Superallowed Beta Transitions in the $N-Z=3$ Series*

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(Received February 28, 1955)

A number of fast beta transitions have been found in the $N-Z=3$ series of radioactive nuclides. King has suggested that some of these transitions (for $A<27$) occur within the lowest $[4\cdots421]$ or $[4\cdots432]$ supermultiplet; hence are superallowed (or favored). Matrix elements of $|\int\boldsymbol{\sigma}|^2$ within the [21] and [32] supermultiplets are computed with the aid of two-way displacement operators on the eigenvalues of T_3, S_z, and Y_{3z}. In the application to $_8O_{11}$ the only spin assignments consistent with an LS coupling interpretation of the fast transition are $I_i=5/2$, $I^*=\frac{3}{2}(S^*=\frac{1}{2})$. These are the value favored by the available experimental information. The jj coupling value of $|\int\boldsymbol{\sigma}|^2$ for the $_8O_{11}$ transition is too large by a factor of five.

1. INTRODUCTION

ALLOWED components with unusually small ft values have been found in the beta activity of $_3Li_6$, $_6C_9$, $_7N_{10}$, $_8O_{11}$, $_{10}Ne_{13}$, $_{11}Na_{14}$, and $_{12}Mg_{15}$. The available experimental information on the fast components is shown in Table I. King[1] has suggested that some of these transitions are superallowed[2] (or favored) in the sense that both final and initial states belong to the same supermultiplet (characterized by the partition symbol $[4\cdots421]$ or $[4\cdots432]$). Figure 1 exhibits the essential qualitative features of King's interpretation based on Wigner's analysis of the supermultiplet into isobaric spin multiplets.[3] The orbital angular momentum L (a constant within the supermultiplet) couples with the spin angular momentum to give the following isobaric spin multiplets:

$$\begin{array}{llll} T=\frac{3}{2}, & S=\frac{1}{2}, & I=L-\frac{1}{2}(L>0), & L+\frac{1}{2}, \\ T=\frac{1}{2}, & S=\frac{1}{2}, & I=L-\frac{1}{2}(L>0), & L+\frac{1}{2}, \\ T=\frac{1}{2}, & S=\frac{3}{2}, & I=L-\frac{1}{2}(L>0), & L+\frac{1}{2}, \\ T=\frac{1}{2}, & S=\frac{3}{2}, & I=L-\frac{3}{2}(L>1), & L+\frac{3}{2}. \end{array} \tag{1}$$

The quantum numbers $T=\frac{3}{2}$, $T_3=\frac{3}{2}$, $S=\frac{1}{2}$, $I_i=L+\frac{1}{2}$ or $L-\frac{1}{2}$ characterize the initial state. Spin-dependent forces are held responsible for displacements in energy making possible a transition within the supermultiplet from $T=T_3=\frac{3}{2}$ to $T=T_3=\frac{1}{2}$ as pictured in Fig. 1. In the following discussion the Gamow-Teller (G-T) matrix elements for all possible final states are evaluated using the pure supermultiplet description of nuclear states. Interesting correlations are observed when these results are compared with experiment. A comparison with corresponding matrix elements computed under the assumption of jj coupling throws light on the type of intermediate coupling existing in light nuclides.

2. UPPER LIMITS ON THE GAMOW-TELLER MATRIX ELEMENTS

The Gamow-Teller matrix element,

$$|\int\boldsymbol{\sigma}|^2=\sum_{m_f}|(\alpha_f I_f m_f;\tfrac{1}{2},\tfrac{1}{2}|\sum\boldsymbol{\sigma}_k Q_k|\alpha_i I_i m_i;\tfrac{3}{2},\tfrac{3}{2})|^2, \tag{2}$$

is a function of the initial and final values of S, L, and T. An upper limit on $|\int\boldsymbol{\sigma}|^2$ for any choice of these quantum numbers can be computed by forming the sum

$$\sum_{\text{all}}|\int\boldsymbol{\sigma}|^2=\sum_{f}|(\alpha_f I_f m_f;T_f,\tfrac{1}{2}|\sum\boldsymbol{\sigma}_k Q_k|\alpha_i I_i m_i;\tfrac{3}{2},\tfrac{3}{2})|^2, \tag{3}$$

in which *all* denotes a summation over all possible final states within the supermultiplet. The application of closure to the right-hand member of Eq. (3) yields

$$\begin{aligned}\sum_{\text{all}}|\int\boldsymbol{\sigma}|^2&=(\alpha_i I_i m_i;\tfrac{3}{2},\tfrac{3}{2}|\sum\boldsymbol{\sigma}_k Q_k^*\cdot\sum\boldsymbol{\sigma}_l Q_l|\alpha_i I_i m_i;\tfrac{3}{2},\tfrac{3}{2})\\&=6(\alpha_i I_i m_i;\tfrac{3}{2},\tfrac{3}{2}|T_3|\alpha_i I_i m_i;\tfrac{3}{2},\tfrac{3}{2})\\&=9.\end{aligned} \tag{4}$$

Equation (A4) of the appendix is used to reduce the matrix element occurring in Eq. (4) to the explicit numerical value.

TABLE I. Low ft transitions in the $N-Z=3$ series.

Transitions	Energy (Mev)	Partial half-life	ft
$_3Li_6\rightarrow{}_4Be_5^*$ [a]	(7.3 ± 1)	0.17 sec	(2000−10 000)
$_6C_9\rightarrow{}_7N_8^*$ [b]	3.5	2.4 sec	3000–4000
$_7N_{10}\rightarrow{}_8O_9^*$ [c]	3.8	4.2 sec	6300
$_8O_{11}\rightarrow{}_9F_{10}^*$ [d]	2.9	29.4 sec/0.70	21 000
$_9F_{12}\rightarrow{}_{10}Ne_{11}^*$ [e]	$\cdots$	5 sec	$\cdots$
$_{10}Ne_{13}\rightarrow{}_{11}Na_{12}^*$ [f]	1.18	40.2 sec/0.07	6800
$_{11}Na_{14}\rightarrow{}_{12}Mg_{13}^*$ [d]	2.7	58.2 sec/0.45	56 000
$_{12}Mg_{15}\rightarrow{}_{13}Al_{14}^*$ [g]	1.59	9.5 min/0.414	60 000
	1.75	9.5 min/0.582	56 000

[a] Gardner, Knable, and Moyer, Phys. Rev. **83**, 1054 (1951); Holt, Thorn, and Waniek, Phys. Rev. **87**, 378 (1952); R. K. Sheline, Phys. Rev. **87**, 557 (1952); W. F. Fry, Phys. Rev. **89**, 325 (1953); D. Reagan, Phys. Rev. **92**, 651 (1943); F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **27**, 77 (1955).
[b] Richard, Hudspeth, and Clendenin, Phys. Rev. **96**, 1272 (1954).
[c] Stephens, Halpern, and Sher, Phys. Rev. **82**, 511 (1951); L. W. Alverez, Phys. Rev. **75**, 1127 (1949); E. Hayward, Phys. Rev. **75**, 917 (1941).
[d] E. Bleuler and W. Zunti, Helv. Phys. Acta **20**, 195 (1947).
[e] E. C. Campbell and C. V. Strain, Oak Ridge National Laboratory Report ORNL, 1496, 1952 (unpublished).
[f] H. Brown and V. Perez-Mendez, Phys. Rev. **78**, 812 (1950).
[g] Daniel, Koester, and Mayer-Kuckuk, Z. Naturforsch. **8a**, 447 (1953).

* Supported in part by the joint program of the U. S. Atomic Energy Commission and the Office of Naval Research.
[1] R. W. King, Bull. Am. Phys. Soc. No. 7, 20 (1954), and preceding paper [Phys. Rev. **99**, 67 (1955)]; see also E. P. Wigner, Proceedings of the International Conference on Theoretical Physics, Kyoto and Tokyo, September, 1953 (Science Council of Japan, Tokyo, 1954).
[2] E. P. Wigner, Phys. Rev. **56**, 519 (1939).
[3] E. P. Wigner, Phys. Rev. **51**, 106 and 948 (1937).

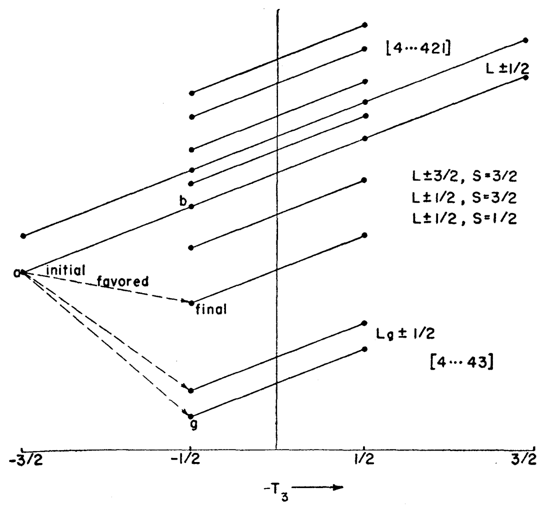

FIG. 1. Supermultiplet interpretation of a fast beta transition by an $N-Z=3$ parent nucleus. The initial state belongs to the lowest [4···421] supermultiplet. Spin-dependent forces split the supermultiplet into widely spaced isobaric spin multiplets. A transition within the supermultiplet then becomes energetically possible.

The $T=\frac{1}{2}$ final states may be selected by introducing the projection operator

$$P=\tfrac{1}{3}(15/4-\mathbf{T}^2) \tag{5}$$

as a factor multiplying $\sum \boldsymbol{\sigma}_k Q_k$ on the left. In this way the sum over all final states with $T=\frac{1}{2}$ is reduced to

$$\begin{aligned}\sum_{T_f=\frac{1}{2}} |\textstyle\int\boldsymbol{\sigma}|^2 &= \sum_{\text{all}} |\textstyle\int P\boldsymbol{\sigma}|^2\\ &=(\alpha_i I_i I_i;\tfrac{3}{2},\tfrac{3}{2}|\sum \boldsymbol{\sigma}_k Q_k^*\cdot P\sum\boldsymbol{\sigma}_l Q_l|\alpha_i I_i I_i;\tfrac{3}{2},\tfrac{3}{2})\\ &=(\alpha_i I_i I_i;\tfrac{3}{2},\tfrac{3}{2}|(15/2)T_3-2(T_3-1)(\mathbf{T}^2-T_3)\\ &\quad-(4/3)(Y_{3x}^2+Y_{3y}^2+Y_{3z}^2)|\alpha_i I_i I_i;\tfrac{3}{2},\tfrac{3}{2}) \qquad (6)\\ &=8,\end{aligned}$$

with the help of Eqs. (A4) and (A5) of the appendix and the relation

$$Y_{3x}^2+Y_{3y}^2=(Y_{3x}-iY_{3y})(Y_{3x}+iY_{3y})+S_z. \tag{7}$$

To select quartet final states, the preceding calculation may be repeated employing the projection operator

$$P=\tfrac{1}{3}(\mathbf{S}^2-\tfrac{3}{4}). \tag{8}$$

Then

$$\begin{aligned}\sum_{S_f=\frac{3}{2}} |\textstyle\int\boldsymbol{\sigma}|^2 &= \sum_{\text{all}} |\textstyle\int P\boldsymbol{\sigma}|^2\\ &=(\alpha_i I_i I_i;\tfrac{3}{2},\tfrac{3}{2}|\sum \boldsymbol{\sigma}_k Q_k^*\cdot P\sum\boldsymbol{\sigma}_l Q_l|\alpha_i I_i I_i;\tfrac{3}{2},\tfrac{3}{2})\\ &=(\alpha_i I_i I_i;\tfrac{3}{2},\tfrac{3}{2}|(6T_3-8)P+4T_3-2| \qquad (9)\\ &\qquad\times\alpha_i I_i I_i;\tfrac{3}{2},\tfrac{3}{2})\\ &=4.\end{aligned}$$

3. EXPLICIT EVALUATION OF THE GAMOW-TELLER MATRIX ELEMENTS

The third line of Eq. (A4) is easily transformed into the matrix element relation:

$$\begin{aligned}&[2(T-1)]^{\frac{1}{2}}(\beta_f;T-1,T-2|Y_{1u}-iY_{2u}|\beta_i;T,T-1)\\ &\quad-[2T]^{\frac{1}{2}}(\beta_f;T-1,T-1|Y_{1u}-iY_{2u}|\beta_i,T,T)\\ &\qquad=2(\beta_f;T-1,T-1|Y_{3u}|\beta_i;T,T-1). \qquad (10)\end{aligned}$$

Also

$$\frac{(\beta_f;T-1,T-2|Y_{1u}-iY_{2u}|\beta_i;T,T-1)}{(\beta_f;T-1,T-1|Y_{1u}-iY_{2u}|\beta_i;T,T)} =\frac{(T\,1\,T-1|T\,1\,T-1\,T-2)}{(T\,1\,T-1|T\,1\,T-1\,T-1)}=\left(\frac{T-1}{T}\right)^{\frac{1}{2}}. \tag{11}$$

Equations (10) and (11) combine to give

$$\begin{aligned}&(\beta_f;T-1,T-1|Y_{1u}-iY_{2u}|\beta_i;T,T)\\ &\quad=-(2T)^{\frac{1}{2}}(\beta_f;T-1,T-1|Y_{3u}|\beta_i;T,T-1). \qquad (12)\end{aligned}$$

Equation (12) may be combined with well-known sum rules[4] to express $|\int\boldsymbol{\sigma}|^2$ in the convenient forms:

$$|\textstyle\int\boldsymbol{\sigma}|^2{}_{I_i=I,\,I_f=I-1}=2TI|(\alpha_f,I-1,I-1;T-1,T-1|Y_{3z}|\alpha_i;I,I-1;T,T-1)|^2, \tag{13a}$$

$$|\textstyle\int\boldsymbol{\sigma}|^2{}_{I_f=I_i=I>0}=2T[(I+1)/I]|(\alpha_f,I,I;T-1,T-1|Y_{3z}|\alpha_i;I,I;T,T-1)|^2, \tag{13b}$$

$$|\textstyle\int\boldsymbol{\sigma}|^2{}_{I_i=I,\,I_f=I+1}=2T\frac{(I+1)(2I+3)}{2I+1}|(\alpha_f,I+1,I;T-1,T-1|Y_{3z}|\alpha_i;I,I;T,T-1)|^2. \tag{13c}$$

Similar formulas have been used to evaluate the G-T matrix element in the important special cases $T_i=T_f=\frac{1}{2}$ and $T_i=1$.[5]

The ${}_2He_4$ transition provides an interesting application for Eq. (13c). Let ψ_a and ψ_l denote the unique normalized wavefunctions belonging to $T=1$, $T_3=0$, $S=S_z=0$ and $T=T_3=0$, $S=1$, $S_z=0$ respectively. The commutator of $\mathbf{S}^2$ and Y_{3z} applied to ψ_a yields the eigenvalue equation:

$$\mathbf{S}^2(Y_{3z}\psi_a)=2(Y_{3z}\psi_a). \tag{14}$$

Similarly, Eq. (A5) of the Appendix requires

$$\mathbf{T}^2(Y_{3z}\psi_b)=2(Y_{3z}\psi_b). \tag{15}$$

[4] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1935), Chap. 3; E. Feenberg and G. E. Pake, *Quantum Theory of Angular Momentum* (Addison-Wesley Press, Cambridge, 1953), Chap. 5.

[5] M. Bolsterli and E. Feenberg, Phys. Rev. **97**, 736 (1955).

Consequently,

$$Y_{3z}\psi_a=\lambda\psi_b,\quad Y_{3z}\psi_b=\lambda'\psi_a. \tag{16}$$

The relation

$$(a|Y_{3z}|b)=\lambda'(a,a)=\lambda(b,b) \tag{17}$$

implies $\lambda'=\lambda$ since ψ_a and ψ_b are normalized. Furthermore the eigenvalues of Y_{3z} are ± 1;[3] the corresponding eigenfunctions are given by suitable linear combinations of ψ_a and ψ_b:

$$Y_{3z}(a\psi_a+b\psi_b)=\pm(a\psi_a+b\psi_b), \tag{18}$$

or

$$a\lambda=\pm b,\quad b\lambda=\pm a \tag{19}$$

yielding

$$\lambda=1 \quad\text{or}\quad -1. \tag{20}$$

Thus, the squared matrix element appearing in the right hand member of Eq. (13c) has the value 1 in the ${}_2He_4$ transition; consequently $|\int\boldsymbol{\sigma}|^2=6^2$.

In the general case, the matrix elements of Y_{3z} occurring in Eq. (13) can be expressed in terms of reduced matrix elements and Racah functions.[6] The Eckart-Wigner theorem yields

$$\begin{aligned}(I_f m;\,T_f,T_3|Y_{3z}|I_i,m;\,T_i,T_3)\\ =(2I_f+1)^{-\frac{1}{2}}(I_f;\,T_f,T_3\|Y_3\|I_i;\,T_i,T_3)\\ \times(I_i\,1\,m\,0|I_i\,1\,I_f\,m),\end{aligned} \tag{21}$$

$$\begin{aligned}(S_f,m_s;\,T_f,T_3|Y_{3z}|S_i,m_s;\,T_i,T_3)\\ =(2S_f+1)^{-\frac{1}{2}}(S_f;\,T_f,T_3\|Y_3\|S_i;\,T_i,T_3)\\ \times(S_i\,1\,m_s\,0|S_i\,1\,S_f\,m_s),\end{aligned} \tag{22}$$

expressing matrix elements of Y_{3z} in terms of reduced matrix elements and vector addition coefficients. The Racah function appears in the relation

$$\begin{aligned}(I_f;\,T_fT_3\|Y_3\|I_i;\,T_iT_3)\\ =(-1)^{L-1-S_i-I_f}[(2I_f+1)(2I_i+1)]^{\frac{1}{2}}\\ \cdot(S_f;\,T_fT_3\|Y_3\|S_i;\,T_iT_3)W(S_f,I_f,S_i,I_i;\,L,1),\end{aligned} \tag{23}$$

and also in the derived formula

$$\begin{aligned}(I_f,m;\,T_f,T_3|Y_{3z}|I_i,m;\,T_i,T_3)\\ =(-1)^{L-1-S_i-I_f}[(2I_i+1)(2S_f+1)]^{-\frac{1}{2}}\\ \cdot(S_f,m_s;\,T_f,T_3|Y_{3z}|S_i,m_s;\,T_i,T_3)\\ \times W(S_f,I_f,S_i,I_i;\,L,1)\frac{(I_i\,1\,m\,0|I_i\,1\,I_f\,m)}{(S_i\,1\,m_s\,0|S_i\,1\,S_f\,m_s)}\end{aligned} \tag{24}$$

expressing the matrix element of Y_{3z} in the Im space in terms of the much simpler matrix element in the Sm_s space.

In the present application we need a number of vector addition coefficients $(I_i\,1\,m\,0|I_i\,1\,I_f\,m)$ for $m=\frac{1}{2}(I_i+I_f-|I_i-I_f|)=I_i$ or I_f whichever is smaller:

$$\begin{array}{lll} I_i & I_f & (I_i\,1\,m\,0|I_i\,1\,I_f\,m) \\ \hline L+\frac{1}{2} & L+\frac{1}{2} & [2L+1/2L+3]^{\frac{1}{2}}, \\ L+\frac{1}{2} & L-\frac{1}{2} & -\left[\dfrac{2L}{(L+1)(2L+1)}\right]^{\frac{1}{2}}, \\ L-\frac{1}{2} & L+\frac{1}{2} & [2/2L+1]^{\frac{1}{2}}. \end{array} \tag{25}$$

The following Racah functions are also needed:

$$W(\tfrac{1}{2},L+\tfrac{1}{2},\tfrac{1}{2},L+\tfrac{1}{2};L,1)=\tfrac{1}{2}\left[\frac{2L+3}{3(L+1)(2L+1)}\right]^{\frac{1}{2}},$$

$$W(\tfrac{1}{2},L-\tfrac{1}{2},\tfrac{1}{2},L+\tfrac{1}{2};L,1)=[3(2L+1)]^{-\frac{1}{2}},$$

$$W(\tfrac{1}{2},L+\tfrac{1}{2},\tfrac{1}{2},L-\tfrac{1}{2};L,1)=[3(2L+1)]^{-\frac{1}{2}},$$

$$W(\tfrac{1}{2},L-\tfrac{1}{2},\tfrac{1}{2},L-\tfrac{1}{2};L,1)=\tfrac{1}{2}\left[\frac{2L-1}{3L(2L+1)}\right]^{\frac{1}{2}},$$

$$\begin{aligned}W(\tfrac{3}{2},L-\tfrac{1}{2},\tfrac{1}{2},L+\tfrac{1}{2};L,1)&=-\tfrac{1}{2}\left[\frac{2L-1}{6(L+1)(2L+1)}\right]^{\frac{1}{2}},\\ W(\tfrac{3}{2},L+\tfrac{1}{2},\tfrac{1}{2},L-\tfrac{1}{2};L,1)&=\tfrac{1}{2}\left[\frac{2L+3}{6L(2L+1)}\right]^{\frac{1}{2}},\\ W(\tfrac{3}{2},L+\tfrac{1}{2},\tfrac{1}{2},L+\tfrac{1}{2};L,1)&=-\left[\frac{L}{6(L+1)(2L+1)}\right]^{\frac{1}{2}},\\ W(\tfrac{3}{2},L+\tfrac{3}{2},\tfrac{1}{2},L+\tfrac{1}{2};L,1)&=-\tfrac{1}{2}[2(L+1)]^{-\frac{1}{2}},\\ W(\tfrac{3}{2},L-\tfrac{3}{2},\tfrac{1}{2},L-\tfrac{1}{2};L,1)&=\tfrac{1}{2}[2L]^{-\frac{1}{2}},\\ W(\tfrac{3}{2},L-\tfrac{1}{2},\tfrac{1}{2},L-\tfrac{1}{2};L,1)&=\left[\frac{L+1}{6L(2L+1)}\right]^{\frac{1}{2}}.\end{aligned} \tag{26}$$

Two quantities remain to be evaluated:

$$\begin{aligned}K&\equiv(\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}|Y_{1z}-iY_{2z}|\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2},\tfrac{3}{2})\\ &=-3^{\frac{1}{2}}(\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}|Y_{3z}|\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2},\tfrac{1}{2}),\\ K'&\equiv(\tfrac{3}{2},\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}|Y_{1z}-iY_{2z}|\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2},\tfrac{3}{2})\\ &=-3^{\frac{1}{2}}(\tfrac{3}{2},\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}|Y_{3z}|\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2},\tfrac{1}{2}).\end{aligned} \tag{27}$$

Let

$$\begin{aligned}\psi_a&\equiv\psi_{\frac{1}{2},\frac{1}{2};\,\frac{3}{2},\frac{3}{2}},\\ \psi_b&\equiv\psi_{\frac{1}{2},\frac{1}{2};\,\frac{3}{2},\frac{1}{2}}=3^{-\frac{1}{2}}(T_1-iT_2)\psi_a,\\ \psi_c&\equiv\psi_{\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}},\\ \psi_d&\equiv\psi_{\frac{3}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}}\\ &=12^{-\frac{1}{2}}(S_x-iS_y[(Y_{1x}-iY_{2x})+i(Y_{1y}-iY_{2y})]\psi_a.\end{aligned} \tag{28}$$

[6] G. Racah, Phys. Rev. **62**, 438 (1942) and **63**, 367 (1943).

TABLE II. Formulas for $|\int\boldsymbol{\sigma}|^2$.

I_i	S_f	I_f	$\|\int\boldsymbol{\sigma}\|^2$
$L+\frac{1}{2}$	$\frac{1}{2}$	$L+\frac{1}{2}$	$\frac{4}{3}\frac{2L+3}{2L+1}$
		$L-\frac{1}{2}$	$\frac{8}{3}\frac{2L}{2L+1}$
$L+\frac{1}{2}$	$\frac{3}{2}$	$L+\frac{3}{2}$	$2\frac{L+2}{L+1}$
		$L+\frac{1}{2}$	$\frac{4}{3}\frac{2L}{2L+1}$
		$L-\frac{1}{2}$	$\frac{2}{3}\frac{L(2L-1)}{(L+1)(2L+1)}$
$L-\frac{1}{2}$	$\frac{1}{2}$	$L+\frac{1}{2}$	$\frac{8}{3}\frac{2L+2}{2L+1}$
		$L-\frac{1}{2}$	$\frac{4}{3}\frac{2L-1}{2L+1}$
$L-\frac{1}{2}$	$\frac{3}{2}$	$L+\frac{1}{2}$	$\frac{2}{3}\frac{(L+1)(2L+3)}{L(2L+1)}$
		$L-\frac{1}{2}$	$\frac{4}{3}\frac{2L+2}{2L+1}$
		$L-\frac{3}{2}$	$2\frac{L-1}{L}$

Now

$$\begin{aligned}(b|Y_{1z}-iY_{2z}|a)&=3^{-\frac{1}{2}}(a|(T_1+iT_2)(Y_{1z}-iY_{2z})|a)\\&=3^{-\frac{1}{2}}(a|2Y_{3z}|a)=3^{-\frac{1}{2}},\\(d|Y_{1z}-iY_{2z}|a)&=12^{-\frac{1}{2}}(a|\{(Y_{1x}+iY_{2x})\\&\quad-i(Y_{1y}+iY_{2y})\}(S_x+iS_y)(Y_{1z}-iY_{2z})|a)\\&=12^{-\frac{1}{2}}(a|4T_3-4S_z|a)=-(4/3)^{\frac{1}{2}}.\end{aligned}\tag{29}$$

Also, by closure,

$$\begin{aligned}|(b|Y_{1z}-iY_{2z}|a)|^2&+|(c|Y_{1z}-iY_{2z}|a)|^2\\+|(d|Y_{1z}-iY_{2z}|a)|^2&\\&=(a|(Y_{1z}+iY_{2z})(Y_{1z}-iY_{2z})|a)\\&=(a|2T_3|a)=3.\end{aligned}\tag{30}$$

Consequently,

$$\begin{aligned}|(c|Y_{1z}-iY_{2z}|a)|^2&=3-\tfrac{1}{3}-4/3\\&=4/3.\end{aligned}\tag{31}$$

Thus,

$$K^2=K'^2=4/3.\tag{32}$$

Explicit calculations with three-particle wave functions provide an instructive check on this bit of operator algebra.

The final formulas for $|\int\boldsymbol{\sigma}|^2$ are collected in Table II. Some numerical values are exhibited in Table III.

4. DISCUSSION

In the decay of ${}_8O_{11}$, the assignments $I_i=5/2$, $I_f=\frac{3}{2}$ are strongly favored by the experimental evidence on beta and gamma transitions involving these states.[7] As the last column of Table III shows, these are the only assignments consistent with an LS coupling interpretation of the initial and final states. Also, $S_f=\frac{3}{2}$ is favored in agreement with King's argument based on energetic considerations. The good agreement may be illusory since two effects tending to change $|\int\boldsymbol{\sigma}|^2$ are certainly present. These are (i) the occurrence of a doublet component in the (assumed) predominently quartet final state and (ii) the mixing of supermultiplets. Both effects are expected to occur as consequences of the spin dependent interaction responsible for the separation of the supermultiplet into energetically distinct isobaric spin multiplets. The first effect may either increase or decrease $|\int\boldsymbol{\sigma}|^2$ while the second is likely to decrease it.

In jj coupling, the $(d_{\frac{5}{2}})^3$ configuration yields[8]

$$|\int\boldsymbol{\sigma}|^2=28/25,\quad I_i=5/2,\quad I_f=\tfrac{3}{2}.$$

Thus

$$ft|\int\boldsymbol{\sigma}|_{jj}^2=23\,000,$$

too large by a factor of 5.

In general, a wide range of values of the nuclear matrix element is possible depending on the relative amplitudes and phases of the doublet and quartet components in the final state wave function.

The transitions at $A=25$ and 27 may be excluded from the superallowed category, because they are only slightly faster than an average allowed transition and also because the energies fall outside the fairly regular

TABLE III. Numerical results.

Transition	L	I_i	S_f	I_f	$\|\int\boldsymbol{\sigma}\|^2$	$ft\|\int\boldsymbol{\sigma}\|^2$
${}_8O_{11}\rightarrow{}_9F_{10}^*$	2	5/2	1/2	5/2	28/15	37 000
$ft=21\,000$				3/2	32/15	45 000
			3/2	7/2	8/3	56 000
				5/2	16/15	22 000
				3/2	4/15	5600
${}_7N_{10}\rightarrow{}_8O_9^*$	1	1/2	1/2	3/2	32/9	22 000
$ft=6300$				1/2	4/9	2800
			3/2	3/2	20/9	14 000
				1/2	16/9	11 000
${}_{12}Mg_{15}\rightarrow{}_{13}Al_{14}^*$	0	1/2	1/2	1/2	4	2.4×10^5
$ft=6\times10^4$			3/2	3/2	4	2.4×10^5

[7] Jones, Phillips, Johnson, and Wilkinson, Phys. Rev. **96**, 547 (1954).

[8] E. Feenberg, *The Shell Theory of the Nucleus* (Princeton University Press, Princeton, 1955), Chap. 8.

decreasing trend exhibited by the transitions in lighter members of the $N-Z=3$ series.

APPENDIX

The function space covered by a supermultiplet is transformed into itself by the set of operators[3]:

$$S_u=\tfrac{1}{2}\sum \sigma_u{}^{(k)},\quad (u=x,y,z),$$
$$T_v=\tfrac{1}{2}\sum \tau_v{}^{(k)},\quad (v=1,2,3), \tag{A1}$$
$$Y_{vu}=\tfrac{1}{2}\sum \tau_v{}^{(k)}\sigma_u{}^{(k)}.$$

In terms of these operators,

$$\begin{aligned}\sum Q_k&=T_1-iT_2,\\ \sum Q_k{}^*&=T_1+iT_2,\\ \sum \sigma_u{}^{(k)}Q_k&=Y_{1u}-iY_{2u},\\ \sum \sigma_u{}^{(k)}Q_k{}^*&=Y_{1u}+iY_{2u}.\end{aligned} \tag{A2}$$

A number of useful commutation relations obeyed by the S, T, Y operators are listed below:

$$\begin{aligned}&[T_1\pm iT_2,\, T_3]=\mp(T_1\pm iT_2),\\ &[T_1\pm iT_2,\, Y_{3u}]=\mp(Y_{1u}\pm iY_{2u}),\\ &[T_1+iT_2,\, T_1-iT_2]=2T_3,\end{aligned} \tag{A3}$$

$$\begin{aligned}&[Y_{1u}\pm iY_{2u},\, T_3]=\mp(Y_{1u}\pm iY_{2u}),\\ &[Y_{1u}\pm iY_{2u},\, Y_{3u}]=\mp(T_1\pm iT_2),\\ &[Y_{1u}\pm iY_{2u},\, T_1\mp iT_2]=\pm 2Y_{3u},\\ &[Y_{1u}+iY_{2u},\, Y_{1u}-iY_{2u}]=2T_3,\\ &[Y_{1x}\pm iY_{2x},\, Y_{1y}\mp iY_{2y}]=2iS_z,\end{aligned} \tag{A4}$$

$$\begin{aligned}&[Y_{1x}\pm iY_{2x},\, S_y]=i(Y_{1z}\pm iY_{2z}),\\ &[\mathbf{S}^2,\, Y_{1x}\pm iY_{2x}]\\ &\quad=-2(Y_{1x}\pm iY_{2x})+2i[S_z(Y_{1y}\pm iY_{2y})-S_y(Y_{1z}\pm iY_{2z})]\\ &\quad=2(Y_{1x}\pm iY_{2x})+2i[(Y_{1y}\pm iY_{2y})S_z-(Y_{1z}\pm iY_{2z})S_y],\end{aligned} \tag{A5}$$

$$\begin{aligned}&[\mathbf{T}^2,\, Y_{1x}\pm iY_{2x}]\\ &\quad=\mp 2(T_1\pm iT_2)Y_{3x}+2(\pm T_3-1)(Y_{1x}\pm iY_{2x})\\ &\quad=\mp 2Y_{3x}(T_1\pm iT_2)+2(Y_{1x}\pm iY_{2x})(\pm T_3+1).\end{aligned}$$

The exact analogy between the 1,2,3 and x,y,z spaces yields an image relation for every one set down in Eqs. (A3) to (A5). Thus, for example, the next to the last line of Eq. (A4) translates into

$$[Y_{1x}\pm iY_{1y},\, Y_{2x}\mp iY_{2y}]=2iT_3.$$

Two-way displacement operators have the property of displacing two eigenvalues in the set T_3', S_z', Y_{3z}' up or down by one unit while leaving the third unchanged. These operators are defined as follows in terms of two valued indices u, v, w which take on values $+1$ and -1 independently:

$$\begin{aligned}M_{uvo}&=(Y_{1x}+iuY_{2x})+iv(Y_{1y}+iuY_{2y}),\\ M_{uow}&=(T_1+iuT_2)+uw(Y_{1z}+iuY_{2z}),\\ M_{ovw}&=(S_x+ivS_y)+vw(Y_{3x}+ivY_{3y}).\end{aligned} \tag{A6}$$

The basic commutation relations are

$$\begin{aligned}&[M_{uvo},\, T_3]=-uM_{uvo},\\ &[M_{uvo},\, S_z]=-vM_{uvo},\\ &[M_{uvo},\, Y_{3z}]=0,\end{aligned} \tag{A7}$$

$$\begin{aligned}&[M_{uow},\, T_3]=-iuM_{uov},\\ &[M_{uow},\, S_z]=0,\\ &[M_{uow},\, Y_{3z}]=-wM_{uow},\end{aligned} \tag{A8}$$

$$\begin{aligned}&[M_{ovw},\, T_3]=0,\\ &[M_{ovw},\, S_z]=-vM_{ovw},\\ &[M_{ovw},\, Y_{3z}]=wM_{ovw},\end{aligned} \tag{A9}$$

$$\begin{aligned}&[M_{u,v,o},\, M_{o,-v,w}]=2uvwM_{uov},\\ &[M_{u,o,w},\, M_{-u,v,o}]=2uvwM_{ovw},\\ &[M_{o,v,w},\, M_{u,o,-w}]=2uvwM_{uvo},\end{aligned} \tag{A10}$$

$$\begin{aligned}&[M_{uvo},\, M_{u'v'o}]=(vT_3+uS_z)(u-u')(v-v'),\\ &[M_{uow},\, M_{u'ow'}]=(wT_3+uY_{3z})(u-u')(w-w'),\\ &[M_{ovw},\, M_{ov'w'}]=(wS_z+vY_{3z})(v-v')(w-w').\end{aligned} \tag{A11}$$

One of these displacement operators is used in constructing ψ_d in Eq. (28). The application of $M_{-1,\,1,\,0}$ to ψ_a transforms a solution with $T_3=\frac{3}{2}$, $S_z=\frac{1}{2}$ into one with $T_3=\frac{1}{2}$, $S_z=\frac{3}{2}$. Since $S_z=\frac{3}{2}$ is associated with $T=\frac{1}{2}$, the solution found in this manner belongs to $T=\frac{1}{2}$, $S=\frac{3}{2}$.