

Reduction of Relativistic Two-Particle Wave Equations to Approximate Forms. III*

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The interaction between two fermions of charge ϵ_I and ϵ_{II} and intrinsic magnetic moment μ_I and μ_{II} is described by a sixteen-component wave equation of the Breit type. The method of reduction of two-particle wave equations, as given in two previous papers by Chraplyvy, is used to convert this equation to an approximate four-component Pauli equation. A simple perturbation calculation is used to determine the contribution of the intrinsic magnetic moments to the fine and hyperfine structure of hydrogen and positronium.

1. INTRODUCTION

IN two previous papers,¹ referred to hereafter as I and II, the Foldy-Wouthuysen canonical transformation² was generalized by Chraplyvy to the two-body problem both for the singular case of equal masses as well as for the case of unequal masses. We consider here several applications using, except where noted, the terminology and notation of I and II.

Four-component one-body relativistic equations of the Dirac type or sixteen-component two-body equations of the Breit type may now be reduced to two- or four-component approximate equations of the Pauli type by the Foldy-Wouthuysen method or by the procedure of expressing the small components of the spinor ψ in terms of its large components.³ In Sec. 2, the differences between these methods are brought out in a discussion of a familiar one-body example which we shall have occasion to refer to later as the limiting case of a two-body problem.

In Sec. 3, we compare the Hermitian part of a three-dimensional Bethe-Salpeter equation with the corresponding Breit equation by applying the two-body transformation of I to both.

In Secs. 4 and 5, the Breit interaction between two fermions of charge ϵ_I and ϵ_{II} is amplified to include intrinsic magnetic moment and virtual annihilation terms. The resulting equations are converted to the Pauli representation by means of the two-body transformation. It is then a simple matter to calculate the contribution of the intrinsic magnetic moments to the fine and hyperfine structure of hydrogen and positronium.

2. THE HYDROGEN ATOM: ONE-BODY TREATMENT

Essentially the same differences exist between the results of the Foldy-Wouthuysen and the "large com-

ponent" methods of reduction when applied to either the one or the two-body problem. Although these differences have been discussed qualitatively before,^{1,2} a simple quantitative comparison may be of interest. We consider the hydrogen atom, approximated as a one-body problem.

The exact solution for the energy E of the Dirac equation⁴ for hydrogen,

$$(\beta m + \alpha \cdot \mathbf{p} - e^2/r)\psi = i\partial\psi/\partial t, \quad (1)$$

may be expanded in ascending powers of the fine structure constant α .

$$E = m - \frac{m\alpha^2}{2n^2} - \frac{m\alpha^4}{2n^4} \left[\frac{n}{j + \frac{1}{2}} - \frac{3}{4} \right] + \dots \quad (2)$$

If the Foldy-Wouthuysen transformation is applied to Eq. (1), one obtains

$$H'\psi' = \left\{ \beta m + \frac{\beta p^2}{2m} - \frac{e^2}{r} - \frac{\beta p^4}{8m^3} + \frac{\pi e^2}{2m^2} \delta(\mathbf{r}) + \frac{e^2}{8m^2} \boldsymbol{\sigma} \cdot \left(\frac{\mathbf{r}}{r^3} \times \mathbf{p} - \mathbf{p} \times \frac{\mathbf{r}}{r^3} \right) + \dots \right\} \psi', \quad (3)$$

where H' is obviously Hermitian and ψ' is still a four-component spinor. This equation may be separated into two two-component equations referring to positive and

TABLE I. One-body relativistic energy level corrections in Eq. (3).

Operator	Expectation value
$-\frac{p^4}{8m^3}$	$-\frac{m\alpha^4}{2n^4} \left(\frac{n}{l + \frac{1}{2}} - \frac{3}{4} \right)$
$\frac{\pi e^2}{2m^2} \delta(\mathbf{r})$	$\frac{m\alpha^4}{2n^3} \delta_{0l}$
$\frac{e^2}{4m^2} \boldsymbol{\sigma} \cdot \frac{\mathbf{r}}{r^3} \times \mathbf{p}$	$\frac{m\alpha^4}{2n^3} \frac{1}{(2l+1)(l+1)} \quad j = l + \frac{1}{2}$ $l \neq 0$
	$-\frac{m\alpha^4}{2n^3} \frac{1}{l(2l+1)} \quad j = l - \frac{1}{2}$

* Units are chosen so that $\hbar = c = 1$.

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¹ Z. V. Chraplyvy, Phys. Rev. **91**, 388 and **92**, 1310 (1953).

² L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

³ Other reduction methods exist. See R. A. Ferrell, thesis, Princeton University, Princeton, New Jersey, 1951 (unpublished).

negative energy states respectively by the simple prescription of replacing β by $+1$ or -1 . The resulting equation for the electron in positive energy states is just the Schrödinger equation for the hydrogen atom plus relativistic correction terms whose contribution to the energy may be readily calculated by perturbation theory.⁵ The three relativistic energy level corrections given in Table I combine to give the $m\alpha^4$ term in the expansion of the Sommerfeld fine structure formula, Eq. (2).

If, on the other hand, one uses the "large component" method⁶ to reduce Eq. (1), one obtains

$$\left\{ \frac{p^2}{2m} - \frac{e^2}{r} - \frac{p^4}{8m^3} + \frac{\pi e^2}{m^2} \delta(\mathbf{r}) + \frac{ie^2}{4m^2} \frac{\mathbf{r}}{r^3} \cdot \mathbf{p} + \frac{e^2}{4m^2} \frac{\mathbf{r}}{r^3} \cdot \mathbf{p} + \frac{ie^2}{4m^2} \frac{\mathbf{r}}{r^3} \cdot \nabla \times \frac{\mathbf{r}}{r^3} \right\} \chi = i \frac{\partial \chi}{\partial t}. \quad (4)$$

Clearly the "Hamiltonian" operating on the two large components χ is not Hermitian. Comparing Eqs. (4) and (3) for positive energy states we find that the kinetic energy relativistic correction and spin-orbit terms appear in both, but instead of one term affecting the S states there are now two. However, since

$$\left\langle \frac{\pi e^2}{m^2} \delta(\mathbf{r}) + \frac{ie^2}{4m^2} \frac{\mathbf{r}}{r^3} \cdot \mathbf{p} \right\rangle = \left\langle \frac{\pi e^2}{2m^2} \delta(\mathbf{r}) \right\rangle = \frac{m\alpha^4}{2n^3} \delta_{0l}, \quad (5)$$

where $\langle A \rangle$ denotes the expectation value of A , we see that the relativistic energy level corrections again add to give exactly the $m\alpha^4$ term in Eq. (2).

In the two-body problem, there is no exact solution available by which to check approximate solutions. One would again expect the two reduction methods to give the same result for the energy levels. Given a choice, however, certainly the method which leads to a Hermitian Hamiltonian is more satisfactory.

3. TWO-BODY EQUATIONS

In I, Chraplyvy has shown that a two-body Hamiltonian written in the form

$$H = \beta^I m_I + \beta^{II} m_{II} + (\mathcal{E}\mathcal{E}) + (\mathcal{O}\mathcal{E}) + (\mathcal{E}\mathcal{O}) + (\mathcal{O}\mathcal{O}), \quad (6)$$

where $(\mathcal{E}\mathcal{E})$, $(\mathcal{E}\mathcal{O})$, $(\mathcal{O}\mathcal{E})$, and $(\mathcal{O}\mathcal{O})$ represent even-even, even-odd, odd-even, and odd-odd terms, may be transformed into an even-even operator⁷

⁵ W. A. Barker and Z. V. Chraplyvy, Phys. Rev. **89**, 446 (1953). See also E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, New York, 1951).

⁶ See, for example, E. L. Hill and R. Landshoff, Revs. Modern Phys. **10**, 87 (1938).

⁷ See I, Eq. (7). The additional terms in (7) which arise if $(\mathcal{O}\mathcal{E})$ and $(\mathcal{E}\mathcal{O})$ do not commute make no contribution to the desired accuracy in any of the cases treated in this paper.

$$\begin{aligned} H_{TR} = & \beta^I m_I + \beta^{II} m_{II} + (\mathcal{E}\mathcal{E}) + \frac{\beta^I}{2m_I} (\mathcal{O}\mathcal{E})^2 \\ & + \frac{\beta^{II}}{2m_{II}} (\mathcal{E}\mathcal{O})^2 - \frac{\beta^I}{8m_I^3} (\mathcal{O}\mathcal{E})^4 - \frac{\beta^{II}}{8m_{II}^3} (\mathcal{E}\mathcal{O})^4 \\ & + \frac{1}{8m_I^2} [[(\mathcal{O}\mathcal{E}), (\mathcal{E}\mathcal{E})], (\mathcal{O}\mathcal{E})] \\ & + \frac{1}{8m_{II}^2} [[(\mathcal{E}\mathcal{O}), (\mathcal{E}\mathcal{E})], (\mathcal{E}\mathcal{O})] \\ & + \frac{\beta^I \beta^{II}}{4m_I m_{II}} [[(\mathcal{O}\mathcal{E}), (\mathcal{O}\mathcal{O})]_+, (\mathcal{E}\mathcal{O})]_+ \\ & + \frac{\beta^I m_I - \beta^{II} m_{II}}{2(m_I^2 - m_{II}^2)} (\mathcal{O}\mathcal{O})^2 + \dots \quad (7) \end{aligned}$$

If $(\mathcal{E}\mathcal{E})$, $(\mathcal{E}\mathcal{O})$, $(\mathcal{O}\mathcal{E})$, and $(\mathcal{O}\mathcal{O})$ are Hermitian, it is evident that H_{TR} is Hermitian.

The Hermitian part of the three-dimensional Bethe-Salpeter equation⁸ written in coordinate space is

$$H\Psi(\mathbf{r}) = (H_I + H_{II} + \frac{1}{2} [\Lambda_+^I \Lambda_+^{II} - \Lambda_-^I \Lambda_-^{II}], U_B(\mathbf{r})]_+) \times \Psi(\mathbf{r}) = E\Psi(\mathbf{r}), \quad (8)$$

where

$$\begin{aligned} H_I & \equiv \beta^I m_I + \alpha^I \cdot \mathbf{p}, & H_{II} & \equiv \beta^{II} m_{II} - \alpha^{II} \cdot \mathbf{p}, \\ \Lambda_{\pm}^I & \equiv \frac{E_I \pm H_I}{2E_I}, & \Lambda_{\pm}^{II} & \equiv \frac{E_{II} \pm H_{II}}{2E_{II}}, \\ E_I & \equiv +(m_I^2 + p^2)^{\frac{1}{2}}, & E_{II} & \equiv +(m_{II}^2 + p^2)^{\frac{1}{2}}, \end{aligned} \quad (9)$$

$$U_B(\mathbf{r}) \equiv \frac{\epsilon_I \epsilon_{II}}{r} \left(1 - \frac{\alpha^I \cdot \alpha^{II}}{2} - \frac{(\alpha^I \cdot \mathbf{r})(\alpha^{II} \cdot \mathbf{r})}{2r^2} \right).$$

If we replace the Casimir projection operators in the curly brackets in Eq. (8) by 1, we obtain the well-known Breit equation.

$$\begin{aligned} H\Psi(\mathbf{r}) = & \left\{ \beta^I m_I + \alpha^I \cdot \mathbf{p} + \beta^{II} m_{II} - \alpha^{II} \cdot \mathbf{p} \right. \\ & \left. + \frac{\epsilon_I \epsilon_{II}}{r} \left(1 - \frac{\alpha^I \cdot \alpha^{II}}{2} - \frac{(\alpha^I \cdot \mathbf{r})(\alpha^{II} \cdot \mathbf{r})}{2r^2} \right) \right\} \\ & \times \Psi(\mathbf{r}) = E\Psi(\mathbf{r}). \quad (10) \end{aligned}$$

Equation (10) is in the form of Eq. (6) with

$$(\mathcal{E}\mathcal{E}) = \epsilon_I \epsilon_{II} / r, \quad (\mathcal{O}\mathcal{E}) = \alpha^I \cdot \mathbf{p}, \quad (\mathcal{E}\mathcal{O}) = -\alpha^{II} \cdot \mathbf{p}, \quad (11)$$

and

$$(\mathcal{O}\mathcal{O}) = -\frac{\epsilon_I \epsilon_{II}}{r} \left(\frac{\alpha^I \cdot \alpha^{II}}{2} + \frac{(\alpha^I \cdot \mathbf{r})(\alpha^{II} \cdot \mathbf{r})}{2r^2} \right).$$

In order to write Eq. (8) in the form of Eq. (6), we evaluate the anticommutator, retaining only those terms in the expansion which contribute to the desired accuracy. The result is

⁸ E. E. Salpeter, Phys. Rev. **87**, 328 (1952).

$$(\mathcal{E}\mathcal{E}) = \frac{\epsilon_I \epsilon_{II}}{2r} (\beta^I + \beta^{II}) - \frac{\epsilon_I \epsilon_{II}}{8} \left(\frac{\beta^I}{m_I^2} + \frac{\beta^{II}}{m_{II}^2} \right) \left[p^2, \frac{1}{r} \right]_+, \quad (12a)$$

$$(\mathcal{O}\mathcal{E}) = \alpha^I \cdot \mathbf{p} + \epsilon_I \epsilon_{II} \left(\frac{1}{4m_I} + \frac{1}{8m_{II}} \right) \times \left[\alpha^I \cdot \mathbf{p}, \frac{1}{r} \right]_+ + \frac{i\epsilon_I \epsilon_{II}}{8m_{II}} \alpha^I \times \sigma^{II} \cdot \left[\mathbf{p}, \frac{1}{r} \right]_+ + \frac{\epsilon_I \epsilon_{II}}{8m_{II}} \times \left[\mathbf{p} \cdot, \frac{\mathbf{r}(\alpha^I \cdot \mathbf{r})}{r^3} \right]_+ + \frac{i\epsilon_I \epsilon_{II}}{8m_{II}} \sigma^{II} \cdot \left[\mathbf{p} \times, \frac{\mathbf{r}(\alpha^I \cdot \mathbf{r})}{r^3} \right]_+, \quad (12b)$$

$$(\mathcal{E}\mathcal{O}) = -\alpha_{II} \cdot \mathbf{p} - \epsilon_I \epsilon_{II} \left(\frac{1}{4m_{II}} + \frac{1}{8m_I} \right) \left[\alpha^{II} \cdot \mathbf{p}, \frac{1}{r} \right]_+ - \frac{i\epsilon_I \epsilon_{II}}{8m_I} \alpha^{II} \times \sigma^I \cdot \left[\mathbf{p}, \frac{1}{r} \right]_+ - \frac{\epsilon_I \epsilon_{II}}{8m_I} \left[\mathbf{p} \cdot, \frac{\mathbf{r}(\alpha^{II} \cdot \mathbf{r})}{r^3} \right]_+ - \frac{i\epsilon_I \epsilon_{II}}{8m_I} \sigma^I \cdot \left[\mathbf{p} \times, \frac{\mathbf{r}(\alpha^{II} \cdot \mathbf{r})}{r^3} \right]_+. \quad (12c)$$

Here

$[\mathbf{A} \cdot, \mathbf{B}]_+ \equiv \mathbf{A} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{A}$ and $[\mathbf{A} \times, \mathbf{B}]_+ \equiv \mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{A}$. The elimination of a leading $(\mathcal{O}\mathcal{O})$ term⁹ is a consequence of

$$[\beta^I, \alpha^I]_+ = [\beta^{II}, \alpha^{II}]_+ = 0. \quad (13)$$

The $(\mathcal{O}\mathcal{O})$ terms in the Breit interaction, $U_B(\mathbf{r})$, are changed into $(\mathcal{E}\mathcal{O})$ and $(\mathcal{O}\mathcal{E})$ terms respectively when acted on by $\alpha^I \cdot \mathbf{p}$ and $\alpha^{II} \cdot \mathbf{p}$ in (8). Expressions (11) and (12) are manifestly Hermitian.

We may now apply the prescription (7) and transform the Hamiltonians in (8) and (10) into even-even operators. Neglecting contributions of order $m\alpha^6$ and higher the result for (8) is

$$(H_{TR}) = \beta^I m_I + \beta^{II} m_{II} + \frac{\beta^I p^2}{2m_I} + \frac{\beta^{II} p^2}{2m_{II}} - \frac{\beta^I p^4}{8m_I^3} - \frac{\beta^{II} p^4}{8m_{II}^3} + (\beta^I + \beta^{II}) \frac{\epsilon_I \epsilon_{II}}{2r} - (\beta^I + \beta^{II}) \frac{\pi \epsilon_I \epsilon_{II} \delta(\mathbf{r})}{4} \times \left(\frac{1}{m_I^2} + \frac{1}{m_{II}^2} \right) - (\beta^I + \beta^{II}) \frac{\epsilon_I \epsilon_{II}}{8r^3} \times \left(\frac{\sigma^I}{m_I^2} + \frac{\sigma^{II}}{m_{II}^2} \right) \cdot (\mathbf{r} \times \mathbf{p}) + (\beta^I + \beta^{II}) \frac{\epsilon_I \epsilon_{II}}{4m_I m_{II}} \times \left\{ \frac{1}{r} p^2 + \mathbf{r} \left(\frac{\mathbf{r}}{r^3} \cdot \mathbf{p} \right) \cdot \mathbf{p} - \frac{1}{r^3} (\sigma^I + \sigma^{II}) \cdot (\mathbf{r} \times \mathbf{p}) \right\} + (\beta^I + \beta^{II}) \frac{\epsilon_I \epsilon_{II}}{8m_I m_{II}} \left\{ \frac{\sigma^I \cdot \sigma^{II}}{r^3} - \frac{3(\sigma^I \cdot \mathbf{r})(\sigma^{II} \cdot \mathbf{r})}{r^5} - \frac{8\pi}{3} \sigma^I \cdot \sigma^{II} \delta(\mathbf{r}) \right\} + \dots \quad (14)$$

⁹ There are $(\mathcal{O}\mathcal{O})$ terms which contribute to higher order.

On the other hand, one obtains the following transformed Hamiltonian¹⁰ from the Breit equation (10):

$$(H_{TR}) = \beta^I m_I + \beta^{II} m_{II} + \frac{\beta^I p^2}{2m_I} + \frac{\beta^{II} p^2}{2m_{II}} - \frac{\beta^I p^4}{8m_I^3} - \frac{\beta^{II} p^4}{8m_{II}^3} + \frac{\epsilon_I \epsilon_{II}}{r} - \frac{\pi \epsilon_I \epsilon_{II} \delta(\mathbf{r})}{2} \left(\frac{1}{m_I^2} + \frac{1}{m_{II}^2} \right) - \frac{\epsilon_I \epsilon_{II}}{4r^3} \times \left(\frac{\sigma^I}{m_I^2} + \frac{\sigma^{II}}{m_{II}^2} \right) \cdot (\mathbf{r} \times \mathbf{p}) + \frac{\beta^I \beta^{II} \epsilon_I \epsilon_{II}}{2m_I m_{II}} \times \left\{ \frac{1}{r} p^2 + \mathbf{r} \left(\frac{\mathbf{r}}{r^3} \cdot \mathbf{p} \right) \cdot \mathbf{p} - \frac{1}{r^3} (\sigma^I + \sigma^{II}) \cdot (\mathbf{r} \times \mathbf{p}) \right\} + \frac{\beta^I \beta^{II} \epsilon_I \epsilon_{II}}{4m_I m_{II}} \left\{ \frac{\sigma^I \cdot \sigma^{II}}{r^3} - \frac{3(\sigma^I \cdot \mathbf{r})(\sigma^{II} \cdot \mathbf{r})}{r^5} - \frac{8\pi}{3} \sigma^I \cdot \sigma^{II} \delta(\mathbf{r}) \right\} + \frac{\beta^I m_I - \beta^{II} m_{II}}{2(m_I^2 - m_{II}^2)} \epsilon_I^2 \epsilon_{II}^2 \times \left\{ \frac{3}{2r^2} - \frac{\sigma^I \cdot \sigma^{II}}{r^2} + \frac{(\sigma^I \cdot \mathbf{r})(\sigma^{II} \cdot \mathbf{r})}{2r^4} \right\} + \dots \quad (15)$$

The spinor on which H_{TR} operates in (14) and (15) has 16 components. We may separate out two four-component wave equations in which both particles are in positive or negative energy states by setting $\beta^I = \beta^{II} = 1$ or $\beta^I = \beta^{II} = -1$ respectively. The other two possibilities corresponding to $\beta^I = -\beta^{II} = \pm 1$, yield four-component wave equations in which one particle is in a positive energy state and the other is in a negative energy state.

Several differences between (14) and (15) should be emphasized:

(a) The e^4 terms in (15) responsible for the disagreement between the predicted and observed values of the fine structure of helium¹¹ are not present in (14). These terms in (15) come from the $(\mathcal{O}\mathcal{O})^2$ term in (7). There are no $(\mathcal{O}\mathcal{O})$ terms in (12) as a result of the fundamental anticommutation relations (13) and hence no e^4 terms appear in (14).

(b) When both particles are in negative energy states ($\beta^I = \beta^{II} = -1$) all the terms in Eq. (14) are equal in magnitude but opposite in sign to the terms obtained when both particles are in positive energy states ($\beta^I = \beta^{II} = 1$). If one particle is in a positive energy state and the other in a negative energy state, all interaction terms in (14) vanish. The same behavior is not exhibited by Eq. (15).

(c) In particular when $\epsilon_I \epsilon_{II} = -e^2$, there are bound negative energy states in the case of Eq. (14), but there are no bound negative energy states in the case of Eq. (15).

¹⁰ There is one incorrect numerical factor and several misprints in I, Eq. (8), which have been rectified in Eq. (15). See Appendix A for Errata to I and II.

¹¹ See G. Breit, Phys. Rev. **36**, 383 (1930) and **39**, 616 (1932).

(d) If the mass of one of the particles is assumed to be so large that it becomes an immovable supplier of field for the other particle, i.e. $m_I \rightarrow \infty$ or $m_{II} \rightarrow \infty$, then Eq. (15) goes over into Eq. (3). The Breit equation (10) from which (15) is obtained goes over into the Dirac equation (1) if we let the velocity operator (α^I or α^{II}) of the immovable particle go to zero. These limiting processes do *not* cause (14) and (8) to go over into (3) and (1).

All of the differences between (14) and (15) are due to the presence of the Casimir projection operators in (8) and their absence in (10). This difference between these two two-body equations stems from the manner in which they were derived. Equation (8) is a specialized form of the Bethe-Salpeter equation derived from hole theory. The Breit equation (10) was derived from the single-electron form of quantum electrodynamics.

The results of this section confirm and elaborate points which have been made previously by Bethe,¹² Brown and Ravenhall,¹³ and Salpeter.¹⁴

4. THE HYDROGEN ATOM: TWO-BODY TREATMENT INTRINSIC MAGNETIC MOMENT TERMS

The possibility of amplifying the Breit interaction to include intrinsic magnetic moment terms has previously been considered by Breit and Meyerott.¹⁵ They found that there is no objection in principle to this generalization of Pauli's idea¹⁶ to the two-body problem.

In the treatment which follows, intrinsic magnetic moment terms are added to the Breit interaction for the electron as well as for the proton. A quantum electrodynamic explanation exists for the electron's intrinsic magnetic moment, but no similar fundamental explanation has yet been advanced to account quantitatively for the proton's intrinsic magnetic moment. Here the intrinsic moments are both placed on the same phenomenological footing.

Our fundamental wave equation is of the form of Eq. (8) with $U_B(\mathbf{r})$ replaced by $U_B(\mathbf{r}) + U_{IM}(\mathbf{r})$ where

$$U_{IM}(\mathbf{r}) = \frac{\epsilon_I \epsilon_{II}}{2m_I} \mu_I \left(i\beta^I \alpha^I \cdot \frac{\mathbf{r}}{r^3} - \beta^I \sigma^I \cdot \alpha^{II} \times \frac{\mathbf{r}}{r^3} \right) - \frac{\epsilon_I \epsilon_{II}}{2m_{II}} \mu_{II} \left(i\beta^{II} \alpha^{II} \cdot \frac{\mathbf{r}}{r^3} - \beta^{II} \sigma^{II} \cdot \alpha^I \times \frac{\mathbf{r}}{r^3} \right) + \beta^I \beta^{II} \frac{\epsilon_I \epsilon_{II} \mu_I \mu_{II}}{4m_I m_{II}} \left\{ \frac{\sigma^I \cdot \sigma^{II}}{r^3} - \frac{3(\sigma^I \cdot \mathbf{r})(\sigma^{II} \cdot \mathbf{r})}{r^5} - \frac{8\pi}{3} \sigma^I \cdot \sigma^{II} \delta(\mathbf{r}) \right\}. \quad (16a)$$

¹² H. Bethe, *Handbuch der Physik* (Verlag Julius Springer, Berlin, 1933), Vol. 24, 1, p. 375.

¹³ G. E. Brown and D. G. Ravenhall, Proc. Roy. Soc. (London) **A208**, 552 (1951).

¹⁴ Reference 8. See especially the discussion on p. 331.

¹⁵ G. Breit and R. Meyerott, Phys. Rev. **72**, 1023 (1947).

¹⁶ W. Pauli, *Handbuch der Physik* (Verlag Julius Springer, Berlin, 1933), Vol. 24, Part 1, p. 232.

Applying the same procedure as before, we find that Eq. (14) is augmented by terms of precisely the same form as terms already appearing in that equation; namely,

$$(H_{IM})_{rr} = -\frac{(\beta^I + \beta^{II})}{4} \pi \epsilon_I \epsilon_{II} \delta(\mathbf{r}) \left(\frac{2\mu_I}{m_I^2} + \frac{2\mu_{II}}{m_{II}^2} \right) - \frac{(\beta^I + \beta^{II})}{8r^3} \epsilon_I \epsilon_{II} \left(\frac{2\mu_I \sigma^I}{m_I^2} + \frac{2\mu_{II} \sigma^{II}}{m_{II}^2} \right) \cdot (\mathbf{r} \times \mathbf{p}) - \frac{(\beta^I + \beta^{II})}{4m_I m_{II} r^3} \epsilon_I \epsilon_{II} (\mu_I \sigma^I + \mu_{II} \sigma^{II}) \cdot \mathbf{r} \times \mathbf{p} + \frac{(\beta^I + \beta^{II}) \epsilon_I \epsilon_{II}}{8m_I m_{II}} (\mu_I + \mu_{II} + \mu_I \mu_{II}) \times \left\{ \frac{\sigma^I \cdot \sigma^{II}}{r^3} - \frac{3(\sigma^I \cdot \mathbf{r})(\sigma^{II} \cdot \mathbf{r})}{r^5} - \frac{8\pi}{3} \sigma^I \cdot \sigma^{II} \delta(\mathbf{r}) \right\}. \quad (16b)$$

If we amplify (10) rather than (8) by $U_{IM}(\mathbf{r})$ we obtain the same result as (16b) for both particles in positive energy states ($\beta^I = \beta^{II} = +1$). It is just the reduced Hamiltonian obtained by setting $\beta^I = \beta^{II} = +1$ in Eqs. (14) and (16b) which we are concerned with in determining the bound state energy levels of hydrogen.

It is convenient¹⁷ to write the operator for the binding energy $W = H_{RED} - m_I - m_{II}$ explicitly in terms of the reduced mass μ . If we identify m_I with the electron mass m and m_{II} with the proton mass M , then $1/\mu = 1/m + 1/M$ and $\epsilon_I \epsilon_{II} = -e^2$. Let

$$W = W_0 + W_1 + W_2 + W_3 + W_4, \quad (17)$$

where

$$W_0 = p^2/2\mu - e^2/r, \quad (18)$$

$$W_1 = -\frac{p^4}{8\mu^3} + \frac{\pi e^2}{2\mu^2} \delta(\mathbf{r}) + \frac{e^2}{4\mu^2} \frac{\sigma^I \cdot \mathbf{r}}{r^3} \times \mathbf{p}, \quad (19)$$

TABLE II. Contributions to $\langle W_2 \rangle$ in Eq. (20).

Operator	Expectation value
$-\frac{e^2}{2mM} \frac{1}{r^2}$	$\frac{\mu^3 \alpha^4}{2mM n^3} - \frac{\mu^3 \alpha^4}{mM n^3 (l + \frac{1}{2})}$
$-\frac{e^2}{2mM} \mathbf{T} \left(\frac{\mathbf{r}}{r^3} \cdot \mathbf{p} \right) \cdot \mathbf{p}$	$\frac{\mu^3 \alpha^4}{2mM n^3} - \frac{\mu^3 \alpha^4}{2mM n^3 (l + \frac{1}{2})} + \frac{\mu^3 \alpha^4}{mM n^3} \delta_{01}$
$\frac{3p^4}{8m^2 M}$	$-\frac{9\mu^4 \alpha^4}{8m^2 M n^4} + \frac{3\mu^4 \alpha^4}{2m^2 M (l + \frac{1}{2}) n^3}$
$\frac{3p^4}{8mM^2}$	$-\frac{9\mu^4 \alpha^4}{8mM^2 n^4} + \frac{3\mu^4 \alpha^4}{2mM^2 (l + \frac{1}{2}) n^3}$
$-\frac{\pi e^2}{mM} \delta(\mathbf{r})$	$-\frac{\mu^3 \alpha^4}{mM n^3} \delta_{01}$

¹⁷ This treatment follows that of W. E. Lamb, Jr., Phys. Rev. **85**, 259 (1952) and K. Bechert and J. Meixner, Ann. Physik **22**, 525 (1935).

TABLE III. Lamb shift contributions from W_3 , Eq. (21).

Operator	Expectation value	$2^2S_{\frac{1}{2}}$	Energy increase in Mc/sec	$2^2P_{\frac{1}{2}}$
$\mu_{\text{I}} \frac{\pi e^2}{m^2} \delta(\mathbf{r})$	$\mu_{\text{I}} \frac{\mu^3 \alpha^4}{m^2 n^3} \delta_{0l}$	+50.065		
$\mu_{\text{II}} \frac{\pi e^2}{M^2} \delta(\mathbf{r})$	$\mu_{\text{II}} \frac{\mu^3 \alpha^4}{M^2 n^3} \delta_{0l}$	+0.023		
$\mu_{\text{I}} \frac{e^2}{2m\mu r^3} \boldsymbol{\sigma}^{\text{I}} \cdot (\mathbf{r} \times \mathbf{p})$	$\mu_{\text{I}} \frac{\mu^2 \alpha^4}{m n^3 (l+1)(2l+1)}$ $j=l+\frac{1}{2}$ $l \neq 0$			+8.349
	$-\mu_{\text{I}} \frac{\mu^2 \alpha^4}{m n^3 l(2l+1)}$ $j=l-\frac{1}{2}$		-16.697	
$-\frac{e^2}{4M^2 r^3} \boldsymbol{\sigma}^{\text{I}} \cdot (\mathbf{r} \times \mathbf{p})$	$-\frac{\mu^3 \alpha^4}{2M^2 n^3 (l+1)(2l+1)}$ $j=l+\frac{1}{2}$ $l \neq 0$			-0.001
	$+\frac{\mu^3 \alpha^4}{2M^2 n^3 l(2l+1)}$ $j=l-\frac{1}{2}$		+0.002	

$$W_2 = -\frac{e^2}{2mM} \frac{1}{r} p^2 - \frac{e^2}{2mM} \mathbf{r} \left(\frac{\mathbf{r} \cdot \mathbf{p}}{r^3} \right) \cdot \mathbf{p} + \frac{3p^4}{8m^2 M} + \frac{3p^4}{8mM^2} - \frac{\pi e^2}{mM} \delta(\mathbf{r}), \quad (20)$$

$$W_3 = \frac{\pi e^2}{m^2} \mu_{\text{I}} \delta(\mathbf{r}) + \frac{\pi e^2}{M^2} \mu_{\text{II}} \delta(\mathbf{r}) + \frac{e^2}{2m^2} \mu_{\text{I}} \boldsymbol{\sigma}^{\text{I}} \cdot \frac{\mathbf{r}}{r^3} \times \mathbf{p} + \frac{e^2}{2mM} \mu_{\text{I}} \boldsymbol{\sigma}^{\text{I}} \cdot \frac{\mathbf{r}}{r^3} \times \mathbf{p} - \frac{e^2}{4M^2} \boldsymbol{\sigma}^{\text{I}} \cdot \frac{\mathbf{r}}{r^3} \times \mathbf{p}, \quad (21)$$

$$W_4 = -\frac{e^2}{4mM} (1+\mu_{\text{I}})(1+\mu_{\text{II}}) \left\{ \frac{\boldsymbol{\sigma}^{\text{I}} \cdot \boldsymbol{\sigma}^{\text{II}}}{r^3} - \frac{3(\boldsymbol{\sigma}^{\text{I}} \cdot \mathbf{r})(\boldsymbol{\sigma}^{\text{II}} \cdot \mathbf{r})}{r^5} - \frac{8\pi}{3} \boldsymbol{\sigma}^{\text{I}} \cdot \boldsymbol{\sigma}^{\text{II}} \delta(\mathbf{r}) \right\} + \frac{e^2}{2mM} (1+\mu_{\text{II}}) \boldsymbol{\sigma}^{\text{II}} \cdot \frac{\mathbf{r}}{r^3} \times \mathbf{p} + \frac{e^2}{4M^2} (1+2\mu_{\text{II}}) \boldsymbol{\sigma}^{\text{II}} \cdot \frac{\mathbf{r}}{r^3} \times \mathbf{p}. \quad (22)$$

W_0 is the Schrödinger Hamiltonian for a particle of mass μ , charge $-e$ in a Coulomb field whose potential is e/r . We now tabulate all the individual contributions to the energy levels, determined by treating W_1 , W_2 , W_3 , and W_4 as perturbations of the Balmer energy levels: $\langle W_0 \rangle = -\mu\alpha^2/2n^2$.

W_1 consists of the relativistic correction terms which are familiar in the reduction of the Dirac equation (1). The individual contributions to W_1 may be obtained from Table I simply by replacing m there by μ .

The last three terms of W_2 are corrections which occur as a result of writing W in terms of the reduced mass. The expectation value of W_2 is obtained without approximation from the addition of the individual contributions in Table II:

$$\langle W_2 \rangle = -\mu^3 \alpha^4 / 8mMn^3. \quad (23)$$

The first four terms of W_3 yield the contributions of the intrinsic electron and proton magnetic moments to the Lamb shift. The last term of W_3 arising as a result of writing W in terms of μ , makes a very small contribution to the Lamb shift. The expectation values of these operators as well as the numerical values for the $2^2S_{\frac{1}{2}}$, $2^2P_{\frac{1}{2}}$, and $2^2P_{\frac{3}{2}}$ states of hydrogen are given in Table III.

In a review paper¹⁸ on the Lamb shift, Salpeter pointed out that the contribution of the electron's intrinsic magnetic moment ($\sim \alpha/2\pi$) for an infinitely heavy nucleus is given by $\frac{1}{2}L$, where the Lamb constant $L = \alpha^3 \text{Ry}_{\infty} c / 3\pi$, and that the correct factor accounting for the finite mass of the nucleus is not $\sim (1-3m/M)$ and had not as yet been calculated. From Table III, we see that the contribution from $\mu_{\text{I}} = (\alpha/2\pi)(1-5.946\alpha/\pi)$ may be written $\sim +\frac{3}{8}L(1-3m/M)(1-5.946\alpha/\pi)$ from the term affecting the S states and $\sim -\frac{1}{8}L(1-2m/M) \times (1-5.946\alpha/\pi)$ from the term affecting the P states. These combine to increase the separation of the $2^2S_{\frac{1}{2}} - 2^2P_{\frac{1}{2}}$ states by $\sim +\frac{1}{2}L(1-11m/4M)(1-5.946\alpha/\pi) = 66.78 \text{ Mc/sec}$, of which -0.10 Mc/sec is due to the mass correction and -0.94 Mc/sec to the fourth order intrinsic magnetic moment of the electron.¹⁹ Corrections of this order are now of interest since the experimental accuracy in Lamb shift measurements is $\pm 0.10 \text{ Mc/sec}$.²⁰

The contribution of the proton's intrinsic magnetic moment to the Lamb shift agrees with that calculated by Salpeter¹⁸ and is still not measurable. It should be pointed out that the normal Dirac moment of the proton as well as that of the electron have been taken into account in these calculations and that their combined contribution to the Lamb shift is represented by the fifth term in W_3 and amounts to -0.002 Mc/sec .

In the case of hydrogen, the hyperfine structure splitting is small compared to the fine structure splitting. The operator for the hyperfine energy, W_4 , includes all

¹⁸ E. E. Salpeter, Phys. Rev. **89**, 92 (1953).

¹⁹ R. Karplus and N. M. Kroll, Phys. Rev. **77**, 536 (1950).

²⁰ Triebwasser, Dayhoff, and Lamb, Phys. Rev. **89**, 98 (1953).

terms which depend on the nuclear spin operator $\mathbf{i} = \boldsymbol{\sigma}^{\text{II}}/2$. Because of the tensor interaction, l is not a good quantum number. We restrict our attention, however, to the S and P states. Since there are no matrix elements of W_4 connecting these two states, we may consider the diagonal elements for a given l .

S states.—The expectation value of W_4 for the S states is given simply by

$$\langle W_4 \rangle = \frac{2\pi\alpha}{3mM} (1+\mu_{\text{I}})(1+\mu_{\text{II}}) \langle \boldsymbol{\sigma}^{\text{I}} \cdot \boldsymbol{\sigma}^{\text{II}} \delta(\mathbf{r}) \rangle, \quad (24)$$

which leads to the S state hyperfine splitting,

$$\Delta W_S = -\frac{8}{3} \frac{\mu^3 \alpha^4}{mMn^3} (1+\mu_{\text{I}})(1+\mu_{\text{II}}). \quad (25)$$

This is in agreement with Lamb,¹⁷ but does not, of course, take into consideration a number of effects treated in quantum electrodynamic calculations.²¹

P states.—It is convenient to write W_4 for $l \neq 0$ in terms of \mathbf{i} , $\mathbf{s} = \boldsymbol{\sigma}^{\text{I}}/2$ and $\mathbf{l} = \mathbf{r} \times \mathbf{p}$.

$$W_4 = \frac{e^2}{mM} (1+\mu_{\text{I}})(1+\mu_{\text{II}}) \frac{1}{r^3} \left\{ \frac{3\mathbf{i} \cdot \mathbf{r} \mathbf{s} \cdot \mathbf{r}}{r^2} - \mathbf{i} \cdot \mathbf{s} \right\} + \frac{e^2}{mM} (1+\mu_{\text{II}}) \frac{1}{r^3} \mathbf{i} \cdot \mathbf{l} + \frac{e^2}{2M^2} (1+2\mu_{\text{II}}) \frac{1}{r^3} \mathbf{i} \cdot \mathbf{l}. \quad (26)$$

Equation (26) is of the same form as the hyperfine energy operator for $l \neq 0$ as treated in Bethe's Handbuch article²² and recently reviewed by Lamb,¹⁷ but it includes additional small contributions. It is, therefore, a simple matter to determine the expectation value of W_4 for the $P_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ states.

$$P_{\frac{1}{2}}: \langle W_4 \rangle = \frac{\mu^3 \alpha^4}{9n^3} \left\{ \frac{(1+\mu_{\text{I}})(1+\mu_{\text{II}})}{mM} + \frac{1+\mu_{\text{II}}}{mM} + \frac{1+2\mu_{\text{II}}}{2M^2} \right\} \langle 2f(f+1)-3 \rangle, \quad (27)$$

$$P_{\frac{3}{2}}: \langle W_4 \rangle = \frac{\mu^3 \alpha^4}{18n^3} \left\{ -\frac{(1+\mu_{\text{I}})(1+\mu_{\text{II}})}{5mM} + \frac{1+\mu_{\text{II}}}{mM} + \frac{1+2\mu_{\text{II}}}{2M^2} \right\} \langle 2f(f+1)-9 \rangle, \quad (28)$$

where $\mathbf{f} = \mathbf{i} + \mathbf{j}$. The $P_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ state hyperfine splittings

²¹ R. Arnowitt, Phys. Rev. **92**, 1002 (1953), has made the latest calculation on the hyperfine splitting of the S states. This and earlier calculations by R. Karplus and A. Klein, Phys. Rev. **85**, 972 (1952); N. M. Kroll and F. Pollock, Phys. Rev. **86**, 876 (1952) and E. E. Salpeter and W. A. Newcomb, Phys. Rev. **87**, 150 (1952), have been used together with the experimental value, known to an accuracy of ± 0.0003 Mc/sec, to calculate precision values of the fine structure constant.

²² Reference 12, p. 386.

are

$$\Delta W_{P_{\frac{1}{2}}} = \frac{4\mu^3 \alpha^4}{9n^3 M m} \left\{ (1+\mu_{\text{II}})(2+\mu_{\text{I}}) + \frac{m(1+2\mu_{\text{II}})}{2M} \right\} = 59.096 \text{ Mc/sec for } n=2, \quad (29)$$

and

$$\Delta W_{P_{\frac{3}{2}}} = \frac{4\mu^3 \alpha^4}{9n^3 M m} \left\{ \frac{(1+\mu_{\text{II}})(4-\mu_{\text{I}})}{5} + \frac{m(1+2\mu_{\text{II}})}{2M} \right\} = 23.626 \text{ Mc/sec for } n=2. \quad (30)$$

These results are to be compared with earlier theoretical expressions given by Lamb²³:

$$\Delta W_{P_{\frac{1}{2}}} = \frac{8\mu^3 \alpha^4}{9mMn^3} (1+\mu_{\text{II}}); \quad (31)$$

$$\Delta W_{P_{\frac{3}{2}}} = -\frac{16}{45} \frac{\mu^3 \alpha^4}{mMn^3} (1+\mu_{\text{II}}),$$

which are the leading terms in (29) and (30). The electron's intrinsic magnetic moment increases the hyperfine splitting of the $2^2P_{\frac{1}{2}}$ and $2^2P_{\frac{3}{2}}$ states by $+0.034$ Mc/sec and -0.007 Mc/sec respectively. The Dirac and intrinsic magnetic moment of the proton (2nd term in curly brackets in Eqs. (29) and (30)) increase the hyperfine splitting of both P states by 0.013 Mc/sec. Corrections of this order should now be considered, inasmuch as $2P$ -state hyperfine splittings are now known experimentally to within 0.20 Mc/sec,²⁰ and more precise results are to be expected.

5. POSITRONIUM

Again we consider a wave equation of the form of Eq. (8) with the Breit interaction amplified to include the first-order single quantum virtual annihilation exchange interaction,²⁴ $U_A(\mathbf{r}, \mathbf{r}')$ as well as the intrinsic magnetic moment terms Eq. (16a). For positronium $m_{\text{I}} = m_{\text{II}} = m$; $\mu_{\text{I}} = \mu_{\text{II}} \sim \alpha/2\pi$, and

$$U_A(\mathbf{r}, \mathbf{r}') = -(\pi e^2/m^2) (\boldsymbol{\gamma}_j c) \delta(\mathbf{r}) \delta(\mathbf{r}') (c^{-1} \boldsymbol{\gamma}_j). \quad (32)$$

The pair creation matrices $\boldsymbol{\gamma}_j c$ and the pair annihilation matrices $c^{-1} \boldsymbol{\gamma}_j$ are defined by the relation:

$$\bar{\psi}(\boldsymbol{\gamma}_j c) (c^{-1} \boldsymbol{\gamma}_j) \psi = \bar{\psi}_{kK} (\boldsymbol{\gamma}_j c)_{Kk} (c^{-1} \boldsymbol{\gamma}_j)_{K'k'} \psi_{k'K'}, \quad (33)$$

$$j = 1, 2, 3,$$

$$k, K = 1, 2, 3, 4,$$

with $\bar{\psi} = \psi^* \boldsymbol{\gamma}_0$; $c = \boldsymbol{\gamma}_0 \boldsymbol{\gamma}_2 = c^{-1}$; $\boldsymbol{\gamma}_j = \beta \boldsymbol{\alpha}_j$; $\boldsymbol{\gamma}_0 = \beta$. Clearly U_A is even-even.

²³ Reference 17, Sec. 56, and W. E. Lamb, Jr., and R. C. Retherford, Phys. Rev. **79**, 549 (1950).

²⁴ R. Karplus and A. Klein, Phys. Rev. **87**, 848 (1952); see Eqs. (3.5) and (5.3).

In I, Chraplyvy has shown that for $m_I = m_{II}$, there is no finite transformation which will convert a two-body Hamiltonian containing $(\Theta\Theta)$ terms into an $(\mathcal{E}\mathcal{E})$ operator. We may still use Eq. (7) for positronium, however, since the Casimir projection operators eliminate $(\Theta\Theta)$ terms to the order considered. [See Eq. (12).] If we amplify the Breit equation (10) rather than Eq. (8) by $U_A(\mathbf{r}, \mathbf{r}')$, we must apply the more general prescriptions of II. In either case we obtain the same result for both particles in positive energy states (apart from the e^4 terms discussed in Sec. 2).

Separating out the four-component wave equation for both particles in positive energy states allows us to write the energy corresponding to the interaction (32) in two equivalent ways:

$$\begin{aligned} V_A &= (\pi e^2/m^2)\psi^*(0)_{uU}\{(\sigma_3)_{Uu}(\sigma_3)_{U'u'} + \delta_{Uu}\delta_{U'u'} \\ &\quad + (\sigma_1)_{Uu}(\sigma_1)_{U'u'}\}\psi(0)_{u'U'} \\ &= (\pi e^2/m^2)\psi^*(0)_{uU}\{\frac{3}{2}\delta_{uu'}\delta_{UU'} \\ &\quad + \frac{1}{2}\boldsymbol{\sigma}_{uu'} \cdot \boldsymbol{\sigma}_{UU'}\}\psi(0)_{u'U'}, \end{aligned} \quad (34)$$

where $u, U = 1, 2$. In the second form of Eq. (34), the Pauli matrices refer to the individual particles I, II. Hence we may say that in the Pauli representation, the virtual annihilation exchange interaction is given by

$$(U_A)_{RED} = \frac{\pi e^2}{m^2} \left(\frac{3}{2} + \frac{1}{2} \boldsymbol{\sigma}^I \cdot \boldsymbol{\sigma}^{II} \right) \delta(\mathbf{r}). \quad (35)$$

In the case of positronium (unlike hydrogen), the hyperfine and fine structure splittings are of the same order of magnitude and l is a good quantum number in spite of the tensor interaction.²⁵ Therefore, we regroup the various terms contributing to the positronium binding energy $V = H_{RED} - 2m$ in a slightly different way than was done for hydrogen. Let

$$V = V_0 + V_1 + V_2 + V_3 + V_4, \quad (36)$$

where

$$V_0 = p^2/m - e^2/r, \quad (37)$$

$$V_1 = -\frac{p^4}{4m^3} + \frac{5\pi e^2}{2m^2} \delta(\mathbf{r}) - \frac{e^2 p^2}{2m^2 r} - \frac{e^2}{2m^2} \mathbf{r} \left(\frac{\mathbf{r}}{r^3} \cdot \mathbf{p} \right) \cdot \mathbf{p}, \quad (38)$$

$$V_2 = \frac{3e^2}{2m^2 r^3} \mathbf{S} \cdot \mathbf{I}, \quad \text{where } \mathbf{S} = \frac{1}{2}(\boldsymbol{\sigma}^I + \boldsymbol{\sigma}^{II}), \quad (39)$$

$$V_3 = \mu_I \left\{ \frac{2\pi e^2}{m^2} \delta(\mathbf{r}) + \frac{e^2}{m^2 r^3} \mathbf{S} \cdot \mathbf{I} \right\}, \quad (40)$$

²⁵ A detailed proof is given in reference 3.

$$\begin{aligned} V_4 &= -\frac{e^2}{4m^2} (1 + \mu_I)^2 \left\{ \frac{\boldsymbol{\sigma}^I \cdot \boldsymbol{\sigma}^{II}}{r^3} - \frac{3(\boldsymbol{\sigma}^I \cdot \mathbf{r})(\boldsymbol{\sigma}^{II} \cdot \mathbf{r})}{r^5} \right. \\ &\quad \left. - \frac{8\pi}{3} \boldsymbol{\sigma}^I \cdot \boldsymbol{\sigma}^{II} \delta(\mathbf{r}) \right\} + \frac{\pi e^2}{2m^2} \boldsymbol{\sigma}^I \cdot \boldsymbol{\sigma}^{II} \delta(\mathbf{r}). \end{aligned} \quad (41)$$

We now write the contributions of V_1, V_2, V_3 , and V_4 to the energy levels, determined, as in the case of hydrogen, by treating them as perturbations on the Balmer energy levels $\langle V_0 \rangle = -m\alpha^2/4n^2$.

V_1 , which includes a part of the virtual annihilation exchange interaction, consists of kinetic energy and orbital terms. It is spin-independent and hence contributes equally to the singlet and triplet states.

$$\langle V_1 \rangle = \frac{11m\alpha^4}{64n^4} + \frac{7m\alpha^4}{16n^3} \delta_{0l} - \frac{m\alpha^4}{4n^3(l + \frac{1}{2})}. \quad (42)$$

The spin-orbit interaction V_2 depends on the total spin S . It does not contribute to either the singlet or S states.

$$\begin{aligned} \langle V_2 \rangle^t &= \frac{3m\alpha^4}{16n^3 l(l + \frac{1}{2})(l + 1)} (1 - \delta_{0l}) \\ &\quad \times \begin{cases} l, & j = l + 1 \\ -1, & j = l \\ -(l + 1), & j = (l - 1). \end{cases} \end{aligned} \quad (43)$$

V_3 gives the contribution of the intrinsic magnetic moments of the electron and positron to the Lamb shift. Using $\mu_I = \mu_{II} \sim \alpha/2\pi$, we get

$$\begin{aligned} \langle V_3 \rangle &= \frac{m\alpha^5}{8\pi n^3} \delta_{0l} + \frac{m\alpha^5(1 - \delta_{0l})}{16\pi n^3 l(l + \frac{1}{2})(l + 1)} \\ &\quad \times \begin{cases} l, & j = l + 1 \\ -1, & j = l \\ -(l + 1), & j = (l - 1). \end{cases} \end{aligned} \quad (44)$$

All spin-spin interactions are given by V_4 . The $\delta(\mathbf{r})$ terms affect only the S states

$$\begin{aligned} \langle V_4 \rangle_{l=0} &= \frac{7\pi\alpha}{6m^2} \left(1 + \frac{4\alpha}{7\pi} \right) \langle (2S^2 - 3) \delta(\mathbf{r}) \rangle \\ &= -\frac{7m\alpha^4}{16n^3} \left(1 + \frac{4\alpha}{7\pi} \right) \text{ for } {}^1S_0 \text{ states} \\ &= \frac{7}{48} \frac{m\alpha^4}{n^3} \left(1 + \frac{4\alpha}{7\pi} \right) \text{ for } {}^3S_1 \text{ states.} \end{aligned} \quad (45)$$

The remaining terms for $l \neq 0$ are zero for the singlet

states.

$$\begin{aligned} \langle V_4 \rangle_{l \neq 0} &= -\frac{e^2}{4m^2} \left(1 + \frac{\alpha}{\pi}\right) \left\langle \frac{\boldsymbol{\sigma}^I \cdot \boldsymbol{\sigma}^{II}}{r^3} - \frac{3(\boldsymbol{\sigma}^I \cdot \mathbf{r})(\boldsymbol{\sigma}^{II} \cdot \mathbf{r})}{r^5} \right\rangle \\ &= \frac{m\alpha^4(1-\delta_{0l})}{16\pi^3 l(l+\frac{1}{2})(l+1)} \left(1 + \frac{\alpha}{\pi}\right) \\ &\quad \times \begin{cases} -\frac{l}{2l+3}, & j=l+1 \\ 1, & j=l \\ -\frac{(l+1)}{2l-1}, & j=(l-1). \end{cases} \end{aligned} \quad (46)$$

Adding $\langle V_0 \rangle$, $\langle V_1 \rangle$, $\langle V_2 \rangle$, $\langle V_3 \rangle$, and $\langle V_4 \rangle$ the binding energy for the singlet and triplet states is

$$\langle V \rangle^s = -\frac{m\alpha^2}{4n^2} + \frac{11m\alpha^4}{64n^4} - \frac{m\alpha^4}{4n^3(l+\frac{1}{2})} + \frac{m\alpha^5}{8\pi n^3} \delta_{0l}, \quad (47)$$

$$\begin{aligned} \langle V \rangle^t &= -\frac{m\alpha^2}{4n^2} + \frac{11m\alpha^4}{64n^4} - \frac{m\alpha^4}{4n^3(l+\frac{1}{2})} + \frac{7m\alpha^4}{12n^3} \delta_{0l} \\ &\quad + \frac{m\alpha^4(1-\delta_{0l})}{8n^3(l+\frac{1}{2})} \begin{cases} \frac{3l+4}{(2l+3)(l+1)}, & j=l+1 \\ -1, & j=l \\ \frac{-3(l-1)}{l(2l-1)}, & j=l-1 \end{cases} \\ &\quad + \frac{m\alpha^5}{8\pi n^3} \delta_{0l} + \frac{m\alpha^5(1-\delta_{0l})}{16\pi n^3 l(l+\frac{1}{2})(l+1)} \\ &\quad \times \begin{cases} \frac{2l(l+1)}{(2l+3)}, & j=l+1 \\ 0, & j=l \\ -\frac{2l(l+1)}{(2l-1)}, & j=l-1. \end{cases} \end{aligned} \quad (48)$$

The $m\alpha^4$ terms in Eq. (47) and Eq. (48) agree with those of Ferrell.²⁶ The only $m\alpha^5$ terms considered in our

²⁶ R. A. Ferrell, Phys. Rev. **84**, 858 (1951). The details are given in Dr. Ferrell's thesis, reference 3.

calculation are those arising from the intrinsic magnetic moments of the electron and positron. Recently Fulton and Martin²⁷ have calculated all the corrections of order α^5 to the $n=2$ energy levels of positronium using a quantum electrodynamic method. It is possible to identify²⁸ the intrinsic magnetic moment contribution in their final expressions. These are found to agree with the results of our phenomenological treatment.

We are indebted to Professor Z. V. Chraplyvy for suggesting this problem and for numerous valuable discussions. One of us (W. A. B.) would like to express his appreciation to Professor W. Pauli for his encouragement and comments, and to Dr. A. Thellung, Dr. R. Jost, and Dr. P. Martin for helpful discussions.

APPENDIX A

Errata to I

In Eq. (4), the first minus sign should be replaced by a plus sign.

On p. 389, the second display formula in the middle of the left column should read:

$$(\Omega\psi)_{kK} = \sum_{n,N} \eta_{kn} \omega_{KN} \psi_{nN}.$$

In Eq. (7j), the numerical coefficient should be $\frac{1}{8}$ (rather than $\frac{3}{16}$).

In Eq. (8b), the denominator of the last term should be $2m_{1c}$.

In Eq. (8e), first line, the minus sign between the terms should be replaced by a plus sign; the curly brackets of the second line should read

$$\left\{ -\frac{i(\mathbf{r} \cdot \mathbf{p}_I)}{r^3} + 4\pi i \delta(\mathbf{r})(\mathbf{r} \cdot \mathbf{p}_I) + \frac{2\boldsymbol{\sigma}^{II} \cdot (\mathbf{r} \times \mathbf{p}_I)}{r^3} \right\}.$$

The last term of Eq. (8e) should be $(8\pi/3)(\boldsymbol{\sigma}^I \cdot \boldsymbol{\sigma}^{II})\delta(\mathbf{r})$.

In Eq. (8f), the curly brackets should read

$$\left\{ \frac{3}{r^2} - \frac{2(\boldsymbol{\sigma}^I \cdot \boldsymbol{\sigma}^{II})}{r^2} + \frac{(\boldsymbol{\sigma}^I \cdot \mathbf{r})(\boldsymbol{\sigma}^{II} \cdot \mathbf{r})}{r^4} \right\}.$$

Errata to II

On p. 1131, the last formula of the left column should begin with $\alpha_y^{II} \alpha_z^I$.

In Eq. (6c), the subscript at the end of the second term should be oo (rather than oe).

²⁷ T. Fulton and P. C. Martin, Phys. Rev. **93**, 903 (1954); **95**, 811 (1954).

²⁸ We are indebted to Dr. Martin for an interesting discussion on the relationship between their work and ours.