Independent Particle Model of the Nucleus. I. Interparticle Forces and Configuration Mixing*

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A simplified method of obtaining the direct interaction between two identical nucleons in the nuclear shell model is given for the special case of singlet forces. Configuration interaction is included in the method. A semi-empirical analysis of simple two and three particle nuclear spectra is outlined which enables one to determine properties of the two-body perturbing interaction provided many body forces are negligible and coupling to the nuclear surface is weak. Corrections to the singlet force formalism due to triplet central and tensor forces are discussed. Formulas are given for magnetic dipole and quadrupole moments and magnetic dipole transition rates for mixed three-particle configurations. The effect of weak surface coupling on multi-particle configuration is given in paper II of this series. A detailed discussion of the spectra of two isotopes of calcium is given in paper III.

I. INTRODUCTION

ACCORDING to the simplest picture of the nuclear shell model, nucleons move independently within the nucleus subject only to the force of a common central potential and to a spin-orbit force.[1] When the degeneracies of this idealized model have been removed by an assumed "pairing energy,"[2] then the model has surprisingly great success in affording a qualitative explanation for most of the low-energy properties of nuclei[3] (with some notable exceptions associated with collective nuclear motion).[4] This qualitative success has led to efforts to refine the shell model to the point where it may yield quantitative agreement with experiment. This refinement has at first taken the form of adding to the independent particle Hamiltonian a direct two-body interaction among the nucleons outside of closed shells. Kurath,[5] Talmi,[6] Flowers and Edmonds,[7] and Pryce[8] have studied the effects of various assumed forms of this perturbing interaction on the energy levels of two-, three-, and four-particle configurations. Although configuration interaction was neglected in their work, the predicted level ordering was in accord with experiment where comparison could be made.

As one might expect from the strength of the nuclear forces, the inclusion of perturbing inter-particle forces in the shell model leads to the prediction that configuration interaction is of substantial importance. de-Shalit and Goldhaber[9] have emphasized some important consequences of configuration interaction, and Redlich[10] has shown that a mixture of configurations considerably improves the calculated ft value of the transition $F^{18}(\beta^+)O^{18}$, as compared to any single configuration calculation. Among the many efforts to explain the deviations of nuclear magnetic moments from the Schmidt lines, one of the simplest and most unforced is embodied in the work of Blin-Stoyle and Perks,[11] who find that the direction and order of magnitude of the deviations can be accounted for in terms of simple interconfigurational mixing. Collective effects may also play an important role in the magnetic moment deviations,[4] but a general survey of this question[12] has led to the conclusion that in addition to surface effects, there must be another and at least as large an effect producing an inward deviation of the moments from the Schmidt lines. This extra effect was first interpreted in terms of a quenching of the anomalous moment of the nucleon, but now appears to be more reasonably attributed to the Blin-Stoyle and Perks effect of configuration mixing. Other recent attempts have been made to take into account both collective and configuration mixing effects[4,13] on nuclear magnetic moments. More recently Elliott and Flowers[14] have had detailed quantitative success in the analysis of simple configurations just beyond the O^{16} core, using a Rosenfeld exchange type force and including configuration interaction. Volkov[15] has carried out calculations with the "statistical shell model" based on very great configuration mixing, but this limit does not appear to be justified by more detailed shell-model calculations.

A detailed treatment of two-body interactions among nucleons outside of closed shells need not, of course,

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lead to a quantitatively correct nuclear model. Either many-body forces or collective nuclear motions, for example, could require large corrections. Indeed the comprehensive work of Bohr and Mottelson[4] and others[16] has shown that at least in certain regions of the periodic table, coupling of nucleons to the nuclear surface and the contribution of the nuclear core to the motion are of dominant importance. These collective effects can be taken account of, however, as a further refinement of the shell model, and indeed the form of the particle-to-surface interaction used by Bohr[17] depends essentially on the zero-order assumption of independent particle motion in the nucleus.

Possible many-body forces cannot be taken into account with equal ease, but the "pair damping" postulated by Drell and Henley,[18] and Brueckner and Watson,[19] in the pseudoscalar meson theory would tend to make many-body forces in the nucleus much less important than two-body forces. At the present stage of understanding of the nuclear forces, however, the role of many-body forces in determining low-energy nuclear properties must be deduced primarily from the indirect evidence of the validity of calculations which ignore such forces. The work of Brueckner, Levinson, and Mahmoud[20] on nuclear saturation with two-body forces, for example, provides indirect evidence of this kind. The shell model calculations of Elliott and Flowers[14] and our calculations on the isotopes of calcium (paper III of this series[21]) further strengthen the conclusion that many-body forces are not of much importance for the low-energy properties of nuclei.

The theoretical[22] and experimental evidence accumulated to date favors the idea that the independent-particle model is a good starting point for any attempted quantitative theory of nuclei in the ground state and low excited states. The work reported in this and the succeeding two papers was motivated by a desire to see to what extent the independent particle model, modified by spin-orbit coupling, by direct interparticle coupling, and by particle-to-surface coupling, could be made to yield quantitative results. These three added interactions have previously been shown separately to be necessary in special cases. Altogether, the model to be tested is represented by the Hamiltonian

$$H=H_p+\sum_{i>j} V_{ij}(p)+H_s(\alpha)+\sum_i V_i(s),$$

in which H_p represents the average central field together with a one-body spin-orbit coupling, $V_{ij}(p)$ represents the direct two-body interactions among particles outside closed shells, $H_s(\alpha)$ represents the energy of collective motion (restricted in practice to the five degrees of freedom of ellipsoidal deformations), and $V_i(s)$ represents the particle-to-surface interaction derived from the linear dependence of particle kinetic energy on the deformation of the average potential well.

The primary interest in this work was divided between the nature of the interparticle forces themselves within condensed nuclear matter and the general validity of the independent-particle model as a basis for quantitative calculations.

Some important questions that naturally arise are:

(1) To what extent can interparticle forces in nuclear matter be treated as two-body forces?

(2) What is the relative importance of surface corrections and interparticle force corrections to the independent-particle model states?

(3) What is the magnitude of the energy shifts due to configuration interaction?

(4) Are the "interparticle forces" that act between particles in a nucleus equivalent to the nuclear force between two nucleons in a vacuum?

In this paper (I), a general discussion is presented of the shell-model assumptions and the nature of the particle and core forces neglecting the surface forces. In paper II,[23] there is included a discussion of weak surface coupling and idealized calculations of the effect of one and two excited phonons on two- and three-particle configurations. In paper III,[21] the techniques developed in the previous papers are applied to the energy levels of Ca^{42} and Ca^{43} and to the magnetic moment of Ca^{43}. On the basis of this limited calculation one can give tentative answers to the above-listed questions.

In this paper, we discuss properties of the shell model modified by direct two-body forces only. Previous applications of such a model have utilized specific assumptions about the two-body forces which in general do not accord with current ideas about the nature of the nuclear force as deduced from theory[24,19] or from nucleon-nucleon scattering experiments.[25] In any case, the question whether the forces in the nucleus are the same as those in the "vacuum" is not answered for certain. Consequently we emphasize a semiempirical approach in which very few details of the nuclear forces are assumed. But the model discussed in this paper does assume the following: (a) A double closed-shell nucleus, or nuclear core, may be replaced by a one-particle potential well plus spin-orbit coupling. The levels in this well are determined empirically by the nucleus containing only one nucleon beyond the closed-shell core. (b) Extra-core nucleons moving in this well interact with each other via two-body forces. (c) These

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interactions mix configurations as well as split configurations, and this mixing must be taken into account. (d) For configurations of identical particles (all neutrons or all protons) the forces are predominantly singlet. This assumption can be tested since the nucleon-nucleon interaction is sufficiently well known from analysis of scattering experiments at the energies important in a nucleus. Of course, in applying such a test one must verify that question four has an affirmative answer.

The calculation of the matrix elements of the interparticle interaction is considerably simplified if one uses methods developed by Bacher and Goudsmit,[26] Racah,[27] and others. It can be shown that the matrix elements involving n particles can be computed from those involving $n-1$ particles, where $n \geqslant 3$. Therefore, the main physical problem is the determination of the two-particle matrix elements. If the two particles are both neutrons or protons, then two simplifying features enter the calculation: (1) the forces are approximately singlet, and (2) the Pauli principle restricts the number of possible states. In order to treat the effect of various two-particle configurations of identical particles, the Schrödinger equation is algebraically manipulated so that the calculation can be done simply. This avoids the more clumsy matrix approach, and makes it possible to determine the matrix elements of the interaction from the empirical two-particle level structure including the effects of configuration interaction.

The three-particle problem is then set up as a matrix diagonalization problem, the matrix elements being derived from the two-particle elements. Hence the three-particle energy levels and wave functions are logically tied to the two-particle matrix elements. This enables one to make an empirical analysis of a two-particle configuration, and then make predictions for the three-particle problem. In this manner, one does not have to know the exact nature of the interparticle forces in order to test the model under consideration. If the model can be verified in this way, one can then test various possible interparticle potentials to see if they give the correct matrix elements. The theory in the present paper is therefore directly applicable only to special configurations of identical particles, and is not explicitly carried beyond the point required for the applications discussed in paper III.[21]

II. HAMILTONIAN WITH PARTICLE FORCES

To zero order one assumes that the particles move independently in the potential of the double closed-shell core in states given by the independent-particle model. The "particle" can in fact be a neutron, a proton, a neutron hole, or a proton hole. The levels of these particles are given by the core-plus-one-particle nuclei. The nuclei are listed in Table I, and are not very numerous. Without experimental data on these nuclei one cannot analyze in a completely empirical way the effects of configuration interaction, since these nuclei give the relative spacing of the various configurations. It is possible, of course, to make reasonable guesses in some cases of the zero-order level spacings. Also it should be mentioned that there may exist closed subshells which to a fair approximation may be incorporated in the cores. For example, Zr^{90} with 40 protons and 50 neutrons shows some of the properties of a double closed-shell nucleus.

The zero-order assumption certainly is not rigorously true. For instance, as pointed out by Blin-Stoyle and Perks,[11] the explanation of the deviation of the magnetic moment of Bi^{209} from the Schmidt value ($\Delta\mu = 1.4$ n.m.) is that the odd proton is not in a pure $h_{9/2}$ state, but other configurations involving the $h_{11/2}$ protons in the core are admixed. This effect should be most significant when there are orbital states in the core identical to orbital states of the extra core particles, i.e., when the core is "j–j magic" but not "L–S magic." Single-particle levels in the core field are not quite the pure single-particle states predicted by the independent-particle model. This effect of admixed states of core excitation is not ruled out even for L–S magic nuclei, but if its effect on the one-particle energy spacing is small, then its effect on the two- and three-particle analysis will be small, independent of its effect on the one-particle wave function.

The coherent nuclear model[20,22] has provided a theoretical justification for the concept of the effective core field. Calculations of this average potential are now being performed.

Interparticle Forces

There are two fundamental difficulties concerned with the choice of the two-body forces acting between the particles moving in the core field. (1) It is uncertain how much these forces within the nucleus differ from the vacuum nuclear forces. Some difference is predicted by the coherent nuclear model,[22] but details about the difference are still under study. (2) If the effective forces within nucleus possess repulsive cores as indicated by theory[19,24] and experiment[25] for free nucleons, then perturbation theory with independent particle wave functions is not appropriate. The first difficulty we avoid by not specifying the forces in advance, but letting matrix elements of the interaction potential appear as parameters in the theory. Only through the simplifying assumption of singlet forces between iden-

TABLE I. Double magic cores and single-particle or single-hole configuration nuclei.

	Particle	Hole	Particle	Hole	Particle	Hole	Particle	Hole
Proton	$_9F^{17}$	$_7N^{15}$	$_{21}Sc^{41}$	$_{19}K^{39}$	$_{21}Sc^{49}$	$_{19}K^{47}$	$_{83}Bi^{209}$	$_{81}Tl^{207}$
Neutron	$_8O^{17}$	$_8O^{15}$	$_{20}Ca^{41}$	$_{20}Ca^{39}$	$_{20}Ca^{49}$	$_{20}Ca^{47}$	$_{82}Pb^{209}$	$_{82}Pb^{207}$
Cores	$_8O^{16}$		$_{20}Ca^{40}$		$_{20}Ca^{48}$		$_{82}Pb^{208}$	

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tical particles are the forces in nuclei and in vacuum assumed to be similar. The second difficulty we do not strictly avoid, but rather ignore. The effect of a repulsive core on the radial parts of the wave function and short range correlations of nucleon positions is not taken into account. There is theoretical evidence[22] that the independent particle picture is approximately correct at low energies in spite of such correlations, but the final test of this approximation must come through the success or failure of calculations which start with the independent particle assumption.

Most workers have assumed a Gaussian radial dependence for the interparticle potential for convenience, but have assumed a variety of exchange mixtures. Based on a comparison of the Gaussian and Yukawa radial dependence, Talmi[6] has stressed the fact that predicted relative energy spacings are sufficiently sensitive to the shape of the potential that detailed conclusions, such as the point of crossing of two levels, should not be drawn for any specific assumed shape.

For identical particles, there are only two independent central force exchange mixtures, which can be conveniently taken to be of singlet and triplet character. In Figs. 1 through 3 we give the diagonal matrix elements of the energies of the two-particle configuration $(f_{7/2})^2$ as a function of the range of the forces for three different assumptions about the shape of the potential and for the singlet and triplet exchange character, in order to give a qualitative illustration of the roles of range, shape, and exchange character in the relative level spacings. Figure 1 is based on the Gaussian potential, $V=V_0\exp(-r^2/r_0^2)$. Energies are calculated by the method of Talmi[6] with oscillator wave functions, which contain the factor $\exp(-r^2/r_n^2)$. The relative range is defined by $\lambda=r_0/r_n$, and the abscissa is taken as $\eta=\lambda^2/(1+\lambda^2)$, which varies between 0 and 1 as the range, r_0, varies from 0 to ∞. The "correct" range can be found by fitting the Gaussian potential to an effective

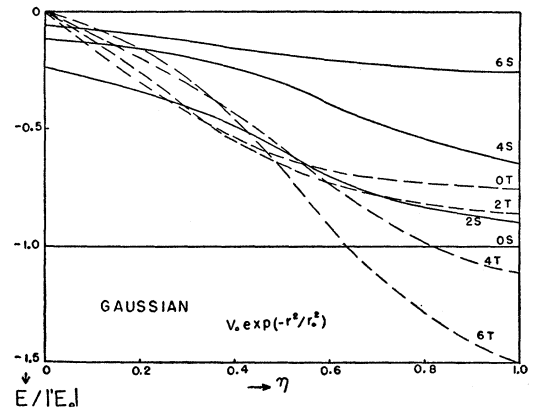

FIG. 1. Diagonal energy matrix elements for an $(f_{7/2})^2$ configuration ($J=0$, 2, 4, 6) for a singlet central (S) and triplet central (T) Gaussian well in units of the singlet spin zero matrix element. η, the range parameter is defined in the text. $\eta=0$ corresponds to a zero range interparticle force and $\eta=1$ corresponds to an infinite range interparticle force.

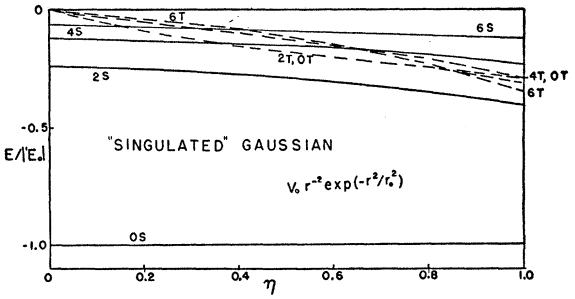

FIG. 2. Diagonal energy matrix elements for a modified Gaussian well plotted as in Fig. 1.

range of 3×10^{-13} cm and to a bound state at zero energy in order to approximate the low-energy singlet well. One finds $V_0=25.8$ Mev and $r_0=2.09\times10^{-13}$ cm. For calcium, Kurath gives as a reasonable value of r_n, 2.9×10^{-13} cm, which leads to $\lambda=0.72$. All energies on the vertical scale are normalized to a value of -1 for the energy of the spin-zero state with an attractive singlet force, and are plotted relative to the configuration energy in the absence of the perturbing forces. The short-range limit shows the characteristic large depression of the spin-zero state (seniority zero) relative to the state of spin 2, 4, and 6 (seniority two) for singlet forces, with of course no energy shifts for triplet forces. In the long-range limit, the singlet energies are proportional to $[\text{const}+J(J+1)]$, while the triplet spacings are equal and opposite to the singlet spacings. For a "reasonable" range, $\lambda\cong0.72$ $\eta\cong0.34$, the singlet 0–2 spacing is larger than the 2–4 spacing, and the triplet splittings are small compared to the singlet splittings. One is therefore qualitatively closer to the short-range than to the long-range limit.

In Fig. 2, the effect of a sharp attractive singularity in the potential is demonstrated. The potential is chosen (for convenience) to have the form,

$$V=V_0r^{-2}\exp(-r^2/r_0^2).$$

The result is that the level splittings remain for all ranges qualitatively very similar to the short range limit, with the singlet spin zero energy depression being large compared to all other energy shifts. As will be brought out in Paper III,[21] this behavior associated with the strong singularity at the origin is inconsistent with the experimental evidence.

Although a repulsive core potential cannot be taken account of properly with independent-particle wave functions, the central repulsion can be simulated by cutting off the potential to zero value at the origin. In Fig. 3, energies are plotted, as in Figs. 1 and 2, for such a potential, chosen to be,

$$V=V_0r^2\exp(-r^2/r_0^2).$$

The short- and long-range limits agree with those of the Gaussian potential, but the intermediate region of "reasonable" ranges is substantially different. The 0–2

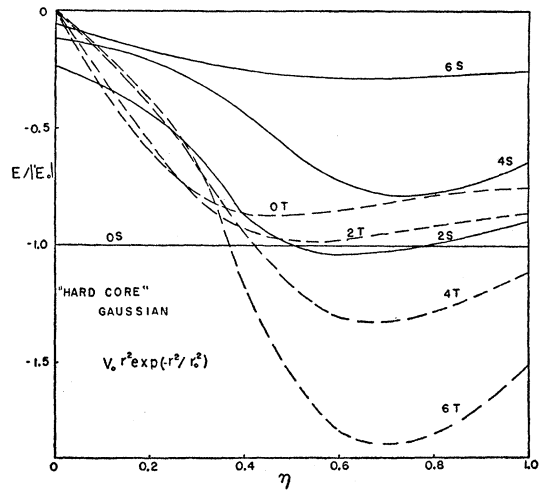

FIG. 3. Diagonal energy matrix elements for a modified Gaussian well plotted as in Fig. 1.

spacing, due to singlet forces, decreases rapidly as the range is increased from zero: the 0–2 and 2–4 spacings become equal at $\eta\cong 0.33$ ($\lambda\cong 0.71$), and the 0–2 singlet spacing becomes zero at $\eta\cong 0.5$ ($\lambda\cong 1.0$). As with the Gaussian (Fig. 1), the triplet energies show cross-over and small splittings in the neighborhood of equal 0–2 and 2–4 singlet splittings. But with this "hard core" potential the triplet splittings become inverted and quite appreciable for $\lambda\gtrsim 0.9$.

The qualitative differences among the diagonal energy spacings of Figs. 1–3 are sufficient to make desirable the kind of empirical approach to the diagonal matrix elements which is outlined in the next section. The figures also suggest that a sufficiently refined treatment of the spectra of some simple nuclear configurations may provide evidence on the shape of the interparticle potential.

It should be stressed that these figures are plotted for equal range and strength of the singlet and triplet potentials. Differences in range and strength may easily be taken account of by reading singlet and triplet splittings from different abcissas, and by multiplying the singlet or triplet energies by a suitable scale factor.

Coulomb Forces

In case one has to deal with proton configurations or to compare the level structure of similar neutron and proton configurations, the contribution of Coulomb forces to the level splitting must be calculated. This may be done readily following the method of Talmi[6] for oscillator wave functions of the nucleons. The Talmi radial integrals for the Coulomb force are

$$I_l=(l!2^{l+1}/(2l+1)!!)\epsilon_0,$$

where $(2l+1)!!=1\cdot 3\cdots(2l+1)$, and the energy unit is

$$\epsilon_0=\pi^{-\frac{1}{2}}e^2/r_n.$$

The nuclear size parameter, r_n, enters only in the energy unit. For a particular configuration, it is necessary to express the Slater integrals, F^k, in terms of the Talmi integrals, I_l, and then to express the energies E_J in terms of the F^k. Coulomb energies obtained in this way for the configurations $(f_{7/2})^2$ and $(f_{7/2})^3$ are given in Table II. The long range Coulomb force splits the levels by a lesser amount than it shifts the center of gravity. For both $(f_{7/2})^2$ and $(f_{7/2})^3$, the total level splitting is about $0.3\epsilon_0$. For $A=40$, $\epsilon_0\cong 0.3$ Mev; therefore, the total level splitting due to Coulomb forces in about 0.1 Mev; an amount small compared to the splitting produced by the nuclear forces.

III. TWO-BODY PROBLEM WITH SINGLET Forces

Let H_0 be the zero-order Hamiltonian of average central core field plus spin-orbit force; V, the sum of two-body interactions among particles outside the core; E, the correct nuclear energy; and $|\psi)$, the "correct" nuclear state vector.

The Schrödinger equation can be written

$$(E-H_0)\,|\psi)=V\,|\psi). \tag{1}$$

Now introduce a representation $|\alpha)$ such that H_0 is diagonal, in this case a jj coupling representation:

$$H_0\,|\alpha)=H_0(\alpha)\,|\alpha). \tag{2}$$

The energies $H_0(\alpha)$ are given empirically by the levels of the nucleus with a single nucleon outside the closed-shell core.

In the $|\alpha)$ representation, (1) can be written:

$$(\alpha|\psi)=\frac{1}{E-H_0(\alpha)}(\alpha|V|\psi). \tag{3}$$

One now introduces a representation $|\beta)$ convenient for expressing the matrix elements of V. Multiplying both sides by $(\beta|\alpha')$ and summing, we get

$$\sum_{\alpha'}(\beta|\alpha')(\alpha'|\psi)=(\beta|\psi)=\sum_{\alpha'}\frac{(\beta|\alpha')(\alpha'|V|\psi)}{E-H_0(\alpha')}$$
$$=\sum_{\alpha',\beta'}\frac{(\beta|\alpha')(\alpha'|V|\beta')(\beta'|\psi)}{E-H_0(\alpha')}. \tag{4}$$

TABLE II. Coulomb energies for configuration $(f_{7/2})^n$ of protons. $\epsilon_0\cong 0.3$ Mev for calcium.

J	$n=2$ E/ϵ_0	J	$n=3$ E/ϵ_0
0	1.357	3/2	3.236
2	1.160	5/2	3.336
4	1.067	7/2	3.410
6	1.033	9/2	3.176
		11/2	3.216
		15/2	3.112

We now specifically choose a singlet potential and let $|\beta)$ be the LS representation. For two particles the α and β labels become

$$|\alpha)=|JM_J j_1 j_2 l_1 l_2), \quad |\beta)=|JM_J LS l_1 l_2). \tag{5}$$

For shortness in notation we shall write $|\alpha)=|j_1 j_2 l_1 l_2)$ and $|\beta)=|l_1 l_2 L)$. The value of J, M_J will be the same throughout and since V is a pure singlet potential we shall only need $S=0$, $L=J$. Hence the above labels will be sufficient. Setting $|\beta)=|l_1 l_2 J)$, Eq. (4) becomes

$$(l_1 l_2 J|\psi) = \sum_{\substack{j_1' j_2' \\ l_1' l_2' \\ l_1'' l_2''}} \frac{(l_1 l_2 J|j_1' j_2' l_1' l_2')(j_1' j_2' l_1' l_2'|V|l_1'' l_2'' J)(l_1'' l_2'' J|\psi)}{E-H_0(j_1' j_2' l_1' l_2')}. \tag{6}$$

Now

$$\begin{aligned}(j_1' j_2' l_1' l_2'|V|l_1'' l_2'' J) &= \sum_{l_1''' l_2'''} (j_1' j_2' l_1' l_2'|l_1''' l_2''' J)(l_1''' l_2''' J|V|l_1'' l_2'' J) \\ &= (j_1' j_2' l_1' l_2'|l_1' l_2' J)(l_1' l_2' J|V|l_1'' l_2'' J),\end{aligned} \tag{7}$$

i.e., the LS to jj transformation coefficient is diagonal in $l_1' l_2'$. The fact that one can factor out the $j_1' j_2'$ dependence in Eq. (7) leads to a great simplification. Equation (6) may now be written:

$$(l_1 l_2 J|\psi) = \sum_{\substack{l_1' l_2' \\ j_1' j_2'}} \frac{|(l_1 l_2 J|j_1' j_2' l_1 l_2)|^2}{E-H_0(l_1 l_2 j_1' j_2')} \times (l_1 l_2 J|V|l_1' l_2' J)(l_1' l_2' J|\psi). \tag{8}$$

We define a new coefficient

$$\alpha(J l_1 l_2) = \sum_{j_1' j_2'} \frac{|(l_1 l_2 J|j_1' j_2' l_1 l_2)|^2}{E-H_0(l_1 l_2 j_1' j_2')}. \tag{9}$$

Then Eq. (8), which defines the expansion coefficients of $|\psi)$ in the LS representation, may be written;

$$\sum_{l_1' l_2'} \{\alpha(J l_1 l_2)(l_1 l_2 J|V|l_1' l_2' J) - \delta(l_1', l_1)\delta(l_2', l_2)\}(l_1' l_2' J|\psi) = 0. \tag{10}$$

The condition that Eq. (10) have nonzero solutions is that for each J the coefficient determinant of Eq. (10) vanish:

$$\det[\alpha(J l_1 l_2)(l_1 l_2 J|V|l_1' l_2') - \delta(l_1', l_1)\delta(l_2' l_2)] = 0, \tag{11}$$

where the rows are labeled by $l_1 l_2$ and the columns by $l_1' l_2'$.

The number of configurations included in Eq. (11) is, of course, arbitrarily great, but in order that the number of unknowns be only as great as the number of equations (equal to the number of levels of the two-particle configuration), it is necessary to make assumptions about the relative magnitudes of the radial integrals (the Slater integrals) for different l values. Near-lying levels of different l but with the same number of radial nodes have clearly similar wave functions, and may to good approximation be regarded as perfectly overlapping. If the number of radial nodes in the admixed wave function is different, one may reduce the interconfiguration matrix elements by an arbitrary factor or one may again assume a perfect overlap in order to obtain an upper limit to the configuration interaction. In either case, all required matrix elements will be expressible in terms of the diagonal matrix elements of V for the zero-order configuration. These assumptions lead to errors, so to speak, only in second order, i.e., in the effect of configuration interaction, not in the diagonal contribution to the energy. The method outlined here will then be appropriate either when configuration interaction is small, or when all important configurations have nearly the same radial dependence of their wave functions.

An especially simple situation results if the only important configurations have the same l value. Then the matrix elements are determined, according to Eq. (11), by

$$(llJ|V|llJ) = [\alpha(Jll)]^{-1}. \tag{12}$$

Using the techniques of matrix diagonalization, one would be confronted instead with a set of third-order matrices to diagonalize, corresponding to the three configurations $(l+\frac{1}{2})^2$, $(l+\frac{1}{2}, l-\frac{1}{2})$, and $(l-\frac{1}{2})^2$.

Equation (4) might be useful for three-body configurations also. One must then introduce extra quantum numbers in the $|\alpha)$ and $|\beta)$ representations. The usefulness of this approach again depends on how one can factor $(\alpha|V|\beta)$.

IV. THREE-BODY CONFIGURATION MATRIX ELEMENTS

In order to evaluate three-body matrix elements in terms of two-body matrix elements, it is necessary to use fractional parentage coefficients, which were introduced by Racah[26] for configurations containing equivalent particles. The extension of Racah's discussion of fractional parentage coefficients to apply to groups of inequivalent particles has been carried out by Redlich[28] and Meshkov.[29] A discussion and tabulation of fractional parentage coefficients for equivalent particles in jj coupling has been given by Flowers and Edmonds.[7] We shall give here Redlich's definition of fractional parentage coefficients for nonequivalent or equivalent particles with jj wave functions and his statement of the matrix element theorems.

Let $\psi_a(j_1 \cdots j_n, \alpha J)$ be an antisymmetrized n-particle wave function made up of states $j_1 \cdots j_n$ coupled together to make a total J. Let α be the other quantum numbers not explicitly written down. One can expand

[28] M. Redlich, thesis, Princeton University, January, 1954 (unpublished).
[29] S. Meshkov, Phys. Rev. **91**, 871 (1953).

ψ_a in terms of functions $\phi[(j_1\cdots j_{r-1}j_{r+1}\cdots j_n\alpha'J')_a \times j_r(n);\alpha J]$ which are constructed by combining functions $\psi_a(j_1\cdots j_{r-1}j_{r+1}\cdots j_n\alpha'J')$ with $\psi(j_r(n))$ by means of the vector coupling coefficients to make a state of angular momentum J. The notation $j_r(n)$ means that the added wave function $\psi(j_r(n))$ be written as a function of the variables $x_n\sigma_n$. ψ_a may be expanded as

$$\psi_a(j_1\cdots j_n\alpha J)=\sum_{r,\alpha',J'}(\alpha'J';j_r][\alpha J) \times\phi[(j_1\cdots j_{r-1}j_{r+1}\cdots j_n\alpha'J')_a j_r(n);\alpha J], \quad (13)$$

where the sum over r is taken only over distinct j_r. The coefficient of ϕ is called a fractional parentage coefficient. Redlich gives formulas for these coefficients for three-particle configurations with two equivalent particles in terms of Racah coefficients [see, e.g., Eq. (30)]. This definition of fractional parentage coefficient reduces to Racah's for the case of equivalent particles.

The formulas for computing matrix elements of one- and two-body operators will be given here: Let $F=\sum_{i=1}^{n}f(i)$, where $f(i)$ operates only on particle i. Then

$$(\psi_a(j_1\cdots j_n\alpha J_1)|F|\psi_a(j_1\cdots j_n\beta J_2)) =n\sum_{\alpha'J'r}(\alpha J_1[\alpha'J';j_r)(\alpha'J';j_rJ_1|f(n)|\alpha'J';j_rJ_2) \times(\alpha'J';j_r]\beta J_2), \quad (14)$$

where $(A[B)=(B]A)^*$. Let $G_n=\sum_{i>k}g(i,k)$ where $g(i,k)$ operates only on particles i and k, and $g(i,k)$ is a scalar operator.

$$(\psi_a(j_1\cdots j_n\alpha J)|G_n|\psi_a(j_1'\cdots j_n'\beta J)) =\frac{n}{n-2}\sum_{r,\alpha',\beta',J'}(\alpha J[\alpha'J';j_r)\delta(j_r,j_r') \times(\psi_a(j_1\cdots j_{r-1}j_{r+1}\cdots j_n\alpha'J')|G_{n-1}|\psi_a \times(j_1'\cdots j_{r-1}'j_{r+1}'\cdots j_n'\beta'J'))(\beta'J';j_r]\beta J). \quad (15)$$

Equation (15) provides a relation between diagonal elements of n-particle and $(n-1)$-particle configurations which makes it possible to test observed nuclear spectra to find out whether or not configuration interaction is important. If configuration interaction is unimportant, then, *independent* of the potential involved (it must be a G-type two-body operator, however), Eq. (15) relates the energies of the n-particle and $(n-1)$-particle configurations. Hence if the observed energies are not related by (15), one knows either that configuration interaction is important, or that forces other than two-body forces are acting, e.g., direct many-body forces, or surface forces. (It is to be noted, however, that the *weak* coupling of particles via the surface can also be represented as two-body forces of type G.)

V. NUCLEAR MOMENTS AND TRANSITION RATES

Nuclear moments and transition rates depend more strongly on configuration admixtures than do energy spacings and hence are a better indication of the amount present. For magnetic moments the admixtures of states with the same l values are most important since they give contributions to the magnetic moments linear in the mixture amplitudes. Blin-Stoyle and Perks,[11] have analyzed the magnetic moments of nuclei from this viewpoint and have shown that, if the configuration interaction is due to attractive central forces, the predicted sign and order of magnitude of the shifts of the magnetic moments away from the Schmidt line are correct. Consider, for example, the magnetic moment of a nucleus with three identical particles beyond closed shells. Let the ground-state wave function be approximated as a three-particle shell-model function:

$$\psi=|IM\rangle=\sum_\alpha a_\alpha|\alpha IM\rangle, \quad (16)$$

where I is the total spin, a_α is a mixture amplitude, and α designates the jj coupling particle configuration. Configurations of three kinds are possible, according as 1, 2, or 3 single-particle states enter: $\alpha_1=[j^3]$, $\alpha_2=[j_1^2(J)j_2]$, and $\alpha_3=[j_1j_2(J)j_3]$, where J labels the intermediate angular momentum. (Antisymmetized functions are assumed.) The magnetic moment of the state is

$$\mu=\langle I,M=I|\mu_z|I,M=I\rangle=\sum_{\alpha,\alpha'}a_\alpha^*a_{\alpha'}\mu(\alpha\alpha'), \quad (17)$$

where

$$\mu(\alpha\alpha')=\langle\alpha I,M=I|\mu_z|\alpha' I,M=I\rangle, \quad (18)$$

and

$$\mu_z=\sum_i g_l l_z(i)+g_s S_z(i). \quad (19)$$

We list here formulas for the components $\mu(\alpha\alpha')$ required in (17) only for configurations of type α_1 and α_2, but components involving α_3 configurations are equally readily derived, by using fractional parentage expansions and the general methods of Racah:

$$\mu(\alpha_1\alpha_1)=\langle j^3I|\mu_z|j^3I\rangle=g_jI, \quad (20)$$

the well-known result that a group of equivalent particles has the same g factor as a single particle in the group,

$$g_j=g_l\pm(g_s-g_l)(2j+1\mp1)^{-1},\quad j=l\pm\tfrac{1}{2}; \quad (21)$$

$$\mu(\alpha_2\alpha_2)=\langle j_1^2(J)j_2I|\mu_z|j_1^2(J)j_2I\rangle =[2(I+1)]^{-1}\{g_1[I(I+1)+J(J+1)-j_2(j_2+1)] +g_2[I(I+1)-J(J+1)+j_2(j_2+1)]\}; \quad (22)$$

and

$$\mu(\alpha_1\alpha_2)=\langle j_1^3I|\mu_z|j_1^2(J)j_2I\rangle =-(J]I)(g_l-g_s)[2(2l+1)(I+1)]^{-1}\delta(l_1,l_2) \times[3(J+I+l+\tfrac{3}{2})(J+I-l+\tfrac{1}{2}) \times(J-I+l+\tfrac{1}{2})(I-J+l+\tfrac{1}{2})]^{\frac{1}{2}}, \quad (23)$$

where $(J]I)$ is shorthand for the fractional parentage

coefficient $(j^2(J)jI\|j^3I)$. The last formula gives the important off-diagonal contribution to the magnetic moment emphasized by Blin-Stoyle and Perks.

Quadrupole moments are sensitive to effects of collective nuclear deformation, and provide a better measure of the strength of surface coupling than does the energy level spectrum. By the same token, however, quadrupole moments are less useful as a measure of the degree of configuration interaction than are magnetic moments. Contributions of weak surface coupling to magnetic and quadrupole moments are discussed briefly in the following paper. In certain nuclei, however (e.g., ${}_8O^{17}$), the direct particle contribution to the quadrupole moment is dominant,[3] and it is therefore of interest to give formulas for the quadrupole moment of mixed configuration analogous to Eqs. (17) through (23) above for magnetic moments. Again let the nuclear ground state wave function be approximated by (16), but with the assumption now that the extra particles are all protons. The quadrupole moment operator is

$$Q=4e\sum_i r_i^2\mathcal{Y}_{20}(\theta_i\phi_i) \tag{24}$$

and $\mathcal{Y}_{20}=(\pi/5)^{\frac{1}{2}}Y_{20}$, Y_{20} being the normalized spherical harmonic. Analogous to Eq. (17), the quadrupole moment of the mixed state is given by

$$Q=\sum_{\alpha,\alpha'} a_\alpha^* a_{\alpha'} Q_{\alpha\alpha'}, \tag{25}$$

where

$$Q_{\alpha\alpha'}=4e\langle r^2\rangle_{\alpha\alpha'}[I(2I-1)(2I+1)(2I+3)]^{\frac{1}{2}} \times(\alpha I\|\textstyle\sum_i \mathcal{Y}_2(i)\|\alpha' I), \tag{26}$$

and $\langle r^2\rangle_{\alpha\alpha'}$ is the radial integral of r_i^2 of the ith proton (any i) between the antisymmetrized states α and α'. This factor is the order of magnitude of the mean square nuclear radius. Equation (26) holds for an arbitrary number of particles, but we list formulas for the reduced matrix elements appearing in (26) only for the special case of the three-particle configurations α_1 and α_2 defined below Eq. (16). For $\alpha=\alpha'=\alpha_1$,

$$(\alpha_1 I\|\textstyle\sum_i \mathcal{Y}_2(i)\|\alpha_1 I)=3(-1)^{-i-I}(2I+1)(j\|\mathcal{Y}_2\|j) \times\sum_{J_1}(-1)^{J_1}|(j^2(J_1)jI\|j^3I)|^2W(jIjI;J_12). \tag{27}$$

Equation (27) is also somewhat more general, holding for the configuration $(j)^n$. For the special case that $I=j$, and seniority, $v=1$, a result of Racah[26] may be applied to yield,

$$(j^n\, v=1\; I=j\|\textstyle\sum_i \mathcal{Y}_2(i)\|j^n\, v=1\; I=j) =[(2j+1-2n)/(2j-1)](j\|\mathcal{Y}_2\|j), \tag{28}$$

a formula already given by Goeppert-Mayer[3] in a discussion of shell-model quadrupole moments. It is to be noted that the absolute value of the quadrupole moment found from Eq. (28) is never greater than the one-particle moment. For the quadrupole moment component diagonal with respect to configuration α_2, one gets

$$(\alpha_2 I\|\textstyle\sum_i \mathcal{Y}_2(i)\|\alpha_2 I) =(2I+1)\{(-1)^{J-j'-I}(j'\|\mathcal{Y}_2\|j')W(j'Ij'I;J2) +(j\|\mathcal{Y}_2\|j)\textstyle\sum_{J_1}|(jj'(J_1)jI\|j^2(J)j'I)|^2 \times(-1)^{J_1-j-I}W(jIjI;J_12)\}, \tag{29}$$

in which the fractional parentage coefficients may be given explicitly[28]:

$$(jj'(J_1)jI\|j^2(J)j'I) =-[\tfrac{2}{3}(2J+1)(2J_1+1)]^{\frac{1}{2}}W(Jj'jJ_1;Ij). \tag{30}$$

For the quadrupole moment component connecting configurations α_1 and α_2 one obtains,

$$(\alpha_1 I\|\textstyle\sum_i \mathcal{Y}_2(i)\|\alpha_2 I)=3^{\frac{1}{2}}(j^2(J)jI\|j^3I)(-1)^{J-j-I} \times(j\|\mathcal{Y}_2\|j')(2I+1)W(jIj'I;J2). \tag{31}$$

The one-particle reduced matrix elements appearing in (27)–(31) are

$$(lj\|\mathcal{Y}_2\|l'j')=\tfrac{1}{2}(2j+1)^{\frac{1}{2}}(j2-\tfrac{1}{2}0|\,j2j-\tfrac{1}{2}) \times|l-l'|=0,\ 2, \tag{32}$$

where $(j2-\frac{1}{2}0|\,j2j-\frac{1}{2})$ is a Clebsch-Gordon coefficient.

The predominant gamma-ray transitions among the low states of nuclei—especially if these states arise primarily from a single multiparticle configuration—are of magnetic dipole or electric quadrupole type. The general remarks above about magnetic and quadrupole moments apply also to $M1$ and $E2$ transition rates. That is, $M1$ transition rates are sensitive to configuration mixing, while $E2$ rates, although also substantially affected by mixing, are more sensitive to collective effects. $E2$ rates are discussed in the following paper.

Magnetic dipole transitions are in fact forbidden between pure states of the configuration $(j)^n$, and therefore occur only through configuration mixing or collective motion when the dominant configuration is of this simple type. Consider, for example, the matrix element,

$$(j^nJ|\,g_lL_z+g_sS_z|\,j^nJ')=(j^nJ|\,g_lJ_z+(g_s-g_l)S_z|\,j^nJ')$$

which governs magnetic dipole transitions. The part g_lJ_z does not connect states of different J. The part $(g_s-g_l)S_z$ is an F-type operator and its matrix elements are given by Eq. (14). $f(n)$ now becomes $(g_s-g_l)S_z(n)$, the single particle spin operator for the nth particle multiplied by (g_s-g_l). The matrix elements of $(g_s-g_l)S_z(n)$ which one needs in order to evaluate $(\alpha'J';\,j(n)J_1|S_z(n)|\alpha'J';\,j(n)J_2)$ are all of the form $(jm_j|S_z|jm_j')$. But as pointed out by Racah and others,[30] $(jm_j|S_z|jm_j')=\mu(j)(jm_j|\,j_z|\,jm_j')$ where $\mu(j)$ is the ratio of the reduced matrix elements of S_z and j_z and only depends on j. Hence the magnetic dipole transition operator can be written as $[g_l+(g_s-g_l)\mu(j)]J_z$ for equivalent particles of spin j. This operator is

[30] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1951).

diagonal with respect to the states of the configuration j^n. It does not connect states of different J, nor does it connect orthogonal states with the same J, e.g., states of different seniority.

We consider now the magnetic dipole transition rate between mixed configurations of three identical nucleons. Let the nuclear wave function be approximated by (16) for the final state and by the analogous expression,

$$\psi_i = |I'M'\rangle = \sum_{\alpha'} b_{\alpha'} |\alpha' I' M'\rangle \tag{33}$$

for the initial state. Then, because the magnetic dipole operator, M_1, is a vector, the methods of Racah may be applied to yield the expression for the reduced transition probability, $B_m(1)$, defined, for example, in reference 4:

$$B_m(1) = (2I'+1)\,|\mathfrak{M}|^2. \tag{34}$$

(The change of I' to I in this formula gives the reduced probability for the inverse transition $I \rightarrow I'$.) $\mathfrak{M}$ is a weighted mean reduced matrix element,

$$\mathfrak{M} = \sum_{\alpha,\alpha'} a_\alpha^* b_{\alpha'} \mathfrak{M}_{\alpha\alpha'}, \tag{35}$$

where

$$\mathfrak{M}_{\alpha\alpha'} = \langle \alpha I \| M_1 \| \alpha' I' \rangle, \tag{36}$$

and the double-bar matrix element is defined as by Racah.[26] The general selection rules, which apply to (36) are $\Delta I = 0, \pm 1$, and configurations α and α' differ by at most the quantum numbers of one particle. For $\alpha = \alpha' = (j)^3$, $\mathfrak{M}_{\alpha\alpha'} = 0$, as shown above. We consider three nonvanishing matrix elements which may contribute to the sum (35).

(a) $\alpha = [j^3]$, $\alpha' = [j^2(J)j']$:

$$\mathfrak{M}_{\alpha\alpha'} = (j^2(J)jI\,\|\,j^3I)(-1)^{J+1-j-I'}[3(2I+1)(2I'+1)]^{\frac{1}{2}} \times W(jIj'I'; J1)(j\|M_1\|j'), \tag{37}$$

where $(j^2(J)jI\,\|\,j^3I)$ is the fractional parentage coefficient of the state α, W is a Racah coefficient, and $(j\|M_1\|j')$ is a reduced one-particle magnetic dipole matrix element.

(b) $\alpha = [j^2(J)j']$, $\alpha' = [j^2(J')j']$:

$$\mathfrak{M}_{\alpha\alpha'} = 3(-1)^{1-I'}[(2I+1)(2I'+1)]^{\frac{1}{2}} \times \{\tfrac{1}{3}(-1)^{J-j'}\delta JJ' W(j'Ij'I'; J1)(j'\|M_1\|j') + \sum_{J_1}(jj'(J_1)jI\,\|\,j^2J\,j'I')(jj'(J_1)jI'\,\|\,j^2J'j'I') \times (-1)^{J_1-j} W(jIjI'; J_1 1)(j\|M_1\|j)\}, \tag{38}$$

in which the sum over J_1 includes the fractional parentage coefficients of the states α and α'.

(c) $\alpha = [j^2(J)j']$, $\alpha' = [j'^2(J')j]$:

$$\mathfrak{M}_{\alpha\alpha'} = 3[(2I+1)(2I'+1)]^{\frac{1}{2}}(j\|M_1\|j') \times \sum_{J_1}(-1)^{J_1-j-I'}(jj'(J_1)jI\,\|\,j^2(J)j'I) \times (j'j(J_1)j'I'\,\|\,j'^2(J')jI')W(jIj'I'; J_1 1). \tag{39}$$

The reduced one-particle matrix elements appearing in Eqs. (37)–(39) are

$$(j\|M_1\|j) = g_j[(3/4\pi)j(j+1)(2j+1)]^{\frac{1}{2}} \tag{40}$$

and

$$(j\|M_1\|j') = (g_s - g_l)\{(3/4\pi)[2l(l+1)/(2l+1)]\}^{\frac{1}{2}}\delta(l,l'). \tag{41}$$

The matrix elements $\mathfrak{M}_{\alpha\alpha'}$ are of the same order of magnitude as the magnetic moments of the states involved. A substantial inhibition of the transition rate (as compared with the one-particle rate) is therefore to be expected only if the dominant configurations of initial and final states are not connected by M_1—e.g., if both are $(j)^3$ or if they differ in the quantum numbers of more than one particle. In either such case, the averaged matrix element, $\mathfrak{M}$, will be smaller than the one-particle matrix element by the order of the mixture amplitudes of configurations which are connected to the dominant configurations, and the transition rate will be smaller by the order of the square of these mixture amplitudes. It is to be noted that these effects could lead to a large inhibition of an $M1$ transition rate, but would not be expected ever to lead to an appreciable enhancement. This fact coupled with the known enhancement of $E2$ radiation due to collective effects might account for the observed $E2$–$M1$ competition in a number of cases where the one-particle transition rate formulas predict that the $M1$ rate should be much greater than the $E2$ rate. An example of such a case, the isotope $_{20}Ca^{43}$, is discussed in paper III.

Consider the following idealized example of the possible utility of transition rate measurements in determining nuclear configurations. Suppose a nucleus has ground state 5/2+ and first excited state 3/2+, and the shell-model assignment of the ground state is $d_{5/2}$. One wishes to know whether the one-particle assignments $d_{5/2}$ and $d_{3/2}$ for the two states or the multiparticle assignments $(d_{5/2})^3$ for both states are more nearly correct. If the absolute $E2$ rate between the states is, for example, determined by a Coulomb excitation process, and the $M1/E2$ ratio is inferred, or limited, by internal conversion or angular correlation experiments, then the $M1$ rate is in principal determined. It may be compared with the known $d_{3/2} \rightarrow d_{5/2}$ prediction: if the same order of magnitude, the one particle configurations are preferred; if much smaller, the $(d_{5/2})^3$ configuration is preferred. Factors of difference of 100 or more are to be expected.

Beta-decay matrix elements are especially sensitive to details of nuclear structure—the more so since states of two different nuclei are involved—and quantitative agreement between theoretical and experimental transition rates has been obtained only for mirror nuclei. Fermi matrix elements, and one-particle Gamow-Teller matrix elements are well known; they have been summarized, for example, by Bohr and Mottelson,[4]

whose notation we follow. Talmi[31] has considered three-particle Gamow-Teller matrix elements in jj coupling and shown that the distinction between favored and unfavored transitions does not exist for pure jj configurations. From his work, the reduced transition probability for $[j^3J'T']\rightarrow[j^3JT]$ follows easily:

$$D_{\mathrm{GT}}=27(2J+1)(2T'+1)\,|\,(l_j\|B^{(1)}\|l_j)\,|^2 \times(T'1T_z'T_z-T_z'|\,T'1TT_z)^2S^2, \quad (42)$$

where $(T_1T_2T_{1z}T_{2z}|\,T_1T_2TT_z)$ is a Clebsch-Gordon coefficient; $(l_j\|B^{(1)}\|l_j)$ is the reduced Gamow-Teller matrix element, sometimes denoted by $\int\sigma$,

$$(l_j\|B^{(1)}\|l_j)=[(j+1)(2j+1)/j]^{\frac{1}{2}}, \quad j=l+\tfrac{1}{2}$$
$$=[j(2j+1)/(j+1)]^{\frac{1}{2}}, \quad j=l-\tfrac{1}{2} \quad (43)$$

and S is a sum over fractional parentage coefficients,

$$S=\sum_{\alpha_1,I_1,T_1}(-1)^{I_1+T_1}W(jJjJ';J_11)W(\tfrac{1}{2}T_{\frac{1}{2}}T';T_11) \times(j^3\alpha TJ\|[j^2(\alpha_1T_1J_1)jTJ) \times(j^2(\alpha_1T_1J_1)jT'J'\|j^3\alpha'T'J'). \quad (44)$$

This result could readily be generalized to other three-particle configurations by expanding them also in terms of fractional parentage coefficients. Applicable selection rules are $\Delta T=0,\ \pm1,\ \Delta J=0,\ \pm1$ and configurations differ at most in the quantum numbers of one particle.

For beta decay (nonmirror transitions), the most important configuration mixing is that between particle states of the same l value. Small deviations from either pure jj or pure LS coupling can cause large changes in the beta decay rate. This can easily be understood, since an "unfavored" transition has zero probability for pure LS coupling and goes at a "favored" rate for pure jj coupling. Mirror transitions are not so sensitive to configuration mixing.

This relative sensitivity to mixing has been shown in calculations by Redlich.[32] Starting with a zero order jj scheme, and computing mixing in the $2s-3d$ shell, he finds that the mirror transition $_{10}Ne^{19}\rightarrow{}_9F^{19}$ goes at a rate greater by a factor of 230 than the nonmirror transition $_8O^{19}\rightarrow{}_9F^{19}$, in accord with experiment, and in disagreement with theory for pure jj states. A similarly important distinction between the mirror transition $_{22}Ti^{43}\rightarrow{}_{21}Sc^{43}$ and the nonmirror transition $_{21}Sc^{43}\rightarrow{}_{20}Ca^{43}$ has been pointed out by Peaslee.[33] The mirror transition goes at a superallowed rate, while the nonmirror transition is slower by a factor of 10 than for pure configurations. As Peaslee and Talmi point out, these unfavored decays are conclusive proof against pure jj configurations. They need not, however, imply that the true coupling scheme is much closer to LS. The particular decay $Sc^{43}\rightarrow Ca^{43}$ is discussed further in paper III, where it is shown that the calculated configuration mixing is sufficient to explain the order of magnitude of the observed slowdown. This calculated mixing results in a coupling scheme about equally removed from both jj and LS coupling.

VI. TRIPLET FORCES AND SCATTERING DATA

It is expected that the nuclear forces derived from scattering data will approximate those acting between the extra-core particles in nuclei. One objection to this point has been that the potentials adjusted to give correct scattering results do not lead to saturation of nuclear forces. However, it has been shown that if one uses repulsive core potentials a fit to both scattering and saturation requirements can be made.[34] For our purposes it is most important to fix the spin and exchange properties of the nuclear force since our approach is independent of the shape of the well. The spin and exchange properties of nuclear potentials adjusted to fit scattering are nearly independent of whether cores were used in the wells or not. We therefore argue that it is sufficient to examine potentials which fit nuclear scattering data but do not necessarily have repulsive cores.

Analysis of n-p and p-p scattering up to 100 Mev has been carried out by Christian and Hart,[35] Christian and Noyes,[36] and Jastrow.[37] They have concluded that:

(a) There is no appreciable central force in odd states.

(b) There is a weak odd-state tensor force present in addition to the even-state tensor force which is needed to give the deuteron quadrupole moment. In their analysis of p-p scattering at 32 Mev, Christian and Noyes fit the differential cross section with a singlet central force and an odd-state tensor force. Using wells of the form $V_0\exp(-r/0.75\times10^{-13}$ cm), these authors find that well depths

$$V_0\ (\text{singlet-central}) = -100\ \text{Mev},$$
$$V_0\ (\text{odd state tensor}) = \ \ 50\ \text{Mev},$$

gives a reasonably good fit to the data. Jastrow uses the same odd-state tensor potential as given above, but uses a singlet central potential with a repulsive core, in his discussion of p-p scattering. In addition he takes an even state tensor force with $V_0=127$ Mev and a triplet central force with $V_0=69$ Mev:

For an n-n or p-p system we need to consider only the following possibilities for the forces: (a) singlet-even, central; (b) triplet-odd, central; (c) triplet-odd, tensor; other forces being excluded by the Pauli principle. Force (a) is the largest and its range and depth are given above for an exponential well. Force (b) has not manifested itself in nucleon-nucleon scattering. Even if it were present between extra core particles its effect on level splitting would be diminished because, as shown in Figs. 1–3, the level depressions

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[37] R. Jastrow, Phys. Rev. **81**, 165 (1951).

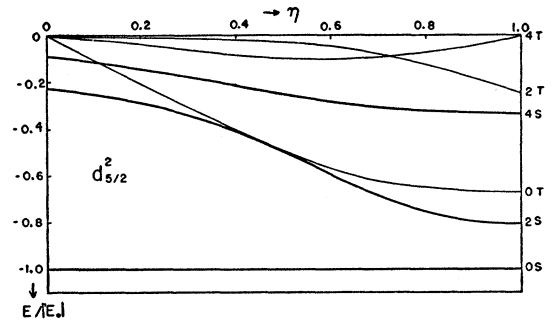

FIG. 4. Diagonal energy matrix elements for a $(d_{5/2})^2$ configuration ($J=0$, 2, 4) for a singlet central Gaussian well (S) and a tensor force with a Gaussian well (T). The tensor force well is half as deep as the singlet well. Elements plotted as in Fig. 1.

due to central triplet interactions are almost independent of J for assumed reasonable force ranges. Force (c), however, is about one-half as large as force (a) in nucleon-nucleon scattering. In order to show qualitatively the effect of such a tensor force on level structure, we plot in Fig. 4 the diagonal energy matrix elements of an attractive central singlet force and of an attractive odd-state tensor force for the configuration $(d_{5/2})^2$. (This calculation utilizes results derived by Talmi[38] for tensor forces.) Gaussian radial dependence is assumed for both potentials with singlet to tensor well depths in the ratio 2:1. The effect of the tensor force on the spin zero state is quite appreciable, as shown in Fig. 4. Because the odd-state tensor force is in fact repulsive,[19,37] the perturbation method of calculation with independent particle wave functions overestimates the effect of the tensor force relative to the singlet force. In addition calculations based on meson theory[19] indicate that the tensor well shape is quite different from the central well shape. Therefore, the results shown in Fig. 4 are at best qualitative. It seems probable that the assumption of pure singlet forces between identical particles is a good first approximation, but the magnitude of the corrections due to odd-state tensor forces may be important.

In the discussion of the Schrödinger equation in Sec. II it was important that one could factor the matrix in Eq. (7). This matrix can also be factored if V is a central triplet force. Then

$$\langle j_1 j_2 l_1 l_2 | V | LSl_1' l_2' \rangle = \langle j_1 j_2 l_1 l_2 | L1 l_1 l_2 \rangle \langle L1 l_1 l_2 | V | L1 l_1' l_2' \rangle \delta(s,1). \quad (45)$$

If V is a triplet tensor force $\mathbf{L}^2$ no longer commutes with V and the factorization cannot be carried out. However, even in this case it may be possible to write a tensor force as an effective central force.

If the semiempirical approach to nuclear spectra outlined here proves to be fruitful, it would appear best first to fix the singlet parameters of the potential by analyzing configurations of identical particles, then to analyze mixed neutron-proton configurations with only the triplet matrix elements regarded as unknowns. In paper III of this series, explicit application of these techniques will be made to the nuclei Ca^{42} and Ca^{43}.

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[38] I. Talmi, Phys. Rev. **89**, 1065 (1952).