Particle States in Spheroidal Nuclei*

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Particle motion in a spheroidal box has been studied by making an appropriate coordinate transformation so that the potential appears as spherically symmetric. The effect of the nonsphericity then appears as a change in kinetic energy for which the perturbation matrix elements can be evaluated. Matrix elements between states in different major shells are small. The energy levels and wave functions (defined in the transformed coordinate system) can then be calculated by the customary techniques of matrix diagonalization.

This method was used to obtain energy levels of a spheroidal box both without and with an additional spin-orbit term of the magnitude suggested by nuclear shell systematics.

Corresponding results for distorted wells of finite depths appear to be rather simply related to those for a box.

Nuclear deformations predicted by this picture are in qualitative agreement with experimental values, and are, in general, considerably smaller than the values implied by a hydrodynamical model.

I. INTRODUCTION

THE striking success of the collective model in accounting for many features of nuclear structure[1,2] has stimulated interest in some of the underlying problems. One of these is the study of the properties of particle states in nonspherical potentials. The strong-coupling version of the collective model applies best for mass numbers between 155 and 185 and above 225.[1,3] For these cases, the nucleus, under the deforming influence of the particles outside of closed shells, acquires a semipermanently deformed shape, which to a good degree of approximation is a spheroid of revolution. As is well known, the collective motion of such nuclei gives rise to rotational and vibrational states and the nonsphericity results in large electric quadrupole moments, both static and dynamic. For lighter nuclei, the strong-coupling approximation is not realized, but collective effects resulting from deviation of the nuclear shape from sphericity are still important. One of the important problems in this connection is to calculate the magnitude of nuclear deformations. Previously some estimates have been made on the basis of a simple hydrodynamical model.[4–7] In general the hydrodynamical model gives deformations considerably larger than empirical values, sometimes by an order of magnitude. However any model which considers particles moving independently in a deformed potential, gives much smaller deformations.[8–13] At least part of this difference is a consequence of the inherent shell-structure regularities in spacings between particle levels for a spherical well. The opposite limit would be a strictly statistical distribution of particle states, for which hydrodynamical estimates would be more nearly valid.[1,2]

The basic assumptions made in the present paper are very extreme. The particles are assumed to move independently in a box ($V=0$ inside, $V=\infty$ outside) whose wall is elastic and deformable into a spheroid of fixed volume. As far as the particle motion is concerned, the wall is considered as static. Dynamic effects, such as rotational motion of the wall, are thus not taken into account. The results depend on the position of particle states in a spherical well. The order of particle levels in nuclei is approximately that for a spherical well of finite depth, with rounded edges and a spin-orbit coupling. The order of levels for such a well is nearly the same as for a box with an additional spin orbit term. The effect of making the well finite and rounding it can, of course, be taken into account. However, the general features of the results probably will not depend very much on the detailed potential. In any case, other effects not considered here are probably of far greater significance. Thus direct interactions between particles outside of closed shells are very important. Furthermore, the nuclear distortion is time-dependent and it must be treated as a quantum variable. Nevertheless, one might hope that the present calculations reproduce at least the qualitative trends of nuclear deformations as function of nucleon number.

In the following discussion, it is shown how wave functions and energy levels in a static nonspherical well may be found to a reasonable degree of accuracy.

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Section II of this paper deals with calculations of the wavefunctions and energy levels for particles in a spheroidal box. The effect of a spin orbit term is discussed in Sec. III. In Sec. IV, some estimates are made for energy levels in a distorted well of more general shape. The deformations calculated in this way and their relation to empirical values are discussed in Sec. V.

II. ENERGY LEVELS AND WAVE FUNCTIONS OF A PARTICLE IN A SPHEROIDAL BOX

Particle states in a spheroidal well do not have a well-defined angular momentum because of the absence of spherical symmetry. However, the parity is a good quantum number, and so is the component of angular momentum about the axis of deformation, denoted here by m. The spatial $(2l+1)$-fold degeneracy of levels in a spherical well is broken up in such a way that only twofold degeneracies remain, those for levels differing solely with respect to their sign of m.

The solution of the wave equation for a spheroidal box can, in principle, be carried through in several ways.

First, one can solve the wave equation by separation into spheroidal coordinates. Such calculations, which have been performed by several authors,[4,13,14] are extremely complicated, and it may be difficult to adapt them to take into account interparticle forces and the time dependence of the nuclear distortion. The spheroidal coordinates ξ and η have no apparent direct significance for problems which involve wavefunctions *inside* the nucleus, since no special distinction attaches to the focal points of a spheroidal nucleus. On the other hand, the separation into spheroidal coordinates may be extremely useful for the study of particle motion *outside* the nucleus; e.g., for α decay. Another promising approach consists of expanding the wave functions for a deformed box in terms of wave functions for a spherical box. The coefficients of this expansion, as well as the wave number, are determined by the requirement that the wave function must vanish at the wall of the box. Such calculations have been made for deformations of arbitrary shape.[2,13]

The basis of the method described in the present paper is that a spheroid (and more generally any ellipsoid) can be transformed into a sphere in a very simple way. The coordinates are merely multiplied by a strain tensor. By use of this transformation, one can calculate energy levels and wave functions for a spheroidal box to any desired degree of accuracy. It is also possible to get results quite simply without losing much precision.

Consider a box whose wall is defined by the following spheroid:

$$R_0^2=Dx^2+Dy^2+D^{-2}z^2. \tag{1}$$

This spheroid has one axis of length DR_0 and two axes of length $D^{-\frac{1}{2}}R_0$. The volume is, of course, independent of D. The deformation d is defined[15] as

$$d=D-1. \tag{2}$$

Now introduce "primed" coordinates as follows:

$$x=D^{-\frac{1}{2}}x', \quad y=D^{-\frac{1}{2}}y', \quad z=Dz'. \tag{3}$$

Then the potential is given by

$$\begin{aligned} V(r')&=0 \quad \text{for} \quad r'<R_0 \\ &=\infty \quad \text{for} \quad r'>R_0. \end{aligned} \tag{4}$$

It is seen that the spheroidal box (and in fact any spheroidal potential), is central in $\mathbf{r}'$ space. This suggests that a formulation of the wave equation in $\mathbf{r}'$ space might be of value. Thus, in the primed coordinate system, the Schrödinger equation reads:

$$(2m)^{-1}(Dp_{x'}^2+Dp_{y'}^2+D^{-2}p_{z'}^2)\psi(\mathbf{r}') +V(r')\psi(\mathbf{r}')=E\psi(\mathbf{r}'). \tag{5}$$

This equation is, apart from the coefficients of the kinetic energy terms, the same as the wave equation for a particle in a spherical potential, but with all quantities defined in $\mathbf{r}'$ space. The energy levels and wave functions can then be found by constructing the secular determinant of Eq. (5) using as zero-order functions the wave functions $\psi_i(\mathbf{r}')$ for the Hamiltonian when $D=1$ (the spherical case). For a spheriodal box, the boundary conditions require that $\psi(\mathbf{r}')$ vanish for $r'=R_0$.

The zero-order functions for a box are given as follows:

$$\psi_i(\mathbf{r}')=j_l(\omega_{nl}r'/R_0)Y_{l,m}(\theta',\phi'). \tag{6}$$

Here $\omega_i=\omega_{nl}$ denotes the nth root of the spherical Bessel function of order l. Since both the Hamiltonian and the zero-order wave functions are expressed in terms of primed coordinates, it may avoid confusion to drop the primes for calculations of matrix elements.

The secular determinant reads:

$$\textstyle\sum_k(H_{ik}-E_i\delta_{ik})\psi_k(\mathbf{r}')=0, \tag{7}$$

where

$$H_{ik}=\tfrac{1}{3}(2D+D^{-2})\frac{p^2}{2m}\Big|_{ik}-\tfrac{2}{3}(D-D^{-2})\frac{T_{20}(\mathbf{p})}{2m}\Big|_{ik}, \tag{8}$$

and

$$T_{20}(\mathbf{p})=(3p_z^2-p^2)/2. \tag{9}$$

Since the potential is zero throughout the box, all matrix elements of V vanish.

For small deformations, Eq. (7) becomes

$$2mH_{ik}=(1+d^2)p^2|_{ik}-2(d-d^2)T_{20}(\mathbf{p})|_{ik}. \tag{10}$$

The matrix elements H_{ik} can be easily evaluated by reference to the well-known properties of spherical Bessel functions.[16]

[14] R. D. Spence, Phys. Rev. **83**, 460 (1951).

[15] The relation of d to some other quantities used to define degrees of distortion is, for small deformations: $d=\frac{2}{3}\epsilon=(5/4\pi)^{\frac{1}{2}}\beta$, where ϵ is defined (see reference 3) as the difference between axes in units of R_0, and β is defined (see reference 1) by:

$$R=R_0[1+\beta Y_2^0(\theta,\phi)].$$

[16] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2, p. 1573.

The results are:

$$\left.\frac{p^2}{2m}\right|_{ik}=E_i\delta_{ik}=\frac{\hbar^2}{2mR_0^2}\omega_i^2\delta_{ik}, \tag{11}$$

$$\left.\frac{T_{20}(\mathbf{p})}{2m}\right|_{ik}=\frac{\hbar^2}{2mR_0^2}\omega_i^2\delta_{n_in_k}\delta_{m_im_k}\frac{l_i(l_i+1)-3m_i^2}{(2l_i-1)(2l_i+3)}$$

$$=\frac{\hbar^2}{2mR_0^2}\omega_i^2P_2(\theta)|_{ik}\quad\text{for}\quad l_k=l_i, \tag{12a}$$

$$\left.\frac{T_{20}(\mathbf{p})}{2m}\right|_{ik}=\frac{\hbar^2}{2mR_0^2}\frac{\omega_i\omega_k}{\omega_k^2-\omega_i^2}\delta_{m_im_k}\times3\left[\frac{(l_i+1+m_i)(l_i+2+m_i)\times(l_i+1-m_i)(l_i+2-m_i)}{(2l_i+1)(2l_i+5)}\right]^{\frac{1}{2}}$$

$$=\frac{\hbar^2}{2mR_0^2}\frac{\omega_i\omega_k}{\omega_k^2-\omega_i^2}2(2l_i+3)P_2(\theta)|_{ik}\quad\text{for}\quad l_k=l_i+2, \tag{12b}$$

where the subscripts i and k denote integration over angles only.

Of course, only states of the same m and of the same parity can be coupled by a spheroidal distortion. It should be noted that the matrix elements between states in different major shells vanish if the orbital angular momenta are the same and they are small for l's different by two, because of the large energy denominator.

On the other hand, the interaction strongly couples states of the same major shell. The energy differences between two states of the same major shell are given approximately by

$$\omega_k^2-\omega_i^2\simeq2(2l_i+3)\quad\text{for}\quad l_k=l_i+2. \tag{13}$$

It follows from Eqs. (8), (11), (12), (13) that matrix elements of H between states in the same shell are given to a good degree of approximation by

$$H_{ik}=(\tfrac{2}{3}D+\tfrac{1}{3}D^{-2})(E_i\delta_{ik})-(\tfrac{2}{3}D-\tfrac{2}{3}D^{-2})(E_iE_k)^{\frac{1}{2}}P_2(\theta)|_{ik}. \tag{14}$$

To calculate energy levels and wave functions to a fair degree of accuracy it is sufficient to diagonalize only the part of the infinite secular determinant (7) between states in the major shell under consideration. This approach neglects the small interactions between different major shells and yields the following average energy for the states in each shell:

$$E_i'=E_i(\tfrac{2}{3}D+\tfrac{1}{3}D^{-2})\xrightarrow[\text{small }d]{}E_i(1+d^2). \tag{15}$$

The perturbation matrix elements $\mathcal{H}_{ik}$ are then given by

$$\mathcal{H}_{ik}=-(\tfrac{2}{3}D-\tfrac{2}{3}D^{-2})(E_iE_k)^{\frac{1}{2}}P_2(\theta)|_{ik}. \tag{16}$$

The secular determinant now reads as follows:

$$\sum_k[\mathcal{H}_{ik}-E_i'\delta_{ik}]\psi_k(\mathbf{r}')=0,$$

where the sum now extends only over states k of the same major shell as state i. The smallness of the coupling term between different major shells shows that on the "average," the $\psi(\mathbf{r}')$ are good zero-order functions to use for this problem. On the average, the spheroidal distortion of a box causes the wave functions to "follow" it apart from (a) the strong mixing within a given shell, and (b) the weak coupling between different shells. It should be noted here that for boxes whose walls are rectangular parallelipipeds or right circular cylinders, the wave equation can be solved exactly. For these cases the wave functions follow the distortion exactly without any mixing.

Slightly more accurate values for the energy levels may be obtained by taking into account the small coupling between different shells. Thus, the energy can be expressed as a power series in d, and values of the coefficients up to d^2 obtained exactly from Eqs. (10)–(12) by use of second order perturbation theory.

The sums over intermediate states can be evaluated in terms of the expression

$$\sum_{\omega_l}\frac{1}{\omega_l^2-x^2}=\frac{1}{2x^2}\left[l-\frac{xj_l'(x)}{j_l(x)}\right], \tag{18}$$

and the two other sums which are obtained by differentiation of (18) with respect to x once and twice. Here x is any number and the sum extends over all positive roots of the spherical Bessel function of order l.

The energy of a given state can then be written as follows:

$$E_{nlm}=E_{nlm}{}^{(0)}+dE_{nlm}{}^{(1)}+d^2E_{nlm}{}^{(2)}\cdots, \tag{19}$$

where the coefficients in the power series are:

$$E_{nlm}{}^{(0)}=(\omega_i{}^2\hbar^2)/(2mR_0^2), \tag{20a}$$

$$E_{nlm}{}^{(1)}=E_{nlm}{}^{(0)}[-2P_2(\theta)|_{ll}], \tag{20b}$$

$$E_{nlm}{}^{(2)}=E_{nlm}{}^{(0)}\{1+2P_2(\theta)|_{ll}-[(2l+7)-2\omega_i{}^2(2l+3)^{-1}]P_2(\theta)|_{l,l+2}{}^2+[(2l-5)-2\omega_i{}^2(2l-1)^{-1}]P_2(\theta)|_{l,l-2}{}^2\}. \tag{20c}$$

Values of the coefficients are given in Table I.

It is evident that the power series expansion does not converge well for states which are strongly coupled to other states in the same shell; e.g., $1d0$ and $2s0$. Such an expansion by itself can therefore not be used for these states and a matrix diagonalization must be carried out.

For the numerical calculations of this paper, only matrix elements between states of the same shell were considered using Eqs. (15) to (17). However, rather than neglecting intershell contributions entirely, the

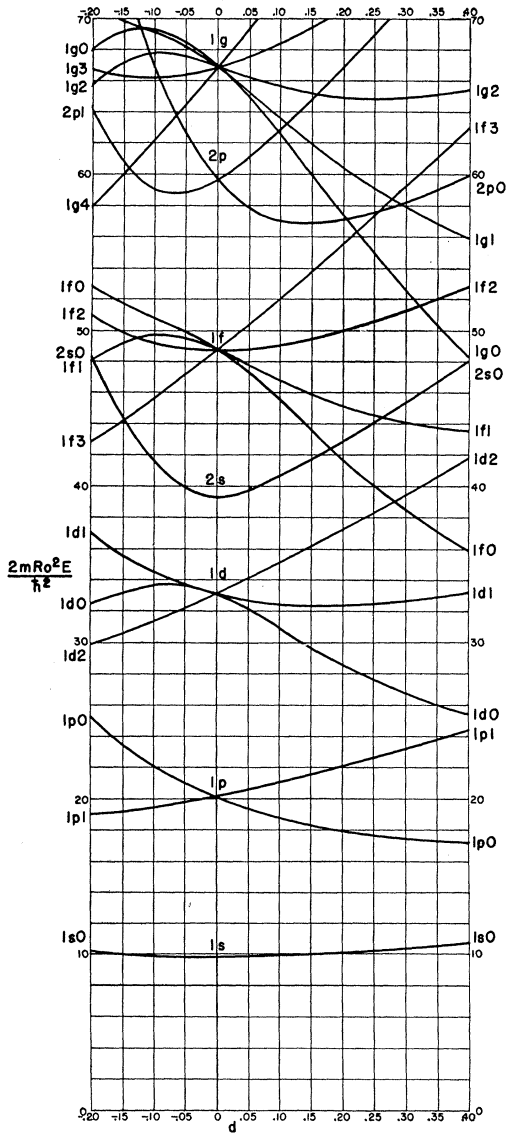

FIG. 1. Energy levels of a spheroidal box as function of deformation. No spin-orbit coupling. Length of symmetry axis $=R_0(1+d)$.

coefficients of $p^2/2m$ were altered slightly from unity to

$$1+(\eta/3)(2D+D^{-2}-3)\xrightarrow[\text{small } d]{}1+\eta d^2, \qquad (21)$$

where η is an adjustable parameter whose values are slightly smaller than unity. Values of η were picked for each state in such a way that the energy would agree exactly with the power series expansion (20) as far as terms in d^2. According to the numbers of Table I, η ranges from 0.92 for the $1s0$ state to 0.66 for the $1g0$ state. Terms of order d^3, d^4, etc., also appear to be quite accurately given with this approximation.

The matrices were diagonalized on the SWAC at the UCLA campus. The Jacobi method of successive rotations was used. Results for states up to the $1g$ shell are shown in Fig. 1. The numerical values obtained agree very well with calculations of Spence[14] and of Hill and Wheeler[13] who used spheroidal coordinates.

III. EFFECT OF A SPIN-ORBIT TERM ON THE ENERGY LEVELS

The above method can be extended to include a spin-orbit coupling term in the Hamiltonian. Strictly speaking, there can be no spin-orbit coupling of the Thomas type $(\boldsymbol{\sigma}\cdot\nabla V\times\mathbf{p})$ for any well with infinitely high walls. However, by adding instead a constant spin-orbit term, it is possible to reproduce the approximate order of single-particle levels in light and medium nuclei.

The particular spin-orbit term used in the present calculations is a constant with an attempted correction for the deformation:

$$H_{\text{s.o.}}=-k[(\mathbf{l}'\cdot\boldsymbol{\sigma})+d(\tfrac{3}{2}l_z'\sigma_z-\tfrac{1}{2}\mathbf{l}'\cdot\boldsymbol{\sigma})], \qquad (22)$$

where all coordinates and operators are defined in $\mathbf{r}'$ space.

The constant k was taken equal to one in units of $\hbar^2/2mR_0^2$ to give approximately the correct doublet splitting in nuclei. The zero-order states were taken as $\psi(\mathbf{r}')\chi(\text{spin})$, where $\chi(\text{spin})$ can be $\uparrow$ or $\downarrow$.

In this approximation only states of the same shell and with the same values of l and m_j are coupled by the spin-orbit term.

The rest of the Hamiltonian was not altered from the form used in the calculations of Sec. 2. The results of including this spin-orbit term are shown in Fig. 2. It is seen that for small deformations, the energy levels agree with the previous calculations using first order perturbation theory.[4–9] For example, the first-order

TABLE I. Energy levels in spheroidal box [in units of $\hbar^2/2mR_0^2$] as function of deformation. First three terms in power series expansion

$$E=E^{(0)}+E^{(1)}d+E^{(2)}d^2+\cdots.$$

n l m	$E^{(0)}$	$E^{(1)}$	$E^{(2)}$
1 s 0	9.870	0	9.041
1 p 0	20.191	−16.153	33.467
1 p 1	20.191	8.076	10.196
1 d 0	33.217	−18.981	−108.935
1 d 1	33.217	−9.491	36.570
1 d 2	33.217	18.981	11.168
2 s 0	39.479	0	192.014
1 f 0	48.831	−26.042	−79.892
1 f 1	48.831	−19.532	−38.347
1 f 2	48.831	0	39.294
1 f 3	48.831	32.554	12.042
2 p 0	59.679	−47.743	244.350
2 p 1	59.679	23.872	127.098
1 g 0	66.955	−34.782	−84.022
1 g 1	66.955	−29.565	−60.983
1 g 2	66.955	−13.913	−5.464
1 g 3	66.955	12.174	41.787
1 g 4	66.955	48.694	12.842

energy shift is given by:

$$E_{nljm}{}^{(1)}=-2E_{nljm}{}^{(0)}\frac{j(j+1)-3m^2}{2j(2j+2)}. \quad (23)$$

IV. SOME RESULTS FOR OTHER FORMS OF THE POTENTIAL

The problem of finding the energy levels in a realistic nuclear potential (e.g., square well with rounded edges) which has been deformed into spheroidal shape is a very formidable one. Moreover, it is not clear that such calculations would yield significant improvements over the results obtained with the spheroidal box. Surely the relation of the effective one-particle potential to the two-body forces must be correctly taken into account. Thus, the well depth is expected to decrease when the well is deformed into spheroidal shape,[17] which results in an additional restoring force of potential origin. Some simple approximate results may, however, be of interest.

One interesting form of the potential which has proven extremely useful for many calculations is that of the harmonic oscillator. It is well known that the energy levels and wave functions for particles in an anisotropic oscillator potential can be obtained exactly.[8–12,18]

The potential can be written as follows:

$$V=\tfrac{1}{2}m\omega^2 r'^2, \quad (24)$$

where the components of r' are defined in [Eq. (3)].

The average energy over a closed shell is given by

$$E'=E^{(0)}(\tfrac{2}{3}D^{\frac{1}{2}}+\tfrac{1}{3}D^{-1})\xrightarrow[\text{small } d]{}E^{(0)}(1+\tfrac{1}{4}d^2), \quad (25)$$

and the average effect of the deformation on the wave functions is to replace $\psi(\mathbf{r})$ by $\psi(\mathbf{r}'')$, where now

$$x=x''D^{-\frac{1}{4}},\quad y=y''D^{-\frac{1}{4}},\quad z=z''D^{\frac{1}{2}}. \quad (26)$$

Thus the wave functions "follow" the deformation to half its extent.

The first-order shift of energy levels in a deformed well of any shape has been calculated previously by several authors.[5,7,9] It can be written as follows:

$$E_{nljm}{}^{(1)}=-E_{nljm}{}^{(0)}2P_2(\theta)\,|_{ii}f, \quad (27)$$

where E is the total energy measured from the bottom the bottom of the well. f is the ratio of kinetic energy (expectation value) to the total energy.

For all states of a harmonic oscillator potential $f=\frac{1}{2}$. A comparison of these results with those obtained in Sec. 2 for a box ($f=1$) suggests that perhaps for an arbitrary rounded well, the effect of changing $V(r)$ to $V(r')$ might be, at least on the average, to distort $\psi(\mathbf{r})$ to $\psi(\mathbf{r}'')$, where

$$z=D^{f}z'',\quad x=D^{-f/2}x'',\quad y=D^{-f/2}y'', \quad (28)$$

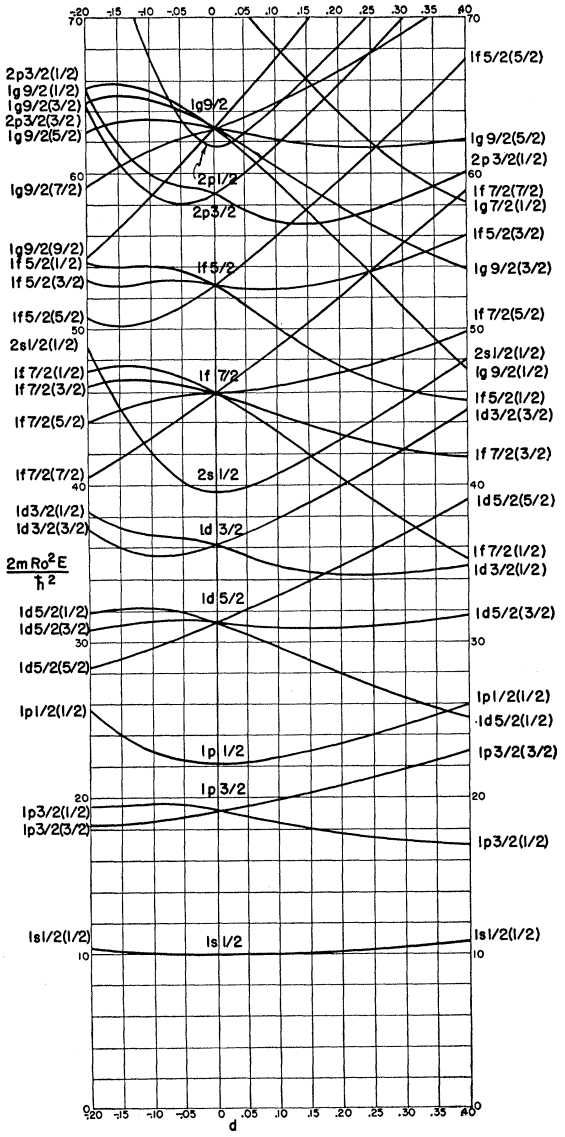

FIG. 2. Energy levels of a spheroidal box as function of deformation A reasonable spin-orbit term has been included.

and that the average energy over a closed shell is:

$$E'=E^{(0)}(\tfrac{2}{3}D^{f}+\tfrac{1}{3}D^{-2f})\xrightarrow[\text{small } d]{}E^{(0)}(1+f^2d^2). \quad (29)$$

The residual interaction would then only mix states in the same shell. Thus [see Eq. (16)],

$$\mathfrak{IC}_{ik}\cong-(E_iE_k)^{\frac{1}{2}}2fdP_2(\theta)\,|_{ik} \quad (30)$$

where f is now a reasonable average value for the shell in question.

The plausibility of this conjecture has been investigated by a variational calculation of the lowest energy level $E_{\min}$ and corresponding wave function. The wave function was assumed to be of the form $\psi(\mathbf{r}'')$ with $\mathbf{r}''$ defined in terms of $\mathbf{r}$ by Eq. (28); however f was taken

[17] K. Brueckner, Phys. Rev. **97**, 1353 (1955).
[18] M. L. Gursky, Phys. Rev. **98**, 1205(A) (1955).

as a variable parameter. The value of f giving minimum energy is denoted here by f_0.

For example, a particular rounded well shape is provided by the potential $V=ar^2-br^4$, at least for not too large values of r. For this case, the variational calculation gives the value of f_0:

$$f_0=1/2-\gamma/5, \tag{31a}$$

where

$$\gamma=(br^4|_{ii})/(ar^2|_{ii}).$$

On the other hand, the ratio of kinetic to total energy is:

$$f=T/E=\tfrac{1}{2}-\tfrac{1}{4}\gamma \tag{31b}$$

which is only very slightly different from f_0.

The corresponding energy is given by

$$E_{\min}=E^{(0)}[1+ff_0d^2+\cdots]. \tag{32}$$

Another special potential of great interest is the square well with walls of finite height. The approximate effect[2,5] of the finite wall height is to reduce all energies (measured from the bottom) to the values they would have for a box with radius $R_0+\lambda_0$ where

$$\lambda_0=[\hbar^2/(2mV_0)]^{\frac{1}{2}}. \tag{33}$$

A more accurate expression for the energy levels[19] is:

$$E_{V_0,R_0}=\epsilon\left\{1-\left(1+\frac{\lambda_0}{R_0}\right)^{-1}\left(\frac{\lambda_0^3}{3R_0^3}\right)\times\left[\frac{2mR_0^2\epsilon}{\hbar^2}-l(l+1)\right]\right\}, \tag{34}$$

where

$$\epsilon=E_{\infty,R_0}(1+\lambda_0/R_0)^{-2}.$$

The energy levels of a spheroidal well with walls of finite height V_0 might then be expected to be close to those of a spheroidal box whose "radius" is larger by λ_0, in other words, of a spheroid with axes: $R_0D+\lambda_0$, $R_0D^{-\frac{1}{2}}+\lambda_0$, $R_0D^{-\frac{1}{2}}+\lambda_0$. In this way, one can obtain approximately the same results as those resulting from application of Eqs. (28)–(30), particularly for tightly bound states.

For a square well, a good approximation to the ratio f is given by

$$f=\frac{\alpha R_0}{\alpha R_0+1}+\frac{l(l+1)}{2(\alpha R_0+1)^3}, \tag{35}$$

where

$$\alpha=[2m(V_0-E)/\hbar^2]^{\frac{1}{2}}.$$

V. SOME APPLICATIONS TO NUCLEAR STRUCTURE AND CONCLUSIONS

The results of the above calculations can be used to estimate the total relative energy of a given number of particles in an elastic well as function of deformation. Although the approximations made here are extreme, it is hoped that some qualitative trends are given correctly. Considering for simplicity only identical nucleons, Fig. 3 shows the total energies as function of deformation for the energetically most favored configuration of 20, 22, 24, 26, and 28 particles. The results are in general agreement with previous calculations[1,2,6] except for the favoring of a prolate shape for the half-filled shell at 24 particles. This is a consequence of the repulsion of the $1f\,7/2(1/2)$ state [the $m=\pm\frac{1}{2}$ substate of the $1f_{7/2}$ shell] by higher-lying states, such as $2p\,3/2(1/2)$ while the $1f\,7/2(7/2)$ state is not pushed downward.

The calculated restoring energies depend somewhat on the model chosen, but are much larger, even by as much as an order of magnitude, than the values calculated with the hydrodynamical model. One important reason for this is that for light and medium nuclei crossing of particle levels is not very extensive.

Consider, for example, the nucleus $_{22}Ti^{44}$. According to the "box model," the most favored deformation $d_{\min}$ is -0.05 corresponding to the complete filling of the $m=\pm 7/2$ substates of $1f_{7/2}$ shell. For a more realistic well of finite depth, $d_{\min}\approx-0.07$ ($f\approx 0.8$). On the other hand, the hydrodynamical model gives $d_{\min}\approx-0.4$. To make this estimate, the hydrodynamical

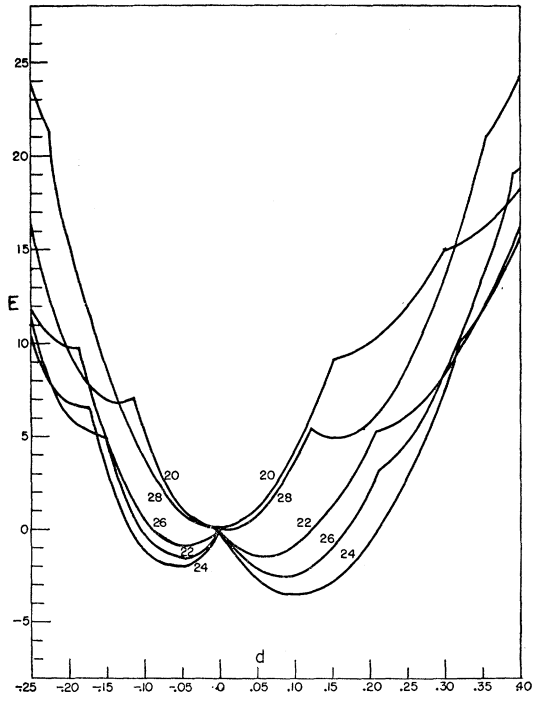

FIG. 3. Total energy as function of deformation for various numbers of identical $f_{7/2}$ particles (plus core of 20 particles) in a spheroidal box. For odd-particle numbers the energies can be obtained by linear interpolation.

[19] For loosely bound states ($V_0-E<0.25V_0$) the "3" in the denominator of Eq. (34) should be replaced by a "2." Equation (34), with this modification when appropriate, gives values of E within 0.1 to 1% of the exact values.

restoring energy was assumed to be given by[20]:

$$19A^{\frac{5}{3}}\times 0.4d^2-0.76Z^2A^{-\frac{1}{3}}\times 0.2d^2, \qquad (36)$$

and $2E^{(0)}f$ in Eq. (27) was taken as equal 45 Mev.

When much level crossing occurs, the effective restoring energy is reduced considerably[1,2] and large deformations can occur. This situation is very probably realized for nuclei with A between 155 and 185 and above 225.

Figure 4 shows calculated equilibrium deformations plotted against numbers of particles. The magnitudes and general trends of deformations calculated with this model seem to agree at least qualitatively with experimental values. The tendency for positive quadrupole moments indicated by Fig. 4 may be significant. Such a trend exists throughout the range of nuclei.[21] For the lighter nuclei, for which the order of particle levels predicted by a box potential is essentially correct, this trend results from the lowering of the $1d\,5/2(1/2)$, $1f\,7/2(1/2)$, and $1f\,7/2(3/2)$ states due to the strong interaction with other states of the same m.

For heavier nuclei, the order of particle levels is significantly different from that in a box, so that no definite conclusions can be drawn on basis of a box model. However, in the hydrodynamical limit, part of the restoring energy is proportional to the surface area[1,2] and this tends to favor positive quadrupole moments.[21] In addition, the Coulomb effect, which also favors positive quadrupole moments, becomes important.

The above calculations show the effect of one important feature of nuclear structure, the deviation of the nuclear potential from sphericity. To obtain detailed agreement with nuclear data; e.g., ground state spins, one must, of course, consider also the direct interactions between extra-core particles. Even if these are treated by themselves, without any core deformation, it is possible to obtain remarkable agreement with experimental data[22–26] for some of the light nuclei near double magic nuclei He^4, O^{16}, and Ca^{40}. Then also the time dependence and quantization of the nuclear distortion must be properly taken into account. This can be readily done for a hydrodynamical model.[1,27–30] In this way, some of the most striking features of nuclear states, such as energy spacings between rotational states for nuclei far away from magic numbers can be accounted for.[1] However, the absolute values of these energies appear to be considerably smaller than the hydrodynamic values deduced from known deformations.[31] Some studies of this problem on basis of the shell model have recently been made.[32,33]

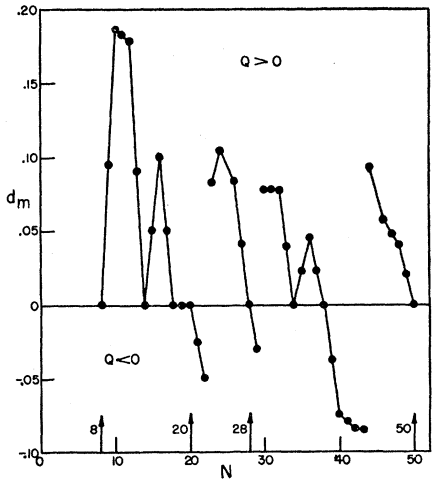

FIG. 4. Energetically favored deformations of a spheroidal box with elastic walls, as function of the number of identical particles.

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