$ and $\alpha=2.31\times10^{12}$ cm^{-1}, and have considered incident meson wave numbers of $k/\mu=1$ and 2.2, and individual scattering phase shifts of $\delta=30°$ and 45°; $\alpha=\exp(i\delta)\sin\delta$. The pertinent results of this study are:

(1) The multiple-scattering corrections are acutely model-dependent when they are at all important.

(2) The *double*-scattering approximation fails to reproduce the *multiple*-scattering results only when applied to Model I [Eqs. (6) and (7)] with the singular propagator. In this case, the presence of the singularity in Eq. (6) at $R=0$ introduces a large specious contribution from the region of close coincidence of the scattering sources, $kR<1$.

We fix these statements with a set of numbers: for backward elastic scattering with $k/\mu=2.2$ and $\delta=45°$, Model I [Eq. (7)] gives a decrease from the impulse approximation by a factor of four; Model II [Eq. (9)], by a factor of two; and Model III [Eqs. (12) and (13)] by a factor of three for source radii $a=1/2\mu$. The results for Models II and III are unchanged in the double-scattering approximation, whereas the reduction factor for model I changes from four to two.

Finally, we note from the equations obtained in this section:

(1) In general there may be appreciable multiple-scattering corrections to the impulse approximation for large individual-particle scattering phase shifts in the case where the magnitude of the scattering length $|\alpha/k|$ is comparable with a mean or average spacing R between scattering sources; i.e., $|\alpha/k|$ must be small compared with the mean spacing between the scatterers if multiple scattering is to be safely neglected. This condition reduces to the requirement of small phase shifts, $|\alpha|<1$, for large-angle elastic scattering since two nucleons must be within a distance $(1/k)<R$ of each other in order to accept a momentum transfer of order k.

(2) The double (as well as all even) scattering contributions are most important for scattering in the backward direction because the double-scattered waves are in phase whatever the relative orientation of the two sources.

IV. CALCULATION OF THE CROSS SECTIONS

We consider now $\pi^{\pm}$-deuteron scattering, using the experimentally observed phase shifts for pion-proton scattering together with the assumption of charge independence. We consider only the double-scattering corrections since the algebra of the spin and isotopic spin matrices becomes rather complicated. The double-scattering calculations which we have carried through for the S-wave scattering cases discussed above indicate the qualitative reliability for usch an approximation.

For this calculation we introduce Eqs. (1) and (5) into Eq. (4), obtaining

$$\begin{aligned}kf(\mathbf{k},\mathbf{k}_0)=[&e^{-i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{R}/2}(\alpha_1+\beta_1\mathbf{k}\cdot\mathbf{k}_0-i\gamma_1\boldsymbol{\sigma}_1\cdot\mathbf{k}_0\times\mathbf{k})\\&+e^{-i(\mathbf{k}_0+\mathbf{k})\cdot\mathbf{R}/2}(\alpha_2+i\beta_2\mathbf{k}\cdot\nabla_R+\gamma_2\boldsymbol{\sigma}_2\cdot\nabla_R\times\mathbf{k})\\&\times(\alpha_1+i\beta_1\mathbf{k}_0\cdot\nabla_R+\gamma_1\boldsymbol{\sigma}_1\cdot\mathbf{k}_0\times\nabla_R)g_0(kR)]\\&+(\text{same expression with }1\leftrightarrow2,\text{ and }R\leftrightarrow-R).\qquad(18)\end{aligned}$$

Different forms can be introduced for the propagator $g_0(kR)$ corresponding to the various models discussed previously. If we introduce Eq. (3) and project onto the isotopic spin singlet state which represents the deuteron, we have, for example,

$$\langle\pi^{\pm},D|2\alpha'+\alpha''(\boldsymbol{\tau}_1+\boldsymbol{\tau}_2)\cdot\mathbf{t}|\pi^{\pm},D\rangle=2\alpha'.\qquad(19)$$

Finally, after averaging over the orientations of R we obtain the general formula for the scattering amplitude. As it is very voluminous, we shall write it in the Appendix, along with the particular cases $f(R;k;0)$ and $f(R;k;\pi/2)$. We present here the formula for $f(R;k;\pi)$ which is the simplest one:

$$\begin{aligned}f(R;k;\pi)=(2/k)\bigg[&(\alpha'-\beta'k^2)\sin(kR)/kR\\&+(\alpha'^2-2\alpha''^2)g_0(kR)-\frac{k^4}{3}(\beta'^2-2\beta''^2)g_0(kR)\\&-\frac{k^4}{3}(\gamma'^2-2\gamma''^2)(\boldsymbol{\sigma}_1\times\mathbf{k}_0)\cdot(\boldsymbol{\sigma}_2\times\mathbf{k}_0)g_0(kR)\bigg]\\&\equiv a_\pi+(b_\pi/k^2)(\boldsymbol{\sigma}_1\times\mathbf{k}_0)\cdot(\boldsymbol{\sigma}_2\times\mathbf{k}_0).\qquad(20)\end{aligned}$$

The cross section follows then when we project f onto the deuteron triplet spin state and take its absolute square:

$$\sigma(R;k;\pi)=|a_\pi|^2+(4/3)|b_\pi|^2+\tfrac{2}{3}(a_\pi b_\pi^*+a_\pi^* b_\pi).$$

Explicit numerical calculations have been carried through for energies of the impinging pion: $E_{\text{lab}}=45$ Mev corresponding to $k=0.75\mu$, and $E_{\text{lab}}=169$ Mev corresponding to $k=1.46\mu$. The corresponding pion-nucleon phase shifts are collected in Table I. At 45 Mev, we extrapolated the Steinberger[11] results. At 169 Mev we

[11] Bodansky, Sachs, and Steinberger, Phys. Rev. **93**, 1367 (1954).

TABLE I. The phase shifts used in the numerical calculations For $E_{lab}=169$ Mev, corresponding to $k=1.46\mu$, we give the two sets proposed by Fermi and by Bethe. For $E_{lab}=45$ Mev, corresponding to $k=0.75\mu$, we have extrapolated the phase shifts from Steinberger's experiments at 65 Mev, assuming that the S and P phase shifts are proportional to k and k^3, respectively. We use the usual notation for the phase shifts, with the first index representing twice the isotopic spin and the second one, twice the angular momentum for the P states.

	δ_3	δ_1	δ_{33}	δ_{31}	δ_{13}	δ_{11}
$k=0.75\mu$	$-5.3°$	9.4°	5.8°	0	0	0
$k=1.46\mu$ (Fermi)	$-42°$	7°	48°	14°	0	0
$k=1.46\mu$ (Bethe)	$-4°$	7°	64°	0	0	0

used both the Fermi[12] and Bethe[13] determinations of the phase shifts.

As stated before, the integration over R was carried out numerically using the Hulthén wave function for the deuteron. The differential elastic cross sections for $\theta=0$, $\pi/2$, and π, and the total cross sections were calculated for the different scattering models discussed in the previous section. The results are summarized in Table II for $E=45$ Mev and in Table III for $E=169$ Mev.

V. CONCLUSIONS

From this study we conclude that a phenomenological calculation of the multiple-scattering correction to the $\pi^{\pm}-D$ scattering can only lead in general to uncertain results. This is due to our lack of knowledge of the off-the-energy-shell amplitudes for which we must introduce specific models. Different models lead to multiple-scattering corrections which have in common only their orders of magnitude.

Since the phase shifts for pion-nucleon scattering are relatively small at pion kinetic energies of 45 Mev, the multiple scattering corrections as exhibited in Table II are of minor importance. However, the phase shifts are large at 169 Mev and the scattering lengths are comparable to the characteristic deuteron size, so that appreciable corrections result from multiple scattering effects. As we see in Table III, these corrections are sensitive to assumptions made concerning the off-the-energy-shell behavior of the scattering amplitudes. They are most important for backward scattering, in which case the double-scattered waves are in phase as discussed previously. However, backward elastic scattering is quite sensitive to the small distance behavior of the deuteron wave function since the large momentum transfer can be effected best when the neutron and proton are in close coincidence. Since the deuteron wave function in this region is both unknown and in itself sensitive to the nuclear force, an interpretation of experimental results on backward elastic scattering must remain to a large extent ambiguous. On the other hand, in that it does depend both on the deuteron wave function and on the scattering amplitudes in a very sensitive manner, backward elastic $\pi^{\pm}-D$ scattering may serve as a critical test of future meson theories.

TABLE II. The double scattering corrections are presented in this table for $E_{lab}=45$ Mev according to Model II and Model III with the two choices of square-well radii corresponding to $a=0.25/\mu$ and $a=0.50/\mu$. Model I, corresponding to $a\to 0$ in Model III, was omitted in these calculations because of unreliability of the double-scattering approximation as discussed in the text.

$k=0.75\mu$ $E_{lab}=45$ Mev	Single scattering	Ratio of the corrected scattering to the single scattering for: Model II	Model III with radius $a=0.25/\mu$	Model III with radius $a=0.50/\mu$
$\sigma(0)$	2.35 mb/sterad	0.99	1.04	1.04
$\sigma(\pi/2)$	0.29 mb/sterad	0.93	1.1	1.05
$\sigma(\pi)$	1.25 mb/sterad	0.97	1.2	1.1
σ_{tot}	25. mb	0.88	0.92	0.94

TABLE III. The double-scattering corrections for $E_{lab}=169$ Mev. The smallness of the cross section at $\pi/2$ is a consequence of near cancellation in the single-scattering calculation. The double-scattering corrections can therefore be relatively very important. However, the smallness of the cross section at $\pi/2$ makes their experimental determination all the more difficult.

$k=1.46\mu$ $E_{lab}=169$ Mev	Single scattering	Ratio of the corrected scattering to the single scattering for: Model II	Model III with radius $a=0.25/\mu$	Model III with radius $a=0.50/\mu$
$\sigma(0)$ (Fermi)	53 mb/sterad	0.89	1.04	0.95
$\sigma(0)$ (Bethe)	55 mb/sterad	0.89	1.07	0.98
$\sigma(\pi/2)$ (Fermi)	0.10 mb/sterad	0.45	1.5	
$\sigma(\pi)$ (Fermi)	5.2 mb/sterad	0.68	0.68	0.63
$\sigma(\pi)$ (Bethe)	5.5 mb/sterad	0.67	0.67	0.57
σ_{tot} (Fermi)	250 mb	0.95	1.01	1.05
σ_{tot} (Bethe)	250 mb	0.97	1.03	1.09

The total cross sections as calculated in Table III can be compared with the experimental results of Ashkin[14] and collaborators. For positive mesons of energy 162 ± 12 Mev they observed a total cross section of 217 ± 14 mb. The predictions of the impulse approximation at 169 Mev is 250 mb and the multiple scattering corrections are calculated to be less than 10 percent. In view of the theoretical and experimental uncertainties, and of the fact that the data were obtained for a broad spread of meson energies in a region of rapidly varying cross section, little more can be inferred from this result than that there is a satisfactory qualitative agreement between the calculated and observed cross sections.

In his analysis of the experiments of Arase, Goldhaber, and Goldhaber[15] on $\pi^{-}-D$ elastic scattering at 140 Mev, Green[16] has pointed out an additional un-

[12] Fermi, Glicksman, Martin, and Nagle, Phys. Rev. **92**, 161 (1953); Fermi, Metropolis, and Alei, Phys. Rev. **95**, 1581 (1954).
[13] de Hoffman, Metropolis, Alei, and Bethe, Phys. Rev. **95**, 1586 (1954).
[14] Ashkin, Blaser, Feiner, Gorman, and Stern, Phys. Rev. **96**, 1109 (1954).
[15] Arase, Goldhaber, and Goldhaber, Phys. Rev. **90**, 160 (1953). See also D. E. Nagle, Phys. Rev. **97**, 480 (1955).
[16] T. Green, Phys. Rev. **90**, 161 (1953).

certainty which the application of the impulse approximation introduces: scattering amplitudes with the momentum energy relations which apply in meson-nucleon collisions are used in the meson-deuteron case for which there are different kinematical conditions. This difference is especially important in this energy region near the resonance where the cross section is rapidly varying with energy.

Finally we note from our results in Table III that it does not seem feasible to use $\pi^{\pm}-D$ scattering experiments alone as a means of distinguishing between the predictions of the Fermi and Bethe sets of meson-nucleon scattering phase shifts since the multiple-scattering corrections are pretty much the same in both cases.

ACKNOWLEDGMENTS

One of us (L. V.) wishes to acknowledge the French "Relations Culturelles" and the C.E.A. of Saclay for financial support.

APPENDIX A

We exhibit in this paragraph a brief deduction of the amplitudes for double and multiple S-scattering from two scattering sources. Writing and formally solving the Schrödinger equation for the case of scattering from a single source described by potential V (in units of $2m/\hbar^2$), we have

$$\begin{aligned}(E-H_0)\psi&=V_1\psi,\\ \psi&=\varphi+GV_1\psi=\varphi+GV_1(1-GV_1)^{-1}\varphi,\end{aligned}\qquad(1\text{a})$$

where $\varphi=\exp(i\mathbf{k}_0\cdot\mathbf{r})$ represents the incident wave, ψ the eigensolution, and $G\equiv(E-H_0)^{-1}$ the integral operator for scattering used by Lippman and Schwinger,[17] with the contour at the singularity chosen to satisfy the scattering boundary conditions. In terms of the Green function $g(r;r')$, we write

$$(Gf)_r=-\int d^3\mathbf{r}'g(\mathbf{r};\mathbf{r}')f(\mathbf{r}'). \qquad(2\text{a})$$

The elastic scattering amplitude in this case is given by

$$-(4\pi)^{-1}\langle\mathbf{k}|V_1(1-GV_1)^{-1}|\mathbf{k}_0\rangle=f_1(\mathbf{k},\mathbf{k}_0). \qquad(3\text{a})$$

For two sources, Eq. (1a) becomes

$$\psi=\varphi+G(V_1+V_2)[1-G(V_1+V_2)]^{-1}\varphi. \qquad(4\text{a})$$

We wish now to express the scattering amplitude from the two sources as given in Eq. (4a) in terms of the scattering amplitudes of the individual sources, Eq. (3a). That is, we shall manipulate the right member of Eq. (4a) into combinations of the form in Eq. (3a). Defining

$$a_i\equiv V_i(1-GV_i)^{-1}\quad\text{for}\quad i=1,2,$$

we find

$$\begin{aligned}GV_1[1-G(V_1+V_2)]^{-1}&\\ &=Ga_1+Ga_1Ga_2+Ga_1Ga_2GV_1[1-G(V_1+V_2)]^{-1}\\ &=(1-Ga_1Ga_2)^{-1}(Ga_1+Ga_1Ga_2).\end{aligned}$$

This gives us then the multiple-scattering expression,

$$\begin{aligned}\psi=\varphi&+(1-Ga_1Ga_2)^{-1}(Ga_1+Ga_1Ga_2)\varphi\\ &+(1-Ga_2Ga_1)^{-1}(Ga_2+Ga_2Ga_1)\varphi,\end{aligned}\qquad(5\text{a})$$

in terms of the scattering amplitudes from the individual sources. If we keep only the single- and double-scattering terms, Eq. (5a) approximates to

$$\begin{aligned}\psi_D&=\varphi+G(a_1+a_2+a_1Ga_2+a_2Ga_1)\varphi\\ &=\varphi+G[V_1(1-GV_1)^{-1}+V_2(1-GV_2)^{-1}\\ &\quad+V_1(1-GV_1)^{-1}GV_2(1-GV_2)^{-1}\\ &\quad+V_2(1-GV_2)^{-1}GV_1(1-GV_1)^{-1}]\varphi,\end{aligned}$$

which together with defining Eqs. (2a) and (3a) gives Eq. (4) of the text for the scattering of waves with wave number k which solve the Helmholtz wave equation with Green's function

$$g(\mathbf{r};\mathbf{r}')=(2\pi)^{-3}\int d^3\mathbf{q}\frac{\exp i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}{q^2-k^2-i\epsilon}. \qquad(6\text{a})$$

In Eq. (6a), the small negative imaginary term is appended in the denominator to define the contour corresponding to outgoing waves scattered from the source.

The multiple scattering can be handled in Eq. (5a) with the assumption of constant scattering amplitudes as follows. Consider a contribution from the denominator expansion,

$$Ga_1Ga_2Ga_1\varphi.$$

Using Eqs. (2a) and (6a), we have

$$\begin{aligned}Ga_1Ga_2Ga_1\varphi&\\ &=Ga_1\left(\frac{4\pi}{(2\pi)^3}\right)^2\int\frac{d^3\mathbf{q}}{q^2-k^2-i\epsilon}\int\frac{d^3\mathbf{q}'}{q'^2-k^2-i\epsilon}e^{i\mathbf{q}\cdot\mathbf{r}'}\\ &\qquad\times\langle\mathbf{q}|a_2|\mathbf{q}'\rangle\langle\mathbf{q}'|a_1|\mathbf{k}_0\rangle.\end{aligned}\qquad(7\text{a})$$

We next note the relation

$$\begin{aligned}\langle\mathbf{q}|a_2|\mathbf{q}'\rangle&\equiv\int d^3\mathbf{r}\exp[i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{r}]a_2(\mathbf{r})\\ &=e^{i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{R}}\langle\mathbf{q}|a_1|\mathbf{q}'\rangle,\end{aligned}\qquad(8\text{a})$$

with R introduced as the separation between the two scattering sources. Introducing Eq. (8a) into (7a) and making use of the above assumption that $-(4\pi)^{-1}\times\langle\mathbf{q}|a_1|\mathbf{q}'\rangle\equiv f_{\mathrm{I}}$ is independent of $\mathbf{q}$ and $\mathbf{q}'$, we obtain

$$Ga_1Ga_2Ga_1\varphi=Ga_1e^{i\mathbf{q}\cdot\mathbf{r}'}(e^{ikR}/R)^2f_{\mathrm{I}}^2, \qquad(9\text{a})$$

which is equivalent to $(e^{ikR}/R)^2f_{\mathrm{I}}^2Ga_1\varphi$ with our as-

[17] B. A. Lippman and J. Schwinger, Phys. Rev. 79, 469 (1950).

sumptions. Equation (9a) leads immediately to the result expressed in the text by Eq. (7).

APPENDIX B

In this appendix we write the formula for $f(R,k,\theta)$, $f(R,k,0)$, and $f(R,k,\pi/2)$, which were too cumbersome to be introduced in the text.

$$f(R,k,\theta)=\frac{2}{k}\Bigg\{\left(\alpha'+\beta'\mathbf{k}\cdot\mathbf{k}_0-i\gamma'\frac{\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2}{2}\cdot\mathbf{k}_0\times\mathbf{k}\right)j_0(z)$$
$$+(\alpha'^2-2\alpha''^2)j_0(y)g_0(x)+\Bigg(2(\alpha'\beta'-2\alpha''\beta'')\mathbf{k}_0\cdot\mathbf{l}$$
$$-i(\alpha'\gamma'-2\alpha''\gamma'')\frac{\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2}{2}\cdot\mathbf{k}_0\times\mathbf{k}\Bigg)\frac{k}{l}j_1(y)g_1(x)$$
$$+[(\beta'^2-2\beta''^2)\mathbf{k}\cdot\mathbf{k}_0-i(\beta'\gamma'-2\beta''\gamma'')(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}_0\times\mathbf{k}$$
$$+\tfrac{1}{2}(\gamma'^2-2\gamma''^2)(\boldsymbol{\sigma}_1\times\mathbf{k}\cdot\boldsymbol{\sigma}_2\times\mathbf{k}_0+\boldsymbol{\sigma}_1\times\mathbf{k}_0\cdot\boldsymbol{\sigma}_2\times\mathbf{k})]$$
$$\times k^2\left(\frac{g_1(x)}{x}j_0(y)-g_2(x)\frac{j_1(y)}{y}\right)+\Bigg((\beta'^2-2\beta''^2)\mathbf{k}_0\cdot\mathbf{l}$$
$$-i(\beta'\gamma'-2\beta''\gamma'')\frac{\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2}{2}\cdot\mathbf{k}_0\times\mathbf{k}(\mathbf{k}_0\cdot\mathbf{l}/l^2)-(\gamma'^2-2\gamma''^2)$$
$$\times\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{k}_0\times\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{k}_0\times\mathbf{k})}{4l^2}\Bigg)\times k^2g_2(x)j_2(y)\Bigg\},\quad(1\text{b})$$

where $x\equiv kR$; $y\equiv lR$; $\mathbf{l}\equiv(\mathbf{k}_0+\mathbf{k})/2$; $z\equiv|\mathbf{k}_0-\mathbf{k}|R/2$; $j_l(z)$ is the spherical Bessel function of order l, with $j_0(z)\equiv\sin z/z$; $g_0(x)$ is one of the propagators discussed in the text, and $g_1(x)=-g_0'(x)$; $g_2(x)=-x[g_1(x)/x]'$.

$$f(R,k,0)=(2/k)\{\alpha'+\beta'k^2+(\alpha'^2-2\alpha''^2)j_0(x)g_0(x)$$
$$+2(\alpha'\beta'-2\alpha''\beta'')k^2j_1(x)g_1(x)+(\beta'^2-2\beta''^2)k^4$$
$$\times[g_1(x)j_0(x)-g_2(x)j_1(x)+xg_2(x)j_2(x)]/x$$
$$+(\gamma'^2-2\gamma''^2)\boldsymbol{\sigma}_1\times\mathbf{k}_0\cdot\boldsymbol{\sigma}_2\times\mathbf{k}_0k^2[g_1(x)j_0(x)$$
$$-g_2(x)j_1(x)]/x\}\equiv a_0+b_0\boldsymbol{\sigma}_1\times\mathbf{k}_0\cdot\boldsymbol{\sigma}_2\times\mathbf{k}_0/k^2.\quad(2\text{b})$$

$$f(R,k,\pi/2)=\frac{2}{k}\Bigg\{\left(\alpha'-i\gamma'k^2\frac{\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2}{2}\cdot\mathbf{n}\right)j_0(u)$$
$$+(\alpha'^2-2\alpha''^2)g_0(x)j_0(u)+[2(\alpha'\beta'-2\alpha''\beta'')$$
$$-i(\alpha'\gamma'-2\alpha''\gamma'')(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{n}]kg_1(x)j_1(u)/\sqrt{2}$$
$$+[-i(\beta'\gamma'-2\beta''\gamma'')(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{n}k^2$$
$$+(\gamma'^2-2\gamma''^2)(\boldsymbol{\sigma}_1\times\mathbf{k}\cdot\boldsymbol{\sigma}_2\times\mathbf{k}_0+\boldsymbol{\sigma}_1\times\mathbf{k}_0\cdot\boldsymbol{\sigma}_2\times\mathbf{k})/2]$$
$$\times k^2[j_0(u)g_1(x)/x-g_2(x)j_1(u)/u]+[(\beta'^2-2\beta''^2)$$
$$-i(\beta'\gamma'-2\beta''\gamma'')(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{n}-(\gamma'^2-2\gamma''^2)\boldsymbol{\sigma}_1\cdot\mathbf{n}\boldsymbol{\sigma}_2\cdot\mathbf{n}]$$
$$\times k^4g_2(x)j_2(u)/2\Bigg\}\equiv a_{\pi/2}+b_{\pi/2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{n}$$
$$+c_{\pi/2}(\boldsymbol{\sigma}_1\times\mathbf{k}\cdot\boldsymbol{\sigma}_2\times\mathbf{k}_0+\boldsymbol{\sigma}_1\times\mathbf{k}_0\cdot\boldsymbol{\sigma}_2\times\mathbf{k})/k^2$$
$$+d_{\pi/2}\boldsymbol{\sigma}_1\cdot\mathbf{n}\boldsymbol{\sigma}_2\cdot\mathbf{n},\quad(3\text{b})$$

where $\mathbf{k}_0\times\mathbf{k}=k^2\mathbf{n}$ and $u\equiv x/2^{\frac{1}{2}}$. The cross sections are deduced as in the text and reduce to

$$\sigma(R;k;0)=|a_0|^2+(4/3)|b_0|^2+\tfrac{2}{3}(a_0^*b_0+a_0b_0^*);\quad(4\text{b})$$
$$\sigma(R;k;\pi/2)=|a_{\pi/2}|^2+(8/3)|b_{\pi/2}|^2+(8/3)|c_{\pi/2}|^2$$
$$+|d_{\pi/2}|^2+\tfrac{1}{3}(a_{\pi/2}^*d_{\pi/2}+a_{\pi/2}d_{\pi/2}^*).\quad(5\text{b})$$

The constants in (4b) and (5b) are defined in (2b) and (3b). The total cross section is expressed by Eq. (17) in the text and requires evaluation of the spin sum:

$$\langle\pi^\pm,D|a_0+b_0\boldsymbol{\sigma}_1\times\mathbf{k}_0\cdot\boldsymbol{\sigma}_2\times\mathbf{k}_0/k^2|\pi^\pm,D\rangle=a_0+\tfrac{2}{3}b_0.\quad(6\text{b})$$