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Nuclear Energy Level Fine Structure and Configuration Mixing

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The departure of shell-model states from independent-particle states is investigated by means of transformation methods developed in previous papers. The starting point in this paper is the many-body Schrödinger equation for the nucleus in which the potential energy is assumed to arise from strong short-range two-body interactions. As a consequence the shell-model wave function cannot be a solution of this equation, however it can be related to the actual nuclear wave function by a suitably chosen transformation operator. This operator preserves the energy and angular momentum of low-lying nuclear states; hence it is possible to examine the splitting of energy levels in the shell model space. For a closed shell plus two or three particles this is shown to originate primarily from perturbations due to two-body interactions between the particles outside the shell. The methods used also give information about the nuclear wave function and provide some justification for the use of configuration mixing in determining nuclear magnetic moments. It is noted that the successes of configuration mixing based on two-body forces provide evidence that two-body correlations dominate over many-body correlations for many properties of the nucleus.

I. INTRODUCTION

THE independent-particle form of the shell model has had many successes in classifying the properties of nuclei in their ground states. These successes have suggested that small deviations from the predictions of the model, such as the departure of magnetic moments from the Schmidt lines and the splitting of energy levels, could be explained by introducing small perturbing interactions which modify the independent particle picture. Suggested interactions have been of two kinds, (1) particle-to-particle interaction usually, but not always, introduced for a small number of particles outside a closed shell, and (2) particle-to-surface coupling associated with collective motion of a core of nucleons.

At first sight it seems strange that nucleon-nucleon interactions, which appear to be strong in scattering experiments, should give rise to almost independent-particle motion in the shell model. It is perhaps even more strange that particle-to-particle interactions should be regarded as perturbations on the motion, since the model appears to require weak interactions and yet the perturbing interactions appear to be strong. In previous papers[1–3] transformation methods have been developed which relate the independent-particle model wave function to the actual wave function. In this paper the methods will be extended so that we can examine deviations from this model. In particular, we shall show that in certain approximations these deviations can be obtained by introducing particle-to-particle interactions to perturb the states of certain particles in the shell model. It should be noted that the approximations which are made do not require a weak interaction between nucleons; in fact the method which we use, even in lowest approximation, takes into account many of the strong correlations which exist in the nuclear wave function.

The general problem we wish to investigate concerns the energy level fine structure and the wave function for a nucleus which contains a few nucleons more than a closed-shell nucleus. In the independent-particle model an inert core is assumed which corresponds to the closed-shell nucleus. Then each energy level of the model will have a degeneracy arising from the different ways the angular momenta of particles outside the core can be combined to give the same total angular momentum. This degeneracy will be removed if a perturbing interaction between these particles is introduced. Such a perturbing interaction will also lead to mixing of different configurations and to modifications in the magnetic moment predicted by the model. The effects of these perturbations on the properties of the independent-particle model have been investigated by several authors,[4,5] and it is clear that these refinements of the independent-particle model represent an important step towards obtaining a model wave function which describes correctly low-energy properties of the nucleus.

The aim of this paper is to try to understand, in terms of the actual nuclear wave function, why it is that this modified independent-particle model is able to predict correctly certain nuclear properties, namely energy level splitting and magnetic moments. The starting point is the many-particle Schrödinger equation for the actual nucleus and we assume strong interactions between nucleons. The method for studying energy

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4 Edmons, Flowers, and Elliott (to be published); M. H. L. Pryce, Proc. Phys. Soc. (London) **A65**, 773 (1952); D. Kurath, Phys. Rev. **91**, 1430 (1953); M. G. Redlich, Phys. Rev. **95**, 448 (1954); A. de-Shalit and M. Goldhaber, Phys. Rev. **92**, 1211 (1953); A. M. Lane (to be published).

5 K. Ford and C. A. Levinson (to be published).

levels consists in expressing the energy of a nucleus which contains $A+2$ nucleons in terms of the energy levels of a nucleus having $A+1$ nucleons together with an additional energy. It is then shown that this additional energy can be expressed as an interaction energy if the following device is used. This device is to consider not the energy eigenvalues of the Schrödinger equation for the actual nuclei but instead the energy eigenvalues of nuclear models which are constructed to have the same energy and angular momentum as the actual nucleus for the states in which we are interested. We find that the models correspond in a certain approximation to particles moving outside an inert core.

Corresponding to a nucleus consisting of $A+1$ nucleons, we obtain a model made up of a single particle moving outside a core. There are correction terms to the energy which are small when the nucleus containing A nucleons is doubly magic (e.g., Ca^{40}), but these correction terms can be included if the energy levels of the model are taken to be the empirical energy levels of the actual nucleus. For the nucleus of $A+2$ nucleons (e.g., Ca^{42}), the model consists of two particles outside the core and we find that these particles interact both directly and via the core. The interaction via the core is found to be much smaller than the direct interaction. The energy levels without these interactions can be related to the energy levels of the single-particle-plus-core model and they are degenerate due to the different ways in which the angular momenta of the single-particle states can be combined. In our model the direct interaction term is diagonal in the states of the model so gives directly an energy shift for each state; these energy shifts serve to split the otherwise degenerate levels. We show that this level splitting is equivalent to the splitting which would be obtained by the methods of configuration mixing using a direct particle-to-particle interaction for the two particles outside the core.

In Sec. II we shall develop the consequences of an application of a simplified form of the general techniques which we have developed which is similar to the approximation used in previous papers[1] on the saturation problem. In Sec. III we shall discuss the structure and interpretation of the wave function and then show the extent to which the results of Sec. II can be related to the approximation methods of Flowers,[4] Pryce,[4] Kurath,[4] Redlich,[4] and Ford and Levinson.[5] In Sec. IV, we shall return to the exact formulation of the problem and show how various correction terms arise, what their physical significance is, and the extent to which they can be included in a simple approximation scheme following closely the methods developed in Sec. II. In Sec. V we calculate the core coupling energy for two particles outside a core interacting via the core particles. In Sec. VI we summarize our results and make some concluding remarks.

Finally we note that throughout the paper we will make use of the effect of the exclusion principle on the behavior of the particles in the nucleus and in the nuclear models. Mathematically, therefore, we should work with occupation numbers for the various states other than in terms of particle coordinates, but this would lead to a more complicated formalism and would obscure the physical discussion. Fortunately there is such a close correspondence[2] between the two methods that it is unnecessary to depart from the formulation in terms of particles, provided that we remember that the exclusion principle operates both in actual states and in intermediate states; this is the procedure adopted in this paper. It must be pointed out, however, that in actual evaluation it is necessary to make a translation from particle description to state description together with the appropriate introduction of second quantized operators.

II. DERIVATION OF ENERGY LEVEL SHIFTS IN THE COHERENT APPROXIMATION

In this section we will consider the energy levels of nuclei containing $A+1$, $A+2$, and $A+3$ nucleons. We will use an approximation which is similar to that used in reference 1, and we shall also assume that the nucleus containing A nucleons is doubly magic. These simplifications are not all necessary but are made so that we can more clearly state the main features of the method. A more detailed analysis of the approximations will be made in Sec. IV where it will be shown that the general conclusions are independent of a number of the simplifying assumptions made in the present section.

A. General Method

The Schrödinger equation for A nucleons is assumed to be

$$[\sum_{i=1}^{A} T_i+\sum_{i<j}^{A} v_{ij}]\Psi(A)=E\Psi(A). \tag{1}$$

We investigate the energy E for various numbers of nucleons, not by comparing the Schrödinger equations for the actual nuclei but by comparing the Schrödinger equations for nuclear models which are chosen to have the same energies as the actual nuclei.

The model wave function satisfies a Schrödinger equation,[3]

$$[\sum_{i=1}^{A} T_i+V_0]\Phi(A)=E\Phi(A), \tag{2}$$

and the relation between the actual nuclear wave function $\Psi(A)$ and the model wave function $\Phi(A)$ is

$$\Psi(A)=M\Phi(A), \tag{3}$$

where

$$M=1+(E-\sum_{1}^{A} T_i)^{-1}\sum_{i<j}^{A}{}' v_{ij}M, \tag{4}$$

and

$$V_0(A)=(\Phi(A),\sum_{i<j} v_{ij}M\Phi(A)). \tag{5}$$

The model wave function can be taken to be a product of single-particle wave functions,[3] each corresponding to independent-particle motion. Thus the ground state of the model will be obtained by filling up the lowest independent-particle states according to the enclusion principle, giving a fully occupied "Fermi sea." We shall later require a model wave function which is the sum of products of independent-particle wave functions which is required to make it an eigenfunction of the total angular momentum, but for the present can be interpreted simply as a product.

To the approximation which we shall use in this section, the model operator M can be replaced by the operator F which is constructed from the following set of equations[2,3]:

$$F=1+\frac{1}{e}\sum_{i<j}^{A}{}'I_{ij}F_{ij}, \tag{6}$$

$$F_{ij}=1+\frac{1}{e}\sum_{m,n\neq i,j}{}'I_{mn}F_{mn}, \tag{7}$$

$$I_{mn}=\text{nondiagonal part of } t_{mn}, \tag{8}$$

$$t_{mn}=v_{mn}+v_{mn}\frac{1}{e}t_{mn}, \tag{9}$$

$$e=E-\sum_{i=1}^{A}T_i-\sum_{i<j}^{A}t_{cij}, \tag{10}$$

$$t_{cmn}=\text{diagonal part of } t_{mn}. \tag{11}$$

(The prime to the left of an operator indicates that matrix elements to the ground state are to be omitted.) The operators t_{ij} are the reaction matrices for the two-body interaction v_{ij} constructed in the nuclear medium; the t_{cij} are analogous to forward scattering amplitudes and the I_{ij} to incoherent scattering. The propagator "e^{-1}" describes the propagation of the nuclear particles in the nuclear medium with the effect of many-particle interactions appearing in the "potential" terms constructed from t_c.

The potential $V_0(A)$ for the model given by (5) can now be written

$$\begin{aligned} V_0(A)&=(\Phi(A),\sum_{i<j}^{A}v_{ij}F(\Phi)(A))\\ &=(\Phi(A),\sum_{i<j}^{A}t_{ij}F_{ij}\Phi(A)). \end{aligned} \tag{12}$$

In deriving (12), terms of order $1/A$ compared with the leading term are neglected.[2,3] This expression for the potential can be expanded by means of (7) as a series in the incoherent reaction matrices I_{ij}. The first two terms of this series are

$$V_0(A)=(\Phi(A),\sum_{i<j}^{A}t_{ij}\Phi(A)) +\sum_{i,j,k}\left(\Phi(A),I_{ij}\frac{1}{e}I_{jk}\frac{1}{e}I_{ki}\Phi(A)\right)+\cdots. \tag{13}$$

The justification for this series expansion comes essentially from the operation of the exclusion principle in the intermediate states of the last term of (13) and of higher order terms in the series; thus the intermediate states must lie above the occupied Fermi sea and this causes large energy denominators in the integrals. A more detailed and quantitative discussion of these "correction" terms will be made in Secs. IV and V. We will proceed now to consider a model for a nucleus containing $A+1$ nucleons and we make the approximation of taking for $V_0(A+1)$ only the leading term in the expansion corresponding to (13).

B. One Particle and Core

We consider the interaction of $A+1$ nucleons and separate the interaction energy into parts associated with one particle, with a core of A particles, and a remainder. The purpose of this separation is to facilitate comparison with the energies of $A+2$ and $A+3$ nucleon systems. As noted previously, we consider the energies of the corresponding nuclear models. The potential energy $V_0(A+1)$ for an $(A+1)$-nucleon system can be written down analogously to Eq. (5). It is (using primes to indicate quantities evaluated for the $A+1$ particle system)

$$V_0'(A+1)=V_0'(A)+V_1', \tag{14}$$

where $V_0'(A)$ is the potential energy of the A-particle core and V_1' is the potential energy of the $(A+1)$th particle in the field of the core.

$$V_1'=\sum_{i=1}^{A}(\Phi(A+1),t_{1i}'\Phi(A+1)). \tag{15}$$

The core potential energy V_0' is closely related to the core potential energy V_0, it is not exactly the same however for the following reasons: the addition of the $(A+1)$th particle changes the properties of the medium in which the core particles move. This change appears through a change in the core propagators e^{-1}, and also through a change in the nuclear volume. The latter effect is not explicitly present in the formalism we use, the former appears explicitly in the reaction matrices through the change

$$t_{ij}'=v_{ij}+v_{ij}\frac{1}{e'}t_{ij}', \tag{16}$$

where e' differs from e due to the presence of the $(A+1)$th particle,

$$e'=e+E_1-T_1-\sum_{i=1}^{A}t_{c1i}. \tag{17}$$

[A subscript "1" shown explicitly always denotes the $A(+1)$th particle.] We can see however that it is a reasonable approximation to neglect the difference between e' and e since the extra term $(E_1-T_1-\sum t_{c1i})$ gives zero when acting on the ground state of the

$(A+1)$th particle, thus any departures will only appear as higher order effects.

A second departure of the $A+1$ model from the A model arises from the change in radius of the model due to the presence of the extra particle. However this radius has to be chosen to minimize the total energy of the system so the core energy will not change to first order, and we will neglect the change here.

The aforementioned approximations will be investigated in more detail in Secs. IV and V where it will be shown that the neglected terms (1) appear to be small, and (2) they can be partly absorbed by taking empirical values for the single-particle energy levels. With these approximations, we find for the energy of the system of $A+1$ nucleons.

$$E(A+1)=E(A)+E_1, \tag{18}$$

where $E(A)$ is the energy of a nucleus containing A nucleons and E_1 is given by

$$E_1=V_1+(\Phi(A+1),\, T_1\Phi(A+1)), \tag{19}$$

$$V_1=\sum_{i=1}^{A}(\Phi(A+1),\, t_{1i}\Phi(A+1)). \tag{20}$$

We note that these formulas have been obtained by neglecting the following effects: (1) direct 3-particle (and higher order) couplings including third-order core polarization, (2) the change in the core propagators, (3) the change in the nuclear volume.

C. Two Particles and Core

We next consider a system of $A+2$ nucleons, which we will represent by two particles outside a core of A particles. We will try to express the energy of this system in a form equal to the sum of the energies of two particles moving without mutual interaction in the field of the core together with interaction terms which couple the two particles together and remove the degeneracy of energy levels of a system containing two noninteracting particles. The value of this form for the energy is that it facilitates comparison with the energy levels given by a single particle and a core.

The potential energy for the system of core-plus-two particles neglecting three- (or more) body correlations and also neglecting "$1/A$" terms, takes the form

$$V_0(A+2)=V_0''(A)+V_1''+V_2''+V_{12}'', \tag{21}$$

where

$$V_0''(A)=\sum_{i<j}^{A}(\Phi(A+2),\, t_{cij}''\Phi(A+2)), \tag{22}$$

and V_1'', V_2'' are given by expressions analogous to (20) but with the modified two-body reaction matrix t_{ij}''. The term V_{12}'' is defined by

$$V_{12}''=(\Phi(A+2),\, t_{12}''\Phi(A+2)). \tag{23}$$

The double primes indicate that the expressions are to be evaluated to include the effects of changes in core volume and core propagators such as those discussed for the system—one particle plus core, and the analogous changes associated with the extra two particles. These effects we will again neglect for the present and limit ourselves to noting the reasons why this is permissible: (1) the terms appear to be small as they involve three-body (or higher order) correlations which are reduced in magnitude by the operation of the exclusion principle and by other properties related to the closed-shell core, (2) some of the terms neglected here will be absorbed in the single-particle energy levels if empirical values of these are used when applying the methods, (3) the small energy shift coming from the core will not affect the relative spacing of the levels we consider. Hence we can drop the primes on the terms in (21), which means that the core energy becomes the energy of the Ath nucleus.

In this approximation the total energy of the system is

$$E(A+2)=E_1+E_2+V_{12}, \tag{24}$$

where the extra term in the energy is

$$V_{12}=(\Phi(A+2),\, t_{12}\Phi(A+2)). \tag{25}$$

This term will depend on the state of the two extra-core particles and thus will remove the degeneracy of the energy levels of the two particles moving without direct interaction in the field of the core.

To make this result more explicit, we next consider the form of the model wave functions $\Phi(A+2)$. Up to this point we have assumed this to be a product function; this, however, is not quite sufficient since it is not possible to form the correct functions of angular momentum in this way. We must instead choose the wave function so that it corresponds to a definite total angular momentum. We will for simplicity take the Ath nucleus to be a closed shell and assume that the model wave function $\Phi(A+2)$ is a product

$$\Phi(A+2)=\Phi(A)\phi(1,2), \tag{26}$$

where $\Phi(A)$ is a wave function for a zero angular momentum core. The outer two particles have a wave function which is the appropriate combination of shell-model wave functions to make up an eigenfunction of the total angular momentum. If for example, the state being filled by particles 1 and 2 have single-particle angular momentum j,j', then the wave function $\phi(1,2)$ will be (neglecting antisymmetrization)

$$\phi_J{}^m(1,2)=\sum_{m_1m_2} C(j,\, j',\, m_1,\, m_2|J,\, j,\, j',\, m)\times\phi_j{}^{m_1}(1)\phi_{j'}{}^{m_2}(2). \tag{27}$$

In this representation, the energy shifts given by V_{12} can be labeled by the total angular momentum, and are

$$\Delta E_{12}(J)=(\phi_J{}^m(1,2),\, t_{12}\phi_J{}^m(1,2)). \tag{28}$$

In Sec. IV we will consider further the interpretation of the energy shifts $\Delta E(J)$.

D. Three Particles and Core

The same techniques as those used in the preceding paragraphs can be applied to a nucleus containing $(A+3)$ nucleons. If the approximations associated with the closed-shell nucleus of A nucleons are still valid for three particles additional to the core, it will still be possible to neglect coupling via the core. However there will now be three-body correlations involving only the three outer particles and these must be taken into account in the fine structure related to the state of these three particles. Then we have to include part of the last term in (13) and the energy $E(A+3)$ will be given by

$$E(A+3)=E(A)+E_1+E_2+E_3+\langle(t_{c12}+t_{c23}+t_{c31})\rangle + \left\langle\left(I_{12}\frac{1}{e}I_{23}\frac{1}{e}I_{31}\right)+\text{permutations}\right\rangle+\cdots. \quad (29)$$

The energy shift $\Delta E_s'$ for a given state $\phi_s(1,2,3)$ of the outer three particles is given by the last two terms of (29), where

$$\langle t_{c12}\rangle=(\phi_s(1,2,3),\, t_{12}\phi_s(1,2,3)), \quad (30)$$

$$\left\langle I_{12}\frac{1}{e}I_{23}\frac{1}{e}I_{31}\right\rangle = \left(\phi_s(1,2,3),\, I_{12}\frac{1}{e}I_{23}\frac{1}{e}I_{31}\phi_s(1,2,3)\right). \quad (31)$$

In the next section we will consider the relation between this energy shift and that given by (28) for two particles outside a core.

III. INTERPRETATION OF THE WAVE FUNCTION; RELATION TO CONFIGURATION MIXING

In this section as in Sec. II we shall restrict ourselves to consideration of the simplified predictions of the coherent model. We shall first consider the nuclear wave function and its relation to the shell-model wave function. We will then consider the energy shifts derived in the previous section and show that these are equivalent to the energy shifts which would be the result of introducing a perturbing potential into the shell-model states.

A. Structure and Interpretation of the Wave Function

For simplicity we will consider a nucleus corresponding to one particle and a closed-shell core in the shell-model picture, since the problems we wish to discuss are all present in such a nucleus. In particular, we will discuss the relation between the nuclear wave function Ψ and the shell-model wave function Φ, and see how far the differences between the wave functions affect our interpretation of the model.

The most profound difficulty which prevents identification of Ψ with the independent-particle wave function Φ arises from the existence of strong short-range interactions between nucleons. As a consequence of these strong interactions a product wave function like Φ is nowhere near to being an approximate solution of the Schrödinger equation for the nucleus. Conversely, even for a closed-shell core it is not possible to write Ψ approximately as a product of a single-particle wave function and a core wave function and still have an approximate solution of the Schrödinger equation. Thus, if we write

$$\Psi(A+1)\doteqdot F_1\psi(1)\Psi(A), \quad (32)$$

we can obtain an approximate solution of the Schrödinger equation only by taking F_1 to be an operator of the form [see Eq. (6)]:

$$F_1=1+\frac{1}{e}\sum_{j=1}^{A} I_{1j}+\cdots. \quad (33)$$

Thus F_1 introduces correlations between the nucleon 1 and all the core nucleons through the incoherent operators I_{1j}. This means that $F_1\psi(1)$ appears to depend strongly on the state of the core nucleons, and cannot be simply regarded as a single-particle state. This contrasts with the shell-model interpretation that a particle outside a closed shell appears to move simply in a single-particle state (for the closed shells O^{16} and Ca^{40}).

The complicated relation between the extra nucleon and the core can be described as a polarizing effect on the core due to the strong short-range forces between the extra nucleon and the core nucleons. Therefore the nucleus would appear to consist not only of a symmetrical core plus a single-particle state but also will contain appreciable admixtures of higher excited states of the core and single particle.

We shall now attempt to relate this result to the predictions of the shell model for a "closed shell plus one" nucleus. There is no difficulty about the assignment of total angular momentum and parity, as these are invariant under the F_1 transformation. The principal difficulty of interpreting our result that the presence of the extra nucleon must lead to appreciable mixing between the single-particle and core states arises in connection with magnetic moments. If we are to avoid qualitative disagreement with experiment we must show at least that this mixing need not appreciably affect the magnetic moment.

We note first an essential difference between the Hamiltonian operator from which we deduced the mixing of states and the magnetic moment operator which seems to indicate the purity of states. The Hamiltonian is a very singular operator and slight departures from the true wave function will cause very large deviations in the energy; in contrast, the magnetic moment operator is not singular and will be relatively insensitive to the use of an approximate wave function.

Since the magnetic moment does not depend on these very fine details of the wave function, it is sufficient to consider why it may be little affected by mixing of shell-model states, rather than trying to look at departures from the actual nuclear wave function. In the framework of the shell model the following argument has been pointed out to us by C. A. Levinson. An examination of the spins and parities of the core states of Ca^{40}, for example, shows that even if the actual wave function for 41 nucleons contains appreciable admixtures of excited core states, there will be no first order effects on the magnetic moment. This can be contrasted with the situation for the doubly magic Pb^{208} plus one nucleon where core admixtures are not only appreciable, but as has been shown by Blyn-Stoyle and Perks,[6] the mixture leads to first order corrections to the independent-particle magnetic moment, bringing it to a position approximately midway between the Schmidt lines. We note also that the foregoing argument is probably reinforced by various other special effects, including (for Ca^{40} or O^{16} plus one nucleon) the greater orthogonality of core and particle states and also the large denominators which reduce the influence of distant excited states.

The foregoing qualitative discussion indicates that it is not unreasonable to assume that the mixing of states which is required for the actual nuclear wave function to satisfy the Schrödinger equation may, for certain nuclei, have only a slight influence on the independent-particle magnetic moment. However, in using the methods discussed in the paper to compare neighboring nuclei, it would be very desirable to reinforce the aforementioned arguments that the magnetic moment of Ca^{41} should lie on the Schmidt lines by direct observation. Then one could use the empirical knowledge of the magnetic moment somewhat analogously to the empirical energy levels for predicting the properties of Ca^{43}.

We shall now proceed to consider the relation of the results of Sec. II to the methods of configuration mixing, and in our discussion we will assume that the single-particle states outside the core can be regarded as nucleon states to the extent that they have the same magnetic moments. *This does not imply that the particle states and nucleon states are not quite different with regard to other observables.*

B. Relation to Configuration Mixing

In Sec. II we have derived formulas for the energies of nuclei containing $A+1$, $A+2$, and $A+3$ nucleons in an approximation depending to some extent for its validity on the nucleus with A nucleons being a closed shell. These approximations have indicated that only the direct particle-to-particle interaction need be considered for the particles outside the core corresponding to the closed-shell nucleus. This particle-to-particle interaction leads to energy shifts in comparing the states of two and three particles outside the core with the corresponding one-particle states. These energy shifts have been derived in a form which we will now proceed to show is equivalent to the energy shift coming from introducing a perturbing potential into shell-model states.

We consider first the energy shift due to the mutual interactions for two particles moving in the average field of a core. The two-particle Schrödinger equation is

$$(H_0+v_{12})\psi=E\psi, \tag{34}$$

where

$$H_0=\sum_{i=1}^{2} T_i+\sum_{i=1}^{2} V_i, \tag{35}$$

and V_i denotes the effective core fields at the points x_i for the model. We compare this with the equation with v_{12} absent,

$$(H_0+\Delta E)\psi_0=E\psi_0. \tag{36}$$

Provided that

$$\Delta E=(\psi_0,v_{12}\psi), \tag{37}$$

the solutions are related by

$$\psi=\Omega\psi_0, \tag{38}$$

where the operator Ω is defined by the equation

$$\Omega=1+(E-H_0)^{-1}{}'v_{12}\Omega. \tag{39}$$

The energy shift ΔE is given by

$$\Delta E=(\psi_0,v_{12}\Omega\psi_0) \tag{40}$$

$$=(\psi_0,t\psi_0), \tag{41}$$

where

$$t=v_{12}+v_{12}(E-H_0)^{-1}t. \tag{42}$$

Consequently the energy shift ΔE given by (41) is the same as the energy shift ΔE_s given by (28) provided that (1) the potentials V_i in (35), are equal to the single-particle potentials given by (20), and (2) the shell model wave function ψ_0 in (41) is taken to be the same as the model wave function $\phi_s(1,2)$ in (28). With these potentials and wave functions we see that the transformation methods of Sec. II lead to an energy shift equal to that given by introducing a two-particle interaction into the shell model for two particles outside a closed shell.

In addition, the wave function ψ given by Eq. (38) can be explicitly constructed by solving Eq. (39) for the transformation operator Ω. This solution will give the admixtures of other states than the lowest independent-particle state on the wave function and can be used to determine for example the expectation value of the magnetic moment.

We consider next the formal derivation of the energy shift coming from the mutual interactions of three

[6] R. J. Blyn-Stoyle and M. A. Perks, Proc. Phys. Soc. (London) **A67**, 885 (1954).

particles moving in the average field of a core. For this system we take

$$H_0=\sum_{i=1}^{3} T_i+\sum_{i=1}^{3} V_i, \tag{43}$$

and we compare the equations

$$(H_0+\sum_{i<j}^{3} v_{ij})\psi=E\psi. \tag{44}$$

$$(H_0+\Delta E)\psi_0=E\psi_0. \tag{45}$$

Then the energy shift ΔE will be given correctly if the wave functions ψ and ψ_0 are related by

$$\psi=\Omega\psi_0=\{1+(E-H_0)^{-1}\sum v_{ij}\Omega\}\psi_0, \tag{46}$$

where Ω is defined by the set of equations

$$\Omega=1+(E-H_0)^{-1}\sum_{i<j}^{3} t_{ij}\Omega_{ij}, \tag{47}$$

$$\Omega_{ij}=1+(E-H_0)^{-1}\sum_{lm\neq ij} t_{lm}\Omega_{lm}, \tag{48}$$

$$t_{ij}=v_{ij}+v_{ij}(E-H_0)^{-1}t_{ij}, \tag{49}$$

and the energy shift is

$$\Delta E=(\psi_0,\sum_{i<j}^{3} v_{ij}\Omega\psi_0), \tag{50}$$

$$=(\psi_0,\sum_{i<j}^{3} t_{ij}\Omega_{ij}\psi_0), \tag{51}$$

$$=(\psi_0,\sum_{i<j}^{3} t_{ij}\psi_0)$$

$$+\sum_{i\neq j\neq k}^{3}\left(\psi_0, I_{ij}\frac{1}{(E-H_0)}I_{jk}\frac{1}{(E-H_0)}I_{ki}\psi_0\right)+\cdots, \tag{52}$$

where I_{ij} is as before the nondiagonal part of t_{ij}.

We see at once that this expression for ΔE corresponds to the series whose first two terms are equal to $\Delta E_s'$ given by the last two terms in (29). The leading term in (57) is simply a sum

$$\Delta E_s'''=(\phi_s(1,2,3),\ \{t_{12}+t_{23}+t_{31}\}\phi_s(1,2,3)). \tag{53}$$

This term can be identified by comparing it with the diagonal elements in the two-particle-plus-core system. The remaining terms in (52) or in (29) can be interpreted as arising from three-particle correlations, they are correction terms to the energy shifts for two-particle correlations which can be obtained from (53). Another way of stating this is that (53) allows for the same amount of configuration mixing in the three-particle system as in the two-particle system. An estimate of the relative importance of the three-particle correlations has been made by C. A. Levinson, who finds that taking diagonal matrix elements only in Ca^{43} [this corresponds to (53)] would lead to an error about 20 percent of the total shift predicted when all matrix elements of the interaction are considered. This can be regarded as some evidence in favor of the conclusion that two-body correlations dominate over the higher order effects.

IV. CORRECTIONS TO THE ENERGY SHIFTS OF THE COHERENT MODEL

In this section we shall make a more careful examination of the energy level fine structure in which we do not omit the various correction terms which were neglected in Sec. II. These corrections are of three types; (1) those which vanish as $1/A$ compared with the main terms in the energy, (2) those which arise from changes in the propagators as particles are added, and (3) those which arise from high-order correction terms in the energy expressions. The latter are the manifestation of third (and higher) order incoherent interaction among three (or more) particles and give the largest perturbation on the energy levels of the coherent model. We shall consider the cases of one, two, and three particles plus core and show how the corrections arise and to what extent they require changes in the formalism and interpretation of the results of Secs. II and III.

A. One Particle and Core

We consider first the interaction of a single particle with a core of A particles. The total potential energy of the system including the correction terms previously omitted is, to third order in the incoherent scattering matrices,

$$V_0(A+1)=(\Phi,\sum_{i<j} t_{cij}\Phi)+(\Phi,\sum_{i<j} I_{ij}F_{ij}\Phi)$$

$$-\left(\Phi,\sum_{i<j} I_{ij}\frac{1}{e}t_{cij}\frac{1}{e}I_{ij}\Phi\right), \tag{54}$$

where the first term is the "coherent" energy previously discussed in Secs. II and III, the second term arises from higher order incoherent terms in the energy, and the last term is a "$1/A$" correction to the total energy which can be neglected in determining the properties of the entire system but not necessarily in the fine structure of the energy levels. The sum over i, j is over all of the $A+1$ nucleons.

For convenience in comparing this expression for the energy of core plus one particle, we next break the sum up into two terms, one involving a sum only over the core particles and the remainder involving the extra particle. To make the separation possible, we take only the first nonvanishing term in the expansion of the second term of Eq. (54), i.e., we use

$$(\Phi,\sum_{i<j} I_{ij}F_{ij}\Phi)\doteq\left(\Phi,\sum_{ijk} I_{ij}\frac{1}{e}I_{jk}\frac{1}{e}I_{ki}\Phi\right), \tag{55}$$

where the restriction is to be imposed on this sum that $i\neq j\neq k$. It is now easy to carry out the desired separation into core and particle energies; we find for the core potential energy:

$$V_0(A)=(\Phi,\sum t_{cij}\Phi)+\sum\left(\Phi, I_{ij}\frac{1}{e}I_{jk}\frac{1}{e}I_{ki}\Phi\right)$$
$$-\sum_{i,j,\mathrm{core}}\left(\Phi, I_{ij}\frac{1}{e}t_{cij}\frac{1}{e}I_{ij}\Phi\right), \quad (56)$$

and for the potential energy of the extra particle in the field of the core

$$V_0(1)=(\Phi, \sum_{i,\mathrm{core}} t_{c1i}\Phi)$$
$$+\sum_{ij,\mathrm{core}}\left[\left(\Phi, I_{1i}\frac{1}{e}I_{ij}\frac{1}{e}I_{j1}\Phi\right)+\text{permutations}\right]$$
$$-\sum_{i,\mathrm{core}}\left(\Phi, I_{1i}\frac{1}{e}t_{c1i}\frac{1}{e}I_{1i}\Phi\right). \quad (57)$$

In this expression for the energy of the extra particle, the first term represents the potential energy of the particle in the field of the core which is associated with coherent propagation without core excitation via the I_{ij} couplings; the second term involving the third-order combination of the I_{ij} couples the motion of particle 1 incoherently to the core with resulting excitation of the i,j pair in the core. These results are correct to all orders in the dominant coherent interaction and to third order in the incoherent interaction terms.

We note that the effects of all such correction terms as those we have considered here are automatically included if we use the empirical energy levels of the single particle plus core. Consequently the change in the formalism of Secs. II and III which arises from proper inclusion of the corrections gives rise to complication only if we wish to make a precise determination of the expected single-particle energy levels from first principles.

B. Two Particles and Core

We next consider two particles and a core and try to bring the expressions for the energy to a form in which the energies of the separate noninteracting nucleons in the field of the core is separated from the interaction terms coupling the two particles together. The latter terms act to remove the degeneracy in the energy levels of the noninteracting particles. This form for the total energy is particularly useful since it allows us to make use of the empirical observations of the one-particle levels to determine the principal contribution to the level positions and spacing.

As in the case of one extra-core particle, we start with the result for the potential energy:

$$V_0(A+2)=(\Phi', \sum_{i,j} t_{cij}'\Phi')+(\Phi', \sum_{i,j}{}' I_{ij}'F_{ij}'\Phi')$$
$$-\left(\Phi', \sum_{ij}{}' I_{ij}'\frac{1}{e}t_{cij}'\frac{1}{e}I_{ij}'\Phi'\right), \quad (58)$$

where the primes indicate that in general (1) the wave function has changed as, for example, by a change in the nuclear volume and shape; (2) the primed sum over i, j includes the additional particle "2"; (3) the operators t_c' and I_{ij}' are changed from the similar operators in the one-particle case. Again we consider the last two terms as perturbations and replace in these the primed t' and I_{ij}' by the unprimed since the difference is of higher order than we are considering.

We evaluate first the principal term in the potential energy, the coherent potential energy. It is apparent in this that part of the energy shift compared with the one-particle case can come from the change in the wave function $\Phi\rightarrow\Phi'$. This effect, arising from volume and shape changes of the nucleus, has been qualitatively discussed in Sec. II. This effect is not obviously small since it can affect all of the A core particles. It also is extremely complicated in origin since it involves an understanding of the precise manner in which the core adjusts itself to the presence of the extra particle. Part of the net effect corresponds to a core polarization, but here we will examine only the effect of the change in the nuclear volume. Let us first consider the somewhat unphysical effect of fixing the nuclear volume of the A particles and adding the $(A+1)$th. The effect on the energy of this $1/A$ change on the volume per particle can be estimated from a knowledge of the rate of change of the total energy of the system with volume. It will be only a second order effect since the system saturates at the minimum of the energy *versus* density curve, and can be estimated using the saturation results of reference 1. For a nucleus of 40 nucleons it gives an energy shift for the entire nucleus which is approximately $\frac{1}{8}$ Mev. Thus the effect in this approximation is very small, furthermore it will result only in a uniform shift in all energy levels (at least those over a narrow region) and does not affect the relative level spacing which is our primary interest here.

The aforementioned argument can be reinforced if we observe the way in which the actual physical nucleus will react to the presence of an extra particle. The saturating character of the forces means that the core will change its density to keep the energy per particle constant; since the addition of the additional particle at fixed volume is equivalent to an increase in the density, the core will adjust itself to a radius sufficiently larger to compensate for the slight change in energy resulting from higher density. Consequently the already small (uniform) shift in the levels can be expected to be reduced further and can be neglected. Thus we shall in the following neglect the effects of changes in the core energy, but it will be noted that this is in no way the same as neglecting the interaction between the extra particle and the core.

In this approximation we need examine only the effects on the energy of the changed propagators in the

t_{ij}. We break the sum up into 3 terms,

$$\sum_{ij}{}' t_{cij}' = \sum_{ij,\text{core}} t_{cij}' + \sum_{i,\text{core}} (t_{i1}' + t_{i2}') + t_{12}'. \quad (59)$$

The first term in this sum is the very large core potential energy; we wish to see to what extent the core energy is affected by the simultaneous presence of two nucleons. A shift in the core energy arising from this effect would be equivalent to an additional particle-particle interaction arising from coupling through the core.

The change in t resulting from the presence of two particles is due to the change in the propagator; we have

$$t_{ij}' = v_{ij} + v_{ij}\frac{1}{e'}t_{ij}', \quad (60)$$

where

$$e' = e + e_1 + e_2 + e_{12}, \quad (61)$$

and

$$e = E_A - \sum_{i,\text{core}} T_i - \sum_{i<j,\text{core}} tc_{ij}, \quad (62)$$

$$e_1 = E_1 - T_1 - \sum_{i,\text{ core}} t_{ci1}, \quad (63)$$

$$e_{12} = E_{12} - t_{c12}. \quad (64)$$

We wish to relate this typical term in the core energy to the energy in the presence of only one particle for which a typical term is

$$t_{ij} = v_{ij} + v_{ij}\frac{1}{e+e_1}t_{ij}. \quad (65)$$

To develop the relation between t_{ij}' and t_{ij}, we note some simple properties of the propagators; first, we have written the total energy E of the system as a sum of energies

$$E = E_A + E_1 + E_2 + E_{12}, \quad (66)$$

where E_A is the energy of the undisturbed core, E_1 and E_2 are the total energies of particles 1 and 2, and E_{12} is the energy resulting from the interaction between particles 1 and 2. A consequence of this separation is that the term e_{12} will always be zero since the operators v_{ij} and t_{ij} act only on core particles, and t_{c12} which commutes with these operators gives E_{12} acting on the wave function of the system. We treat the remaining correction by expanding the propagator about the point where e_2 is zero, i.e., we use the identity

$$\frac{1}{e+e_1+e_2} = \frac{1}{e+e_1} - \frac{1}{e+e_1}e_2\frac{1}{e+e_1} + \cdots. \quad (67)$$

This separation is not yet complete since the second term contains in part the energy of the core in the presence of particle 2. A further expansion gives

$$\frac{1}{e+e_1+e_2} = \frac{1}{e+e_1} + \frac{1}{e+e_2} - \frac{1}{e} + \left\{\frac{1}{e}e_1\frac{1}{e}e_2\frac{1}{e} + (1\to 2)\right\}. \quad (68)$$

The last two terms then give rise to the lowest order shift in the core energy due to the simultaneous presence of two particles. Consequently we can write for the change in the core energy:

$$\Delta E = \sum_{ij}\left[v_{ij}\frac{1}{e}\left(e_2\frac{1}{e}e_1 + e_1\frac{1}{e}e_2\right)\frac{1}{e}t_{ij}\right]. \quad (69)$$

This term is of fourth order and therefore of higher order than the terms we have retained elsewhere; we still need to examine its structure, however, to show that the presence of the double sum over $A^2/2$ pairs of core particles does not result in an appreciable contribution to the energy.

The term e_1 can be written

$$e_1 = E_1 - T_1 - \sum_{k,\text{core}} t_{ck1}, \quad (70)$$

which vanished acting on the ground state. In addition, since v_{ij} and t_{ij} depend only on core particles, T_i commutes with them and has the value T_1^0 acting on the ground state. Thus we can write

$$e_1 = \sum_{k,\text{core}} [t_{ck1}^0 - t_{ck1}], \quad (71)$$

whereby t_{ij}^0 we mean the value that t_{ij} has acting on the ground state. We now observe that in a term such as

$$\sum_{ij} v_{ij}\frac{1}{e}\sum_k (t_{ck1}^0 - t_{ck1}), \quad (72)$$

the contribution will be zero unless i or j is equal to k, the operator t_{k1} otherwise commuting with v_{ij} to act on the ground state. Consequently the sum over i, j, k reduces to (we consider only one of the typical terms)

$$\sum_{ij} v_{ij}\frac{1}{e}(t_{ci2}^0 - t_{ci2})\frac{1}{e}(t_{ci1}^0 - t_{ci1})\frac{1}{e}t_{ij}. \quad (73)$$

The dependence on the total number of particles can now be estimated; each term in v_{ij} or t_{ij} introduces a factor $1/v$ (v the nuclear volume); the summation over a single intermediate state implied in the matrix product gives, when replaced by an integration, a factor of v. The double sum over core states gives a factor of A^2. Thus the dependence on A is of the form

$$A^2(v/v^4) \sim A^2/A^3 = 1/A, \quad (74)$$

which is the same as that of the principal energy splitting term $t_{c12} \sim 1/A$. This additional correction term is of fourth order and in addition is small because of the cancellations occurring between t_c and t_c^0. Consequently, to the order we are calculating, we can neglect the change in the energy of the core due to the simultaneous presence of the two extra nucleons. The same arguments allow us to replace t_{ci1}' by t_{ci1} in Eq. (59).

The remaining smaller correction terms can be easily broken up into terms referring to the core and particle-core and particle-particle interactions. Finally, combining the resulting terms which can be identified with the single-particle energies and calling them E_1 and E_2, we obtain for the energy of the A-plus-2-particle system

$$E(A+2)-E_1-E_2=t_{c12}-\left(\Phi,\,I_{12}\frac{1}{e}t_{c12}\frac{1}{e}I_{12}\Phi\right)+\sum_{i,\mathrm{core}}\left(\Phi,\left[I_{12}\frac{1}{e}I_{i2}\frac{1}{e}I_{i1}+\text{permutations}\right]\Phi\right),\qquad(75)$$

which is correct to third order in the I_{ij} operators. The relative simplicity of this result is a consequence, of course, of the absorption of many of the complicated energy corrections into the single-particle energies. Each of the terms removing the degeneracy of the single-particle levels, i.e., the last three terms of Eq. (75), has the same dependence on the total number of particles, i.e., a $1/A$ dependence.

This result differs from the results of Sec. II almost entirely from the presence of the last term, the second term having a negligible effect on the energy. The last term is a coupling of particles 1 and 2 through the ith core particle, summed over the core. The resulting interaction will depend on the coordinates of particles 1 and 2 and in general be largest when the particles are close together. The interaction will also depend on the state of the core and of the extra-core particles. It can be expected to have a nonlocal character when expressed in coordinate space; in addition its spin and isotopic spin dependence will bear no simple relation to the two-body potential v_{12} which generates t_{12}. This term is in some sense analogous to the particle-particle coupling which, according to Bohr and Mottelson, is the result of interaction with the surface of the core. We shall in the next section carry out an explicit evaluation of the correction to the two-body interaction which arises from this type of particle-core coupling and show that its effect is very small.

We would now like to point out that insofar as the effects of a particle-core particle coupling can be replaced by those of an equivalent short-range two-body interaction, the net effect of the direct and core-particle interactions can be represented by a phenomenological 2-body potential. Consequently, a study of the energy-level fine structure which neglects the particle-core couplings and attempts to fit the experimental data in such an approximation will lead to an effective two-body potential which in general will differ (although possibly only slightly) from the free two-particle interaction. Stating this more recisely, if we combine all of the level splitting terms into one effective reaction matrix defined by

$$G_{c12}=t_{c12}-\left(\Phi,\,I_{12}\frac{1}{e}t_{c12}\frac{1}{e}I_{12}\Phi\right)+\sum_{i,\mathrm{core}}\left(\Phi,\,I_{12}\frac{1}{e}I_{2i}\frac{1}{e}I_{1i}+\text{permutations}\right),\qquad(76)$$

then we can expect to find an approximately local "potential" g_{12} which will generate G_{12} according to the equation

$$G_{12}=g_{12}+g_{12}\frac{1}{e}G_{12}.\qquad(77)$$

The effective potential g_{12} will then be expected to differ somewhat from v_{12}. As we shall see in the next paragraphs, the phenomenological introduction of such an effective potential to describe the net particle-particle coupling including core effects can be expected to lead to a considerable improvement in accuracy if an attempt is made to predict the energy levels of a three-particle-plus-core system making use of empirical knowledge of the core-plus-one-particle and core-plus-two-particle systems.

C. Three Particles and Core

The same techniques as those used in the preceding paragraphs can be used for the case of three extra-core particles. We shall not go through the details of the analysis since these very closely parallel the analysis of the one- and two-particle systems. The result is

$$E(A+3)=E_1+E_2+E_3+G_{c12}+G_{c13}+G_{c23}+\left(\Phi,\,I_{12}\frac{1}{e}I_{23}\frac{1}{e}I_{31}\Phi\right)+\text{permutations},\qquad(78)$$

where the operators G_{cij} are defined by equations similar to Eq. (77). Thus, if we proceed phenomenologically from the case of two extra-core particles to the three extra-core particle case, the only new terms which arise are the last of Eq. (78), those arising from incoherent coupling of the three extra-core particles. The effect of these new terms is relatively small, although not negligible, as has been pointed out in Secs. II and III. It also should be pointed out that these final correction terms in the three-particle case can be computed with sufficient accuracy from the "effective potentials" g_{ij} defined by Eqs. (76) and (77). Thus one can expect to get a high degree of accuracy in the final results if one not only uses the empirical energy levels in the one particle case but also determines the effective particle-particle coupling by examining the fine structure of the two-particle case. The errors in this procedure are quite small since they are of higher order than the leading (and dominant) terms we have considered.

V. EVALUATION OF PARTICLE-CORE COUPLING EFFECTS

We shall in this section determine the size of the correction terms defined in the previous section which originate in particle-core coupling. These terms are

$$\Delta E_{pc}=\sum_{i,\mathrm{core}}\left[\left(\Phi,\,I_{12}\frac{1}{e}I_{2i}\frac{1}{e}I_{1i}\Phi\right)+5\text{ permutations of }(12i)\right].\qquad(79)$$

In evaluating this correction to the interaction of particles 1, 2, we shall make certain simplifying assumptions which will allow us to make a fairly straight forward evaluation of the effect. These are: (1) the exclusion principle will be included only by requring that transitions occur to states above the Fermi limit and that particles interact only in allowed states of relative orbital momentum, (2) the wave function will be approximated by a degenerate Fermi gas of plane waves rather than the actual independent particle states of a spherical nucleus, and (3) the incoherence operators I_{ij} will be approximated by the incoherent Born approximation scattering from a Yukawa well with Serber exchange mixture. We will also approximate to the result by evaluating only a typical one of the six correction terms in Eq. (79) and multiplying the result by six.

We consider the first term of Eq. (79), written now explicitly in momentum space.

$$\Delta E_{pc}=\sum_{i,\text{core}}\sum_{\mathbf{k}_1'}\sum_{\mathbf{k}_2'}\sum_{\mathbf{k}_3'}(\mathbf{k}_1\mathbf{k}_2|I|\mathbf{k}_1'\mathbf{k}_2')\times\frac{1}{E_1+E_2-E_1'-E_2'}(\mathbf{k}_2'\mathbf{k}_i|I|\mathbf{k}_2\mathbf{k}_i')\times\frac{1}{E_1+E_i-E_1'-E_i'}(\mathbf{k}_1'\mathbf{k}_i'|I|\mathbf{k}_1\mathbf{k}_i).\quad(80)$$

For the I's, in the approximation described above, we have

$$(\mathbf{k}_3\mathbf{k}_4|I|\mathbf{k}_1\mathbf{k}_2)=\delta(\mathbf{k}_1+\mathbf{k}_2,\,\mathbf{k}_3+\mathbf{k}_4)\left(\frac{2\pi V_0}{\mu v}\right)\times[f(\mathbf{k}_1+\mathbf{k}_3)+f(\mathbf{k}_1+\mathbf{k}_2)],\quad(81)$$

where

$$f(\mathbf{x})=(\mu^2+x^2)^{-1},\quad(82)$$

and V_0 is the well depth, $1/\mu$ is the well range, and $v=(4/3)\pi r_0^3A$ is the nuclear volume.

For the energies of the particles moving in the nuclear well, we use the result derived in reference 1, that

$$E=k^2/2M^*+V,$$

where V is a constant well and M^* is an effective nucleon mass which is modified from the nucleon mass to include the effects of the quadratic momentum dependence of the actual well. In this approximation, and replacing the sums by integrals (and making use of the Kronecker delta functions over total momentum), we find

$$\Delta E_{pc}=\left[\frac{v}{(2\pi)^3}\right]^2\left(\frac{2\pi V_0}{\mu v}\right)^3(M^*)^2\int d\mathbf{k}_1'd\mathbf{k}_i\times[f(\mathbf{k}_1-\mathbf{k}_1')+f(\mathbf{k}_2-\mathbf{k}_2')]\times[f(\mathbf{k}_2-\mathbf{k}_1')+f(\mathbf{k}_i-\mathbf{k}_2)]\times[f(\mathbf{k}_1-\mathbf{k}_1')+f(\mathbf{k}_i-\mathbf{k}_1')]\times\frac{1}{[k_1^2+k_2^2-k_1'^2-(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_1')^2]}\cdot\frac{1}{[k_1^2+k_i^2-k_1'^2-(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_1')^2]}\quad(83)$$

with the restriction arising from the exclusion principle that

$$|\mathbf{k}_1'|\geqslant k_F,\quad|\mathbf{k}_2'|=|\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_1'|\geqslant k_F,\quad|\mathbf{k}_i'|=|\mathbf{k}_1+\mathbf{k}_i-\mathbf{k}_1'|\geqslant k_F.\quad(84)$$

We will also approximately include the effects of the exclusion principle by multiplying this result by the apriori probability that particles interact in even states of orbital momentum, namely $\frac{1}{2}(\frac{1}{2}\times\frac{3}{4}$ for the n-p pairs plus $\frac{1}{2}\times\frac{1}{4}$ for the like particle pairs).

We are particularly interested in the interaction between particles in the same states; thus we shall in the integral of Eq. (83) set $\mathbf{k}_1=\mathbf{k}_2$. In addition we consider particles moving in the last state filled in the Fermi gas and set $|\mathbf{k}_1|=k_F$. We shall also introduce a further approximation at this stage. We note that the momenta which enter the integrals are all quite large and that since $f(x)$ is a rapidly decreasing function of x for large x, the only term in the integrand which will contribute appreciably is that in which terms of the form $f^3(x)$ appear. A term for example of the form $f^2(\mathbf{x})f(\mathbf{x}+\mathbf{k}_i)$ will be much smaller since it is not simultaneously possible for $f^2(\mathbf{x})$ and $f(\mathbf{x}+\mathbf{k}_i)$ to be large. Consequently we drop such terms and obtain (with a change in variable in the integral)

$$\Delta E_{pc}=\frac{1}{2}\frac{v^2}{(2\pi)^6}\left(\frac{2\pi V_0}{\mu v}\right)^3M^{*2}\int d\mathbf{k}_id\mathbf{x}\frac{f^3(\mathbf{x})}{x^2(x^2-\mathbf{k}_1\cdot\mathbf{x}-\mathbf{k}_i\cdot\mathbf{x})},\quad(85)$$

with the restrictions

$$|\mathbf{x}-\mathbf{k}_1|\geqslant k_F,\quad|\mathbf{x}+\mathbf{k}_1|\geqslant k_F,\quad|\mathbf{x}-\mathbf{k}_i|\geqslant k_F.\quad(86)$$

The integration over the core variable i is, except for these restrictions, extended over the Fermi sphere of momentum. This integral cannot be carried out completely in closed form; an excellent approximation (introducing errors of less than 20 percent), however, leads to the result

$$\Delta E_{pc}=\frac{v^2}{(2\pi)^6}\left(\frac{2\pi V_0}{\mu v}\right)^3(M^*)^2(2\pi)^2\int_0^\infty f^3(x)xdx.\quad(87)$$

We compare this (after multiplication by six to include the other terms arising from permutations of the particles) with the diagonal term for interaction between the two surface particles,

$$\Delta E = 4\pi V_0/\mu^3 v. \tag{88}$$

The ratio is

$$\frac{\Delta E_{pc}}{\Delta E} = \frac{3}{16\pi^2}\left(\frac{M^* V_0}{\mu^2}\right)^2. \tag{89}$$

Taking $M^* = 0\ 54M$, $V_0/\mu = 0.252$ (corresponding to a bound state at zero energy), the final result is

$$\Delta E_{pc}/\Delta E = 0.016. \tag{90}$$

Thus the contribution to the particle-particle interaction energy which arises from particle-core-particle coupling is quite small. As pointed out in the previous section, this can be readily included by a slight modification of the effective two-body interaction or a more detailed analysis of the type carried out in this section can be made to give its effect explicitly.

VI. CONCLUSIONS

We have considered the formalism developed in treating other problems of nuclear structure and nuclear saturation, and shown how it may be extended to the more detailed problem of energy-level fine structure. We have taken as our starting point a nuclear Hamiltonian containing as interaction terms only strong short-range two-body potentials. We have then shown how the relationships among the energy levels of neighboring nuclei can be very accurately determined if use is made of empirical knowledge of energy levels of one of the nuclei and a particle-to-particle coupling is introduced. This coupling is shown to be the result to a very good approximation of direct interaction through the two-body nucleon potential, effects of coupling through the nuclear core being evaluated and shown to be small. Thus we have demonstrated the relationship between the treatment of the original many-particle Hamiltonian and the determination of details of the energy-level structure by the methods of, for example, Flowers[4] and Ford and Levinson.[5]

The remaining problem which we have considered more briefly is the character of the nuclear wave function and the effects of particle-particle and particle-core configuration mixing. These unavoidably appear in the actual nuclear wave function as a consequence of the assumed strong two-body interactions and in general will affect markedly such quantities as the nuclear magnetic moment. The configuration mixing among particles outside a simple doubly-magic core (such as Ca^{40} or O^{16}) is known to provide an explanation of the anomalies in magnetic moments; it is not obvious, however, that additional large effects will not appear due to the particle-core coupling. We have given qualitative reasons for the smallness of such effects on the magnetic moment; these reasons are (1) relative insensitivity of the magnetic moment to short-range correlations in the wave functions, (2) absence in Ca^{40} and O^{16} of core states which when mixed into the single-particle wave function can give linear changes in the magnetic moment, and (3) relatively large excitation energies and consequent nonpolarizability of the core of a doubly magic nucleus. We have also remarked, however, that the apparent purity of a state as shown by a magnetic moment on the Schmidt line cannot be regarded as evidence for a simple independent-particle nuclear wave function, but only that admixtures of other states can in certain quite special circumstances have little effect on the magnetic moment. In these special cases it also appears that arguments can be made in favor of working with a nonsingular two-body potential to perturb the shell-model states since the respulsive core of the actual two-body potential does not affect the parts of the wave function which dominate in determining energy level splitting and magnetic moments. This result is obviously related to a similar situation which exists in nucleon-nucleon scattering where, over sizeable energy intervals, many equivalent potentials give almost identical predictions of scattering.[7]

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[7] See for example the discussion of the effects on high-energy processes of correlations in the nuclear ground-state wave function by Brueckner, Eden, and Francis (to be published).