assumption that the laws connecting the macroscopic variables are Markoffian.

6. CONCLUSION

In this paper we have developed in classical (non-quantum) mechanics a model for a stationary irreversible process that is capable of being treated with the methods of statistical mechanics due to Gibbs. Our model is based on the assumption that no essential features of the real process are lost if the interaction of the system with the driving reservoirs is pictured in terms of instantaneous impulsive interactions. The reservoirs themselves are described as infinitely large composites consisting of identical noninteracting components in canonical distribution. Thus each reservoir has a definite temperature, infinite heat capacity, and vanishing internal heat conductivity. No special assumptions are made concerning the structure of the system; its internal dynamics are governed by some nonsingular Hamiltonian.

We have succeeded in showing that our model, with arbitrary initial ensemble distribution, will approach the canonical distribution if driven by a single reservoir, will approach a stationary (noncanonical) distribution if driven by several reservoirs at different temperatures, and in the stationary state will obey the Onsager relations if the driving temperature gradients are small.

Further work will be devoted to a more detailed investigation of the stochastic kernels that represent the action of the reservoirs, the introduction of more general thermodynamic forces than temperature gradients, and the transition to quantum mechanics.

PHYSICAL REVIEW VOLUME 99, NUMBER 2 JULY 15, 1955

Lagrangians Linear in the "Velocities"*

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(Received March 16, 1955)

Lagrangian linear in the first time derivatives are of sufficient importance in physics (particle fields, general relativity in the Palatini formulation, Einstein-Strauss type unified field theory, etc.) to warrant special consideration. Our treatment is patterned after Dirac's more general exploration of Lagrangians leading to algebraic relations between the canonical variables. In our case, the number of such constraints is at least as large as the number of configuration coordinates. The secondary constraints are free of canonical momentum densities. We have examined all the possibilities that may arise—incompatibility of the field equations, proper Cauchy-Kowalewski problems, and the appearance of arbitrary functions in the solutions. Appropriate quantization procedures for the compatible cases will be indicated.

1. INTRODUCTION

MANY physical theories are derivable from Lagrangians that are linear in the first time derivatives of the field variables. We have investigated in this paper the compatibility of the field equations of such a theory, methods of constructing a Hamiltonian formalism, and quantization procedures. The formalism developed is capable of handling such diverse theories as the Pauli-Fierz equations, gravitational theory in the Palatini form (i.e., considering the components of the affine connection as independent variables), the Einstein-Strauss unified field theory, and Maxwell theory with the vector potentials treated as variables independent of **E** and **H**.

Our treatment is based on Dirac's[1] study of theories for which the momenta canonically conjugate to the field variables are not all algebraically independent of the field variables and their spatial derivatives themselves. Dirac reduces all cases to Lagrangians that are homogeneous of the first degree in the "velocities" (derivatives of the field variables with respect to the chosen time coordinate); any Lagrangian can be given this form by the device of parametrization. Lagrangians that are (inhomogeneously) linear in the velocities do not require this treatment. They can be discussed quite successfully without the introduction of a parameter. They are of sufficient importance in physics that they warrant a specialized treatment.

There are two general types of linear Lagrangians (in our sense). They all lead to differential equations that are free to accelerations and are linear in the velocities. The first type leads to equations in which the matrix of the coefficients of the velocities is nonsingular, the second to equations in which the same matrix is singular. The first case can be treated completely and in full generality. The second case has a number of subcases. Whenever the matrix of the coefficients is singular, then there exists a number of linear combinations of the (Lagrangian field) equations that are free of velocities. These combinations may be empty; if they are not, their time derivatives may be independent of the original field equations. In the latter case, new combinations

* This work was supported by the Office of Naval Research.
† Submitted to the Graduate School of Syracuse University as a thesis for the master's degree.
[1] P. A. M. Dirac, Can. J. Math. 2, 129 (1950); 3, 1 (1951).

free of velocities may be formed, and so forth. Roughly, these subcases may then lead to systems of equations that (a) are incompatible, (b) lead to proper initial value problems (Cauchy-Kowalewski systems), or (c) possess solutions incorporating a number of arbitrary functions of the coordinates.

In all cases, we have endeavored to indicate appropriate procedures for quantizing the theory.

2. BASIC EQUATIONS

We consider Lagrangian densities of the form

$$L=f^{A\rho}y_{A,\rho}+Q^*(y_B)=f^A(y_B)\dot{y}_A+Q,\quad \rho=1\cdots 4$$
$$f^A=f^{A4},\quad Q=Q^*+f^{As}y_{A,s},\qquad (2.1)$$
$$A=1\cdots N,\quad s=1,2,3,$$

where y_A are the field variables and $y_{A,s}$ their spatial derivatives. The Euler equations are

$$L^A=\{\partial^A f^{B\rho}-\partial^B f^{A\rho}\}y_{B,\rho}+\partial^A Q^*$$
$$=\{\partial^A f^B-\partial^B f^A\}\dot{y}_B+\delta^A Q=0,\qquad (2.2)$$

where

$$\partial^A f^B\equiv\partial f^B/\partial y_A,\quad \partial^{As}f^B\equiv\partial f^B/\partial y_{A,s},$$
$$\delta^A Q\equiv\partial^A Q-(\partial^{As}Q)_{,s}.$$

The equations have been written first in four-dimensional notation and then with the time variable singled out. In the remainder of this paper, only the latter will be used. We shall write

$$\partial^A f^B-\partial^B f^A\equiv f^{AB}.\qquad (2.3)$$

Because f^{AB} is skewsymmetric, its determinant will vanish if there are an odd number of field variables. If for this or any other reason, $\det|f^{AB}|=0$, then we cannot solve the Euler equations algebraically for $\dot{y}_A$. Even then it may happen that the field equations are of the Cauchy-Kowalewski type. This statement will be amplified in Sec. 5.

To go over to the Hamiltonian formalism we introduce the momentum densities conjugate to the y_A,

$$\pi^A=\partial L/\partial\dot{y}_A=f^A.\qquad (2.4)$$

Ordinarily, the defining equations for the canonical momentum densities are used in turn to express the time derivatives of the field variables, the $\dot{y}_A$, in terms of the canonical variables and to eliminate them from the standard expression for the Hamiltonian density,

$$\mathcal{H}=\dot{y}_A\pi^A-L.\qquad (2.5)$$

Our Eq. (2.4) contains no reference to the $\dot{y}_A$. Hence the usual procedure is not feasible. However, it may be possible to obtain expressions for the "velocities" directly from the field equations (2.2) and to use these expressions to obtain the Hamiltonian density. Such a determination will be unique if the determinant of the f^{AB} does not vanish. If it does, the lack of uniqueness in the determination of the $\dot{y}_A$ from Eq. (2.2) may give rise to a certain arbitrariness in the Hamiltonian density. This contingency will be taken up in detail in Secs. 4 and 5. At any rate, if

$$\dot{y}_A=\mu_A(y_B,y_{B,s})\qquad (2.6)$$

is a particular solution of the field equations (2.2), we may substitute this solution into the expression for the Hamiltonian and obtain

$$\mathcal{H}=\mu_A\pi_A-f^A\mu_A-Q=\mu_A C^A-Q,\qquad (2.7)$$

with μ_B satisfying

$$f^{AB}\mu_B+\delta^A Q=0.\qquad (2.8)$$

The expressions C^A, which we have introduced in Eq. (2.7),

$$C^A\equiv\pi^A-f^A=0,\qquad (2.9)$$

vanish because of Eq. (2.4). They represent conditions on the canonical field variables that are free of velocities. We call such conditions constraints. More particularly, the C^A are "primary constraints," because they result directly from the defining Eq. (2.4). Even though the C^A vanish, their partial derivatives will appear in the canonical field equations. The canonical field equations have the form

$$\dot{y}_A=\partial\mathcal{H}/\partial\pi^A=\mu_A,$$
$$\dot{\pi}^A=-\delta^A\mathcal{H}=\delta^A Q+\mu_B\partial^A f^B-C^B\delta^A\mu_B.\qquad (2.10)$$

Presumably, Eqs. (2.9) and (2.10) determine the behavior of the field in the Hamiltonian formalism. For this system of equations to be consistent, the time derivatives of the constraints must vanish. In some theories with constraints, this set of conditions leads to additional constraints (the secondary constraints), whose time derivatives must again vanish, etc. In this instance, these consistency conditions are satisfied automatically. We have

$$\dot{C}^A=\dot{\pi}^A-\dot{y}_B\partial^B f^A$$
$$=\delta^A Q+f^{AB}\mu_B-C^B\delta^A\mu_B.\qquad (2.11)$$

The first two terms on the right-hand side will vanish together if the choice (2.6) for μ_A was made correctly, i.e., in accordance with the Lagrangian field equations (2.2). The last term will vanish if the constraints (2.9) are satisfied. Similarly, the time derivatives of the expressions (2.11) will again vanish modulo Eqs. (2.2), (2.6) and (2.9). It follows that if we satisfy the canonical field equations (2.10) at all times and the constraints (2.9) at least at one time t_0, then the constraints will remain satisfied at all times. This set of conditions,—(2.9) at one time, (2.7), (2.10) at all times,—is then equivalent to the Lagrangian field equations (2.2), possibly specialized (if $\det|f^{AB}|=0$) by the choice (2.6).

A relationship which will be useful in the remainder of the paper is the Poisson bracket between two constraints. A short calculation shows

$$(C^A(x),C^B(x'))=f^{AB}\delta(x-x').\qquad (2.12)$$

Attempts to quantize a theory with constraints in the usual manner, i.e., by equating the commutators of the quantum-theoretical observables with the Poisson brackets of the c-number theory run into a characteristic difficulty. One might be tempted to require of the constraints that they are identically zero, i.e., that all matrix elements of the corresponding operators vanish. Such an assumption is: inconsistent with the requirement that some of the commutators of these constraints with other observables (which may also be constraints, though they need not be) are different from zero. Frequently, this difficulty, which also arises in quantum electrodynamics, has been circumvented by the requirement that the constraints do not vanish identically, but only that the wave vectors are eigenvectors of the constraints, belonging to the eigenvalue zero. This alternative is not entirely satisfactory, either. If the constraints have nonvanishing Poisson brackets, as they do in our case [see Eq. (2.12)], then the requirement that the state vector is a null vector of both of the constraints involved leads to the self-contradictory requirement that it also be a null vector of the commutator. Similar though different difficulties arise when the constraints have c-number commutators with other field variables, a situation that arises in electrodynamics.

Dirac[1] has suggested a modification of the usual quantization procedures that can be applied to theories incorporating constraints. He classifies constraints (within the c-number theory) into two categories, those that have vanishing Poisson brackets with all other constraints ("first-class constraints") and those that do not ("second-class constraints"). By replacing a given set of constraints by a new set of linear combinations, he first maximizes the number of first-class constraints. The remaining second-class constraints, say R in number (R in our case equals the rank of the matrix f^{AB}) will then have a set of Poisson brackets whose determinant does not vanish. He then introduces a new type of bracket symbol, which is now usually called a Dirac bracket, defined in terms of the Poisson bracket (M,N) as follows:

$$\{M,N\}=(M,N)-\int\int(M,\theta^{\alpha}(x'))F_{\alpha\beta}(x',x'')\times(\theta^{\beta}(x''),N)d^3x'd^3x''. \quad (2.13)$$

The quantities θ^{α} represent the totality of all new second-class constraints, and have Poisson brackets

$$(\theta^{\alpha}(x),\theta^{\beta}(x'))=\Phi^{\alpha\beta}(x,x'),\quad \Phi^{\alpha\beta}(x,x')=-\Phi^{\beta\alpha}(x',x) \quad (2.14)$$

and $F_{\alpha\beta}$ is the reciprocal to the $\Phi^{\alpha\beta}$,

$$\int\Phi^{\alpha\beta}(x',x)F_{\beta\gamma}(x,x'')d^3x=\delta(x'-x'')\delta_{\gamma}{}^{\alpha}. \quad (2.15)$$

By direct computation, Dirac has shown that his bracket satisfies the Jacobi relationship, i.e.,

$$\{A,\{B,C\}\}+\{B,\{C,A\}\}+\{C,\{A,B\}\}\equiv 0. \quad (2.16)$$

This fact alone suggests that the Dirac bracket represents a group-theoretical commutator, just as the ordinary Poisson bracket does.[2] At any rate, inasmuch as the Dirac bracket between any θ^{α} and any dynamical variable vanishes identically, it is possible to set the θ^{α} identically zero, and thereby replace the whole phase space of $2N$ dimensions (per space point) by the $(2N-R)$-dimensional hypersurface $\theta^{\alpha}=0$. On this hypersurface, our theory will still have $(N-R)$ first-class constraints (per space point). The quantization procedure applies to this reduced theory; Dirac brackets, rather than Poisson brackets, are related to the commutators of the quantum-mechanical observables. Dirac has also shown that for the correct Hamiltonian

$$\dot{y}_A=(y_A,H)=\{y_A,H\},\quad \dot{\pi}^A=(\pi_A,H)=\{\pi^A,H\}. \quad (2.17)$$

In other words, the Dirac bracket is capable of taking over completely the role that is played by Poisson brackets in theories without second class constraints.

The field equations of Lagrangians linear in the velocities belong to one of three main types.

(1) They may determine the "velocities" $\dot{y}_A$ uniquely and algebraically as functions of y_A and $y_{A,s}$ (Sec. 3).

(2) It may be impossible to determine the "velocities" uniquely (Sec. 4).

(3) We may be able to derive conditions of compatibility, which, together with the original field equations, determine the "velocities" (Sec. 5).

3. "VELOCITIES" DETERMINED BY EULER EQUATIONS

The first case is characterized by $\det|f^{AB}|\neq 0$. In this case, the Hamiltonian H is a uniquely defined function of the y_A, $y_{A,s}$ and π^A. Since the rank of f^{AB} is N, all constraints and all the π^A's can be eliminated by means of Dirac's method. Wherever a quantity π^A occurs we replace it by the corresponding f^A. Thus the Hamiltonian reduces to $\mathcal{H}=-Q$. The canonical equations are now obtained in terms of the Dirac brackets. We shall derive some properties of the Dirac bracket for this case.

Choosing for θ^{α} all the C^A's and thus making $\Phi^{\alpha\beta}$ equal to $f^{AB}\delta(x-x')$, and defining a new set of quantities f_{AB} by

$$f^{AB}f_{BC}=\delta_C{}^A, \quad (3.1)$$

[2] P. G. Bergmann, and I. Goldberg, Phys. Rev. 98, 531 (1955).

we get from Eq. (2.13)

$$\{y_A(x),y_B(x')\}$$
$$=(y_A(x'),y_B(x'))-\int\int(y_A(x),C^D(x''))$$
$$\times f_{DE}(x'',x''')\delta(x''-x''')(C^E(x'''),y_B(x'))d^3x''d^3x'''$$
$$=\delta_A{}^D f_{DE}\delta_B{}^E\delta(x-x')=f_{AB}\delta(x-x'). \quad (3.2)$$

We can write the Dirac bracket between any two functionals of the y_A and $y_{A,s}$ by using Eq. (3.2). The result is

$$\{M,N\}=\int\frac{\delta M}{\delta y_A}f_{AB}\frac{\delta N}{\delta y_B}d^3x. \quad (3.3)$$

If we substitute $N=-\int Qd^3x=H$ and $M=y_A$ followed by $M=f_A$, we get from Eqs. (2.17) and (3.3),

$$\dot{y}_A=-\delta^B Q f_{AB} \quad \text{or} \quad \dot{y}_A+\delta^B Q f_{AB}=0, \quad (3.4)$$

and

$$\dot{f}_C=\partial^A f^C \dot{y}_A=\{f^C,H\}=-\partial^A f^C\delta^B Q f_{AB}$$

or

$$(\dot{y}_A+\delta^B Q f_{AB})\partial^A f^C=0.$$

The first of these is exactly equal to the Euler equations when we multiply by f^{CA}.

The quantization procedure is now as follows. Instead of relating the Poisson bracket to the commutator, we relate the Dirac bracket. Therefore the operators y_A satisfy the commutator relation

$$[y_A(x),y_B(x')]=i\hbar f_{AB}\delta(x-x'). \quad (3.5)$$

The Schrödinger equation is

$$i\hbar\frac{\partial\Psi}{\partial t}=-\int Qd^3x\Psi. \quad (3.6)$$

Terms in Q must be ordered so as to make Q Hermitian. No mention need be made of either the constraints or the momentum densities.

Although this formalism can be applied among other examples to the Pauli-Fierz[3] equations, we will illustrate it with a much simpler example. Consider the Lagrangian for the Schrödinger equation, the ψ and ψ^* taken as classical field variables,

$$L=\tfrac{1}{2}i\hbar(\psi^*\dot{\psi}-\psi\dot{\psi}^*)-\tfrac{1}{2}(\hbar^2/m)\psi^*{}_{,s}\psi_{,s}. \quad (3.7)$$

Hence,

$$f^{AB}=i\hbar\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

and

$$f_{AB}=\frac{1}{i\hbar}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

From Eq. (3.5), we see that $[\psi(x),\psi^*(x')]=\delta(x-x')$, the usual commutator in the second-quantized theory (if the particles are bosons).

[3] W. Pauli and M. Fierz, Helv. Phys. Acta 12, 297 (1939).

4. UNDETERMINED VELOCITIES

The more difficult and mathematically more interesting case is that of $\det|f^{AB}|=0$. A general solution to Eq. (2.8) is

$$\mu_A=\bar{\mu}_A+\xi^i u_{iA}, \quad (4.1)$$

where ξ^i are a set of functions of the y_A, $y_{A,s}$ and x^α, for the moment unspecified, with i going from 1 to $(N-R)$ (R is the rank of the matrix f^{AB}); u_{iA} are a complete set of linearly independent vectors satisfying the homogeneous equation $f^{AB}u_{iB}=0$, and $\bar{\mu}_A$ is any particular solution of Eq. (2.8). $\bar{\mu}_A$ is then a known algebraic function of y_A and $y_{A,s}$.

If we multiply Eq. (2.2) by u_{iA}, we see that the following $(N-R)$ conditions must be satisfied if solutions of Eq. (2.8) (or the Euler equations) are to exist:

$$u_{iA}\delta^A Q=0. \quad (4.2)$$

These conditions may be satisfied identically, in analogy to the Bianchi identities of general relativity. But it may also happen, instead, that these conditions restrict solutions of our Euler equations to hypersurfaces in configuration or phase space. They contain only the field variables y_A and their spatial derivatives, $y_{A,s}$. We call constraints of this type secondary constraints and label them

$$C_i=u_{iA}\delta^A Q. \quad (4.3)$$

If Eqs. (4.2) are identities, we shall write them with an identity sign, otherwise with an ordinary equality sign. It may be that in a particular problem both types of vector u_{iA} [those that satisfy Eq. (4.2) identically and those that impose constraints] will appear.

We consider first the case of all the C_i vanishing identically, i.e. $u_{iA}\delta^A Q\equiv 0$ for all i. The ξ^i of Eq. (4.1) are completely arbitrary functions. The equation $u_{iA}\delta^A Q\equiv 0$ is empty and we cannot derive additional conditions for the ξ^i, as we can when $u_{iA}\delta^A Q=0$. (We shall study the use of the secondary constraints for determining the ξ^i in the next section.) The Hamiltonian for the present problem is

$$\mathcal{H}=\bar{\mu}_A C^A+\xi^i u_{iA}C^A-Q. \quad (4.4)$$

We can better understand the appearance of the arbitrary ξ^i in the canonical formalism if we use $\xi^i u_{iA}C^A=C$ as the generator of an infinitesimal canonical transformation,

$$\delta y_A=\partial C/\partial\pi^A=\xi^i u_{iA}. \quad (4.5)$$

The functional change in the Lagrangian is given by[4]

$$\delta' L=-C^\rho{}_{,\rho}-L^A\delta y_A \quad (4.6)$$

where the C^ρ is arbitrary and the L^A are the field equations. Putting Eq. (4.5) into Eq. (4.6) and noting that

$$L^A\delta y_A=L^A\xi^i u_{iA}=(f^{AB}\dot{y}_B+\delta^A Q)u_{iA}\xi^i\equiv 0,$$

[4] P. G. Bergmann and R. Schiller, Phys. Rev. 89, 4 (1953).

we see that $\delta' L = -C^{\rho}{}_{,\rho}$. This is the condition for a continuous group of invariant transformations which depend on the arbitrary functions ξ^i. In other words, for a solution y_A of the field equations satisfying certain initial conditions, $y_A' = y_A + \xi^i u_{iA}$ is also a solution satisfying the same initial conditions if ξ^i and its spatial derivatives vanish at $t=0$. The choice of a particular set of ξ^i is equivalent in Maxwell theory to a particular gauge frame and in general relativity to particular coordinate conditions.

In order to quantize this type of theory by Dirac's procedure, we must maximize the number of first class constraints. As was explained in Sec. 2, this is done by taking linear combinations of the old constraints, the coefficients being the coefficients of the orthogonal matrix which borders the singular matrix f^{AB}. Since f^{AB} is of rank R, there will be R second-class constraints and $(N-R)$ first-class constraints. We use these R new second-class constraints to form the Dirac bracket and to eliminate R variables from the theory. The quantum-mechanical commutation relations are associated with the Dirac bracket rather than with the Poisson bracket, and the $(N-R)$ first-class constraints C^{*a} give rise to $(N-R)$ Schrödinger equations:

$$C^{*\alpha}\Psi = 0, \tag{4.7}$$

in addition to the usual one:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\int Q d^3x \Psi. \tag{4.8}$$

5. COMPATIBILITY CONDITIONS WHEN $\det|f^{AB}| = 0$

When Eq. (4.2) is not an identity (i.e., when we have secondary constraints) there are so many subcases that it has been difficult for us to give a systematic analysis of it. The simplest way to see the various possibilities is to consider a theory with a finite number of degrees of freedom (i.e., a problem in mechanics) rather than one with an infinite number of degrees of freedom (a field). The Lagrangian analogous to Eq. (2.1) is

$$L = f^{\alpha}(x)\dot{x}_{\alpha} + Q(x). \tag{5.1}$$

The Euler equations are

$$L^{\alpha} = f^{\alpha\beta}\dot{x}_{\beta} + \partial^{\alpha}Q = 0, \tag{5.2}$$

$$f^{\alpha\beta} \equiv \frac{\partial f^{\beta}}{\partial x_{\alpha}} - \frac{\partial f^{\alpha}}{\partial x_{\beta}}, \quad \partial^{\alpha}Q \equiv \frac{\partial Q}{\partial x_{\alpha}}.$$

The case we are considering now is that of $\det|f^{\alpha\beta}| = 0$ and $u_{i\alpha}\partial^{\alpha}Q = 0$ where $u_{i\alpha}$ satisfies the homogeneous equation $f^{\alpha\beta}u_{i\beta} = 0$. For compatability the time derivative of the secondary constraint, $C_i = u_{i\alpha}\partial^{\alpha}Q$, must vanish. These time derivatives, which are homogeneous in the velocities, have the form

$$\partial^{\beta}(u_{i\alpha}\partial^{\alpha}Q)\dot{x}_{\beta} = 0. \tag{5.3}$$

These equations (5.3), together with the original Euler equations, form an extended set of field equations. Three possibilities may arise: (a) There may exist no algebraic solutions of these equations for the velocities, in this case the original Euler equations are incompatible; (b) Eqs. (5.3), (2.2) may uniquely determine the velocities in terms of the x's, so that our equations are of the Cauchy type (i.e., we have an initial value problem); (c) algebraic solutions may exist without being unique. In the last case there may be linear combinations of the extended field equations that do not contain velocities, different from the ones we have already obtained, $u_{i\alpha}\partial^{\alpha}Q = 0$. We must then extend the field equations further by taking their time derivatives. This procedure is repeated until the velocities are uniquely determined by all the extended field equations or when the last linear combination of field equations free of velocities is identically zero. In the latter case, there will appear in the Hamiltonian an arbitrary function, and our theory will have similar invariance properties as those of the previous section. The quantization of problems of this type must be handled individually; no general procedure can be set up because of the extreme complexity of the problem.

When the velocities are uniquely determined by the extended field equations, the Hamiltonian is also a unique function. However, to the canonical equations of motion we must adjoint not only the primary constraints, $C^{\beta} = p^{\beta} - f^{\beta} = 0$ but also the secondary constraints, i.e., all the independent linear combinations of the extended field equations that are free of velocities. In order to apply Dirac's procedure for quantization, we must form the matrix whose elements are the Poisson brackets between all the constraints, primary and secondary. It will be of order $N+M$, where N is the number of degrees of freedom of the mechanical system and M is the number of secondary constraints. The matrix will be of the form

$$F = \begin{pmatrix} f^{\alpha\beta} & (C_i, C^{\beta}) \\ (C^{\alpha}, C_i) & 0 \end{pmatrix}. \tag{5.4}$$

The terms in the lower right-hand corner are zero because the C_i contain only configuration space variables. In general the rank of F will be less than $N+M$, though there are interesting cases for which it is equal to $N+M$. In the latter case, all the constraints are, in Dirac's terminology, second-class constraints. The Dirac bracket is formed with these constraints and $N+M$ variables are removed from the canonical formalism. The quantum-mechanical commutation relations are then associated with the Dirac bracket and we have only one Schrödinger equation because there are no first-class constraints.

If the rank, R, is less than $N+M$, we must take a linear combination of all the constraints and maximize the number of first-class constraints, as was done in the previous section. The Dirac bracket is then constructed from the new second-class constraints. When we quantize, there will be $(N+M-R)$ first-class constraints that will go over into Schrödinger equations in addition to the usual one, Eqs. (4.7) and (4.8).

To help clarify the procedures outlined in this section, we will apply them to a simple example. Consider the Lagrangian density L with five independent field variables u and u_α ($\alpha=1, 2, 3, 4$):

$$\begin{aligned} L&=-u_\alpha u_{,\alpha}+\tfrac{1}{2}u_\alpha{}^2-\tfrac{1}{2}\kappa^2u^2 \\ &=u_4\dot{u}-u_su_{,s}+\tfrac{1}{2}u_s{}^2-\tfrac{1}{2}u_4{}^2-\tfrac{1}{2}\kappa^2u^2, \\ c&=1,\quad \kappa=(m/\hbar)^2,\quad u_{,\alpha}\equiv\partial u/\partial x^\alpha,\quad s=1,2,3. \end{aligned} \tag{5.5}$$

The Euler equations are

$$\dot{u}_4-u_{s,s}+\kappa^2u=0,\quad u_s=u_{,s},\quad \dot{u}=u_4. \tag{5.6}$$

These two sets of equations are obviously equivalent to the Klein-Gordon equation, but the Lagrangian is in such a form that we can apply our procedure. A simple calculation shows that there are only three secondary constraints,

$$C_s\equiv u_s-u_{,s}=0. \tag{5.7}$$

The time derivatives of these constraints,

$$\dot{u}_s-\dot{u}_{,s}=0 \tag{5.8}$$

plus the Euler equations, allow us to solve for the velocities, though the Euler equations by themselves are insufficient to determine $\dot{u}_s$:

$$\dot{u}_4=u_{s,s}-\kappa^2u,\quad \dot{u}_s=u_{4,s},\quad \dot{u}=u_4. \tag{5.9}$$

Since the momenta are defined as

$$\pi^\alpha=\partial L/\partial\dot{u}_\alpha=0,\quad \pi=\partial L/\partial\dot{u}=u_4, \tag{5.10}$$

we obtain the five primary constraints:

$$C^\alpha=\pi^\alpha=0,\quad C=\pi-u_4=0. \tag{5.11}$$

The matrix, whose elements are the Poisson brackets of all the 8 constraints, (5.7) and (5.11), is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.12}$$

Since its rank is eight, all the constraints are second-class, and eight variables can be eliminated. The most convenient choice for elimination is u_α and π^α. The Hamiltonian density,

$$\begin{aligned} \mathcal{H}&=\dot{u}_4\pi^4+\dot{u}_s\pi^s+\dot{u}\pi-L \\ &=\dot{u}_4C^4+\dot{u}_sC^s+\dot{u}C+u_{,s}u_s-\tfrac{1}{2}(u_s{}^2-u_4{}^2-\kappa^2u^2), \end{aligned} \tag{5.15}$$

then reduces to

$$\mathcal{H}=\tfrac{1}{2}(\pi^2+u_{1s}{}^2+\kappa^2u^2). \tag{5.16}$$

The Dirac bracket between any two functionals becomes simply

$$\{M,N\}=\int\left(\frac{\delta M}{\delta u}\frac{\delta N}{\delta\pi}-\frac{\delta M}{\delta\pi}\frac{\delta N}{\delta u}\right)d^3x. \tag{5.17}$$

We see that our procedure reduces to the usual quantization procedure for the Klein-Gordon equation.

6. CONCLUSION

Theories that are linear in the derivatives of the field variables lend themselves particularly well to a detailed examination of the Dirac bracket formalism and its applicability to the problem of quantization, because of the comparatively lesser formal complexity of such theories. Despite some simplification, there remain both fundamental and computational difficulties that must be overcome before essentially nonlinear theories (such as the general theory of relativity) can be quantized successfully with the help of the formalism described in this paper.

The principal fundamental difficulty is the formulation of rules that will permit the ordering of non-commuting factors in the Hamiltonian. Presently it appears that the requirements of general covariance will greatly restrict the freedom of choice in this respect, but without leading to a unique determination; we have not yet succeeded in clearing this question up definitely.

The investigation of this fundamental question is closely related to the computational difficulty involved in handling very large systems of simultaneous algebraic equations. In general relativity, in the Palatini formulation, there are altogether 50 field variables, 10 components of the metric tensors and 40 components of the affine connection. Though some degree of separation of the system of equations that determines the null vectors of the matrix f^{AB} is possible, the remaining problem is still formidable. We hope that these difficulties will be overcome in the near future.