into two parts,

$$P\phi^{\text{out}}=P\phi_A{}^{\text{out}}+P\phi_B{}^{\text{out}},\quad \psi^{\text{out}}=\psi_A{}^{\text{out}}+\psi_B{}^{\text{out}},$$

such that the fields $\psi_A{}^{\text{out}}$ and $P\phi_A{}^{\text{out}}$ satisfy the same commutation relations as the incoming fields. Then a unitary matrix, M, can be defined as

$$M^{-1}\psi^{\text{in}}M=\psi_A{}^{\text{out}},\quad M^{-1}P\phi^{\text{in}}M=P\phi_A{}^{\text{out}}.$$

One would like to say that this M matrix corresponds to the Heisenberg S matrix. However, the relation of this M matrix to the classical conservation equations is not very clear, especially since the fields $P\phi_B{}^{\text{out}}$ and $\psi_B{}^{\text{out}}$ do not vanish in general. The situation would be improved if they did vanish in general. Hayashi[6] in his latest investigations of local fields in nonlocal interaction seems to be proceeding in this direction, that is, making the out fields obey the same commutation relations as the in fields by modifying the equations of motion. In general, if the quantization difficulties are cleared up for local fields in nonlocal interaction, one can expect that those of the present theory will also be cleared up.

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Renormalization in the New Tamm-Dancoff Theory of Meson-Nucleon Scattering*†

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The new Tamm-Dancoff equations for meson-nucleon scattering are set up in the lowest approximation and it is shown how explicit nonphysical singularities may be avoided in these equations. The particle self-energies appearing in the integral equation are renormalized, but the resulting modified propagator for the system then has a nonphysical singularity. For the states $T=j=\frac{1}{2}$, the vertex and self-energy expressions generated by the uncrossed graph are considered. The renormalized vertex may be constructed by the successive solution of two one-dimensional integral equations, the finite part of the self-energy then being obtained by quadratures. Vertex renormalization is uncertain to a constant factor in the $S_{\frac{1}{2}}$ state, and the $S_{\frac{1}{2}}$ theory therefore depends on two parameters. No numerical results are obtained, owing to a number of difficulties found in this theory—a comparison is made between these difficulties and those of the corresponding Bethe-Salpeter equation.

I. INTRODUCTION

SOME preliminary calculations of phase shifts for meson-nucleon scattering have recently been reported[1] for the relativistic π-meson theory with pseudoscalar coupling, based on the simplest Tamm-Dancoff approximation to the meson-nucleon system. This approximation omits all amplitudes describing the system except those directly coupled to the one-meson one-nucleon amplitude in consequence of the interaction between the meson and nucleon fields. For this amplitude an integral equation was obtained. In addition to terms describing the interaction between the meson and nucleon of this amplitude, the integral equation contained a number of divergent terms, representing the self-energies of the particles and the change in the energy of the vacuum state which result from the interaction between the meson and nucleon fields. The interaction kernel in this integral equation is clearly the most important term physically since it accounts for the scattering process, while these other terms simply describe various kinds of correction to the motion. Since these corrections could not be evaluated in this Tamm-Dancoff theory, self-energy terms were simply omitted and calculations were carried through for the $T=\frac{3}{2}$, $S_{\frac{1}{2}}$, and $P_{\frac{3}{2}}$ states of the meson-nucleon system. Although this lowest-order Tamm-Dancoff theory represents a drastic approximation to the complete γ_5 theory, very considerable success was obtained in accounting for the striking behavior of the phase-shift[2] δ_{33} in terms of one parameter, the coupling constant $G^2/4\pi$. For the $S_{\frac{1}{2}}$ state, the linear behavior of the phase shift δ_3 for high momenta could be understood in terms of the strong repulsive interaction obtained

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[1] Dyson, Ross, Salpeter, Schweber, Sundaresen, Visscher, and Bethe, Phys. Rev. 95, 1644 (1954). This paper will be referred to as D3.

[2] De Hoffmann, Metropolis, Alei, and Bethe, Phys. Rev. 95, 1586 (1954); R. L. Martin, Phys. Rev. 95, 1606 (1954).

in the Tamm-Dancoff theory, the departures at lower momenta being attributed to some weak attractive interaction not included in the theory at the present stage.

Experiments on meson-nucleon scattering have now determined these and other phase shifts over a wide range of energies. The cross sections are relatively insensitive to certain phase shifts, notably the $T=\frac{1}{2}$, P-phase shifts δ_{11} and δ_{13}, for which it can only be said that they are small. However, in addition to the phases δ_3 and δ_{33}, the behavior of the $T=\frac{1}{2}$, $S_{\frac{1}{2}}$ phase δ_1 now appears to be fairly well established.[3]

In the Tamm-Dancoff theory, the interaction kernel contains one term (the uncrossed graph) which is effective only for the $T=\frac{1}{2}$, $j=\frac{1}{2}$ meson-nucleon states. This is a consequence of the fact that these states have the same isotopic spin, angular momentum, and parity as the state of one stationary nucleon or of one antinucleon. Although the integral equation has a finite form, it has no finite solution for these states, since this additional term generates new self-energy divergences in an implicit way. However this interaction term gives an important contribution to the scattering processes and cannot consistently be omitted. No means were available for the treatment of this problem and no calculations were made for these states in the previous work.

In order to avoid the presence of the vacuum self-energy term in the integral equation, a new Tamm-Dancoff (N.T.D.) method has been proposed[4] in which the amplitudes are based on the physical vacuum state rather than the bare-particle vacuum state. In the integral equation obtained with the corresponding Tamm-Dancoff approximation, no vacuum self-energy term appears, the energy of the system being measured relative to that of the physical vacuum, and the particle self-energies appearing have a form which is now closely related to that of the covariant theory.[5] In this N.T.D. theory, the positive and negative frequency states were treated symmetrically, and a new difficulty appeared, namely the presence of unphysical singularities in the N.T.D. amplitudes. The avoidance of these singularities clearly required the statement of a further boundary condition expressing the fact that the state with respect to which the N.T.D. amplitudes are defined is really the physical vacuum state.

The purpose of the present investigation was to inquire to what extent the existing phase-shift calculations could be improved by the inclusion of particle self-energy corrections, and supplemented by their extension to the $T=\frac{1}{2}$, $j=\frac{1}{2}$ states. In Sec. II, the N.T.D. equations for the problem are set up in the simplest Tamm-Dancoff approximation and it is found possible in this approximation, by the use of the boundary condition mentioned above, to give a set of equations which have no nonphysical singularities. This requires, however, that the theory no longer be symmetrical between positive and negative frequencies. The renormalization of the self-energies appearing in the integral equation itself is carried out in Sec. III, where it is found that a further nonphysical singularity appears in consequence of this achievement. In Sec. IV, the renormalization of the vertex and self-energy parts generated by the uncrossed graph (see Fig. 1) is carried through, and the evaluation of their finite parts by the solution of finite integral equations and by finite integration processes is discussed. The Tamm-Dancoff method has the attraction that it is numerically calculable since all the operations appropriate may be handled by standard numerical techniques. Except at one point (vertex renormalization in the $T=\frac{1}{2}$, $S_{\frac{1}{2}}$ state), the N.T.D. theory suffers no disadvantage from its lack of formal covariance, apart from the algebraic complexity of the equations finally obtained. This is emphasized in the final Sec. V, where the difficulties preventing calculation of phase shifts with this theory are discussed, and a comparison is made with the corresponding situation in the lowest approximation for the Bethe-Salpeter equation.

II. NEW TAMM-DANCOFF EQUATIONS FOR MESON-NUCLEON SCATTERING

For the interaction of the meson and nucleon fields, the Hamiltonian has the form

$$H=H_0+H_I, \tag{1}$$

where H_0 is the free particle Hamiltonian and

$$H_I=G\int \psi^*\gamma\tau_\alpha\psi\phi_\alpha d_3r. \tag{2}$$

In this expression (2), γ denotes the Dirac matrix $i\beta\gamma_5$, and

$$\begin{aligned}\psi(r)&=(2\pi)^{-\frac{3}{2}}\int d_3p\sum_u b_{pu}ue^{ip\cdot r},\\ \psi^*(r)&=(2\pi)^{-\frac{3}{2}}\int d_3p\sum_u b^*_{pu}u^*e^{-ip\cdot r},\\ \phi_\alpha(r)&=(2\pi)^{-\frac{3}{2}}\int d_3k(2\omega_k)^{-\frac{1}{2}}(a_{k\alpha}+a^*_{-k\alpha})e^{ik\cdot r}.\end{aligned} \tag{3}$$

The spinor u satisfies the equation

$$(\alpha\cdot p+\beta M)u=\pm E_pu, \tag{4}$$

with $E_p=(M^2+p^2)^{\frac{1}{2}}$, according as u refers to a nucleon or an antinucleon state. These spinors are normalized

[3] H. A. Bethe and F. de Hoffmann, Phys. Rev. **95**, 1100 (1954); J. Orear, Phys. Rev. **96**, 176 (1954).

[4] F. J. Dyson, Phys. Rev. **90**, 994 (1953) and Phys. Rev. **91**, 1543 (1953). The latter paper will be referred to as D2.

[5] F. J. Dyson, Phys. Rev. **91**, 421 (1953) (this paper will be referred to as D1); W. M. Visscher, Phys. Rev. **96**, 788 (1954).

to the relation

$$u^*u=1. \tag{5}$$

The projection operators $\Omega^{\pm}(p)$ given by

$$\Omega^+(p)=\sum_{u+}uu^*,\quad \Omega^-(p)=\sum_{u-}uu^*, \tag{6}$$

may then be expressed in the form

$$\Omega^{\pm}(p)=[E_p\pm(\alpha\cdot p+\beta M)]/2E_p. \tag{7}$$

Consider now a physical state Ψ of one meson and one nucleon in interaction, and the physical vacuum state Ψ_0. For any product P of the emission and absorption operators b, b^*, a, and a^* of (3), we will use the notation

$$\langle P\rangle\equiv\langle\Psi_0|P|\Psi\rangle. \tag{8}$$

Denoting the physical energy of the state Ψ (relative to the vacuum state) by ϵ, then

$$\begin{aligned}\epsilon\langle P\rangle&=\langle[P,H]\rangle\\&=\langle[P,H_0]\rangle+\langle[P,H_I]\rangle.\end{aligned} \tag{9}$$

By expressing P as a sum of terms in normal order, i.e., of terms in which all creation operators stand to the left of the annihilation operators, every $\langle P\rangle$ may be written as a linear sum of the "new Tamm-Dancoff" amplitudes $\langle C(N)A(N')\rangle$.

Consider first the amplitude

$$\langle b_{-qu}a_{q\alpha}\rangle. \tag{10}$$

For this, the Eq. (9) now leads to

$$(\epsilon-\omega_q-\eta_{-qu}E_q)\langle b_{-qu}a_{q\alpha}\rangle=\langle[b_{-qu}a_{q\alpha},H_I]\rangle, \tag{11}$$

where η_{-qu} takes the value $+1$ for a positive energy spinor and -1 for a negative energy spinor. With H_I replaced by expression (2), the right-hand side of (11) becomes

$$G(2\pi)^{-\frac{3}{2}}\int\int d_3p d_3k(2\omega_k)^{-\frac{1}{2}}\sum_w\sum_v(v^*\gamma\tau_\beta w)\times\langle[b_{-qu}a_{q\alpha},\,(a_{k\alpha}+a^*_{-k\alpha})b^*_{p+k,\,v}b_{pw}]\rangle. \tag{12}$$

The commutator appearing in (12) is now brought to normal form, in order to express (12) in terms of the N.T.D. amplitudes. Equation (11) then becomes

$$\begin{aligned}(\epsilon-\omega_q-\eta_{-qu}E_q)\langle b_{-qu}a_{q\alpha}\rangle\\=G(2\pi)^{-\frac{3}{2}}\bigg[&(2\omega_q)^{-\frac{1}{2}}\theta_{-qu}\sum_w(u^*\gamma\tau_\alpha w)\langle b_{0w}\rangle\\&+(2\omega_q)^{-\frac{1}{2}}\int d_3p\sum_w\sum_v(v^*\gamma\tau_\alpha w)\langle b_{-qu}b^*_{p-q,\,v}b_{pw}\rangle\\&+\int d_3k(2\omega_k)^{-\frac{1}{2}}\sum_w(u^*\gamma\tau_\beta w)\\&\times\langle a_{q\alpha}(a_{k\beta}+a^*_{-k\beta})b_{-q-k,\,w}\rangle\bigg],\end{aligned} \tag{13}$$

where $\theta_{-qu}=(1+\eta_{-qu})/2$ with the values $+1$ for a positive energy spinor u, 0 for a negative energy spinor.

It is now necessary to obtain similar equations for the various N.T.D. amplitudes appearing on the right-hand side of (13). In this way, successively, an infinite set of N.T.D. equations may be built up. At this point however, in analogy to the Tamm-Dancoff approximation made in the previous calculations[1] of meson-nucleon scattering, this set of equations will be approximated by omitting each N.T.D. amplitude depending on four or more creation or absorption operators. With this approximation the two-meson amplitudes of (13) are given by the following equations:

$$\begin{aligned}(\epsilon-\omega_q-\omega_k-E_{q+k})\langle a_{q\alpha}a_{k\beta}b_{-q-k,\,w}\rangle\\=G(2\pi)^{-\frac{3}{2}}\theta_{-q-k,\,w}\sum_u\{&(w^*\gamma\tau_\alpha u)(2\omega_q)^{-\frac{1}{2}}\langle a_{k\beta}b_{-ku}\rangle\\&+(\omega^*\gamma\tau_\beta u)(2\omega_k)^{-\frac{1}{2}}\langle a_{q\alpha}b_{-qu}\rangle\},\end{aligned} \tag{14}$$

$$\begin{aligned}(\epsilon+\omega_q+\omega_k+E_{q+k})\langle a^*_{-q\alpha}a^*_{-k\beta}b_{-q-k,\,w}\rangle\\=G(2\pi)^{-\frac{3}{2}}\bar{\theta}_{-q-k,\,w}\sum_u\{&(w^*\gamma\tau_\alpha u)(2\omega_q)^{-\frac{1}{2}}\langle a^*_{-k\beta}b_{-ku}\rangle\\&+(\omega^*\gamma\tau_\beta u)(2\omega_k)^{-\frac{1}{2}}\langle a^*_{-q\alpha}b_{-qu}\rangle\},\end{aligned} \tag{15}$$

$$\begin{aligned}(\epsilon+\omega_q-\omega_k-\eta_{-q-k,\,w}E_{q+k})\langle a^*_{-q\alpha}a_{k\beta}b_{-k-q,\,w}\rangle\\=G(2\pi)^{\frac{3}{2}}\sum_u\{&\bar{\theta}_{-q-k,\,w}(w^*\gamma\tau_\alpha u)(2\omega_q)^{-\frac{1}{2}}\langle a_{k\beta}b_{-ku}\rangle\\&+\theta_{-q-k,\,w}(w^*\gamma\tau_\beta u)(2\omega_k)^{-\frac{1}{2}}\langle a^*_{-q\alpha}b_{-qu}\rangle\},\end{aligned} \tag{16}$$

where $\bar{\theta}_{pu}=(1-\theta_{pu})$.

For this last Eq. (16), there arises the question discussed in D2, the possibility that an unphysical singularity may occur in the amplitude $\langle a^*_{-q\alpha}a_{k\beta}b_{-k-q,\,w}\rangle$. For a positive spinor w, the factor multiplying this amplitude in (16) may vanish, since there exist infinitely many k, for a given q, for which

$$\epsilon+\omega_q-\omega_k-E_{q+k}=0. \tag{17}$$

Such a singularity in $\langle a^*_{-q\alpha}a_{k\beta}b_{-k-q,\,w}\rangle$ would mean that, in coordinate space, this amplitude would have a finite value at infinity. This would imply the existence of a real process resulting in one meson, one nucleon and minus-one meson. Such a process would only be possible here if a real meson were present in the comparison state Ψ_0. In the present problem, however, this is not the case, since the comparison state Ψ_0 is the physical vacuum.[6] The amplitude $\langle a^*_{-q\alpha}a_{k\beta}b_{-q-k,\,w}\rangle$ must therefore be finite even for momenta satisfying Eq. (17). For fixed q and positive w, the right-hand side of (16) must vanish for every k satisfying (17); thus

$$\sum_u(w^*\gamma\tau_\beta u)\langle a^*_{-q\alpha}b_{-qu}\rangle=0 \tag{18}$$

[6] The N.T.D. equations which have been set up have the same form for any comparison state Ψ_0. It is only at this point that the nature of the comparison state is invoked to provide an additional condition on the amplitudes which are to describe the physical situation. The boundary condition which had been proposed in D2 is not correct, since the principal value singularity still allows a standing wave at infinity.

for an infinite set of positive energy spinors $w_+(-k-q)$. From this it follows[7] that (since γ and τ_β are nonsingular)

$$\sum_u u\langle a^*_{-q\alpha}b_{-qu}\rangle=0. \qquad (19)$$

The spinors u form a complete set, so that

$$\langle a^*_{-q\alpha}b_{-qu}\rangle=0, \qquad (20)$$

for all q, α, u. From (15) and (16), it then follows that

$$\langle a^*_{-q\alpha}a^*_{-k\beta}b_{-q-k,w}\rangle\equiv 0, \qquad (21)$$

and that, for positive w,

$$\langle a^*_{-q\alpha}a_{k\beta}b_{-q-k,w}\rangle=0. \qquad (22)$$

For the three-nucleon amplitude of (13), the N.T.D. equation is

$$\begin{aligned}(\epsilon-\eta_{-qu}E_q+\eta_{p-q,v}E_{p-q}-\eta_{pv}E_p)\langle b_{-qu}b^*_{p-q,v}b_{pw}\rangle\\ =G(2\pi)^{-\frac{3}{2}}\{(2\omega_q)^{-\frac{1}{2}}(w^*\gamma\tau_\alpha v)\langle b_{-qu}(a_{q\alpha}+a^*_{-q\alpha})\rangle\\ \times(\theta_{pw}\bar\theta_{p-q,v}-\bar\theta_{pw}\theta_{p-q,w})+(2\omega_p)^{-\frac{1}{2}}(u^*\gamma\tau_\beta v)\\ \times\langle b_{pw}(a_{-p\alpha}+a^*_{p\alpha})\rangle(\bar\theta_{-qu}\theta_{p-q,v}-\theta_{-qu}\bar\theta_{p-q,v})\}. \qquad (23)\end{aligned}$$

From this, it follows at once that

$$\langle b_{-qu}b^*_{p-q,v}b_{pw}\rangle=0 \qquad (24)$$

when the spinors u, v, w are all positive or all negative. Consider next the equation (23) for u positive and v, w negative [recalling Eq. (20)],

$$\begin{aligned}(\epsilon-E_p-E_{p-q}-E_q)\langle b_{-qu}b^*_{p-q,v}b_{pw}\rangle\\ =-G(2\pi)^{-\frac{3}{2}}(2\omega_p)^{-\frac{1}{2}}(u^*\gamma\tau_\alpha w)\langle b_{pw}a_{-p\alpha}\rangle. \qquad (25)\end{aligned}$$

For this amplitude it is again necessary to invoke the boundary condition that Ψ_0 is the physical vacuum state since, for every p, the energy factor on the left of (25) vanishes for infinitely many q. From reasoning similar to that given above, this leads to the condition that, for w negative,

$$\langle b_{pw}a_{-p\alpha}\rangle=0. \qquad (26)$$

From (23), it now follows that the amplitude $\langle b_{-qu}b^*_{p-q,v}b_{pw}\rangle$ is zero except when one (and only one) of u, v, w refers to a negative energy state. For these nonzero components, Eq. (23) reduces to

$$\begin{aligned}(\epsilon+E_q+E_{p-q}-E_p)\langle b_{-qu}b^*_{p-q,v}b_{pw}\rangle\\ =G(2\pi)^{-\frac{3}{2}}(2\omega_p)^{-\frac{1}{2}}(u^*\gamma\tau_\alpha w)\langle b_{pw}a_{-p\alpha}\rangle, \qquad (27)\end{aligned}$$

$$\begin{aligned}(\epsilon-E_q-E_{p-q}-E_p)\langle b_{-qu}b^*_{p-q,v}b_{pw}\rangle\\ =G(2\pi)^{-\frac{3}{2}}\{(2\omega_q)^{-\frac{1}{2}}(w^*\gamma\tau_\alpha v)\langle b_{-qu}a_{q\alpha}\rangle\\ +(2\omega_p)^{-\frac{1}{2}}(u^*\gamma\tau_\alpha v)\langle b_{pw}a_{-p\alpha}\rangle\}, \qquad (28)\end{aligned}$$

$$\begin{aligned}(\epsilon-E_q+E_{p-q}+E_p)\langle b_{-qu}b^*_{p-q,v}b_{pw}\rangle\\ =-G(2\pi)^{-\frac{3}{2}}(2\omega_p)^{-\frac{1}{2}}(w^*\gamma\tau_\alpha v)\langle b_{-qu}a_{q\alpha}\rangle, \qquad (29)\end{aligned}$$

where (27), (28), and (29) in turn refer to the cases u, v, and w negative.

For the remaining amplitude of (13), the N.T.D. equation is

$$\begin{aligned}(\epsilon-\eta_{0w}M)\langle b_{0w}\rangle=G(2\pi)^{-\frac{3}{2}}\int d_3k\sum_u\theta_{-ku}\\ \times(w^*\gamma\tau_\alpha u)(2\omega_k)^{-\frac{1}{2}}\langle b_{-ku}a_{k\alpha}\rangle, \qquad (30)\end{aligned}$$

taking account of Eqs. (20).

These equations now form the complete set of Tamm-Dancoff equations in the present approximation. For each nonzero amplitude on the right of (13), there is an equation expressing this amplitude in terms of $\langle a_{p\alpha}b_{-pu}\rangle$. Substituting these expressions in Eq. (13) then gives an equation for $\langle a_{p\alpha}b_{-pu}\rangle$ alone. The use made of the fact that Ψ_0 is the physical vacuum state has ensured that this equation will contain no unphysical singularities. The equation obtained is

$$\begin{aligned}(\epsilon-\omega_p-E_p)\langle a_{p\alpha}b_{-pu}\rangle=\Big\{(T_N+T_m)\langle b_{-pu}a_{p\alpha}\rangle\\ +\frac{G^2}{16\pi^3}\int\frac{d_3k}{(\omega_p\omega_k)^{\frac{1}{2}}}(u^*[C(p,k)Q'_{\alpha\beta}\\ +D(p,k)Q_{\alpha\beta}]w)\langle b_{-pw+}a_{p\beta}\rangle\Big\}, \qquad (31)\end{aligned}$$

where S_N, S_m, $C(p,k)$, and $U(p,k)$ are defined as follows:

$$\begin{aligned}T_N=\frac{3G^2}{16\pi^3}\Big\{u^*\int\frac{d_3k}{\omega_k}\Big(\frac{\gamma\Omega^+(-p-k)\gamma}{\epsilon-\omega_p-\omega_k-E_{p+k}}\\ +\frac{\gamma\Omega^-(-p-k)\gamma}{\epsilon-\omega_p+\omega_k+E_{p+k}}\Big)u\Big\}, \qquad (32)\end{aligned}$$

$$\begin{aligned}T_m=\frac{G^2}{16\pi^3}\frac{2}{\omega_p}\int d_3k\,\mathrm{Sp}[\gamma\Omega^+(-k)\gamma\Omega^-(-p-k)]\\ \times\Big(\frac{1}{\epsilon-E_p-E_k-E_{p+k}}-\frac{1}{\epsilon-E_p+E_k+E_{p+k}}\Big), \qquad (33)\end{aligned}$$

$$C(p,k)=\frac{\gamma\Omega^+(-p-k)\gamma}{\epsilon-\omega_p-\omega_k-E_{p+k}}-\frac{\gamma\Omega^-(-p-k)\gamma}{\epsilon-E_p-E_k-E_{p+k}}, \qquad (34)$$

$$D(p,k)=\frac{\gamma\Omega^+(0)\gamma}{\epsilon-M}+\frac{\gamma\Omega^-(0)\gamma}{\epsilon+M}=\gamma\frac{1}{\epsilon-\beta M}\gamma. \qquad (35)$$

The quantities Q', Q occurring in (31) are discussed in D3. In particular $Q_{\alpha\beta}=\tau_\alpha\tau_\beta$, $Q'_{\alpha\beta}=\tau_\beta\tau_\alpha$, and (Q',Q) have the eigenvalues (2,0) for $T=\frac{3}{2}$ states, and $(-1,3)$ for $T=\frac{1}{2}$ states.

It is of interest now to compare this equation with the corresponding equation of the old Tamm-Dancoff theory [Eq. (10) of D3]. Owing to the use of amplitudes relating to the physical vacuum state rather

[7] If $Q=-k-q$, then the vectors Q satisfying (17) lie on a complete surface enclosing the origin. If $(w^*(Q)A)=0$, then $\Omega^+(Q)A=0$. From this, $[(E_Q-\beta M)/Q+\alpha\cdot Q/Q]A=0$. If this expression is integrated over the surface, then the last term vanishes, and one finds $A=0$.

than to the bare vacuum state, there is no vacuum self-energy term in the present equation, the energy of the state being referred directly to the energy of the real vacuum. The graphs corresponding to the terms appearing in Eq. (31) are given in Fig. 1. In these graphs the + sign refers to normal particles and the − sign to the minus-particles, borrowed from the vacuum state. If a + line goes forward, it corresponds to a plus particle, if backward then to a plus antiparticle (of positive energy), while a − line going forward corresponds to a minus antiparticle, backward to a minus particle. Thus graphs (a) and (b) correspond to the nucleon and meson self-energy terms T_N and T_m. In these the (+) terms do not differ from the corresponding terms in the old Tamm-Dancoff equation, but the energy denominators of the (−) terms differ just in the way necessary for a close correspondence with the self-energy expressions of the covariant theory (see references 5 and Sec. III). The crossed graphs (c) and (d) are unchanged; the numerical calculations previously reported for the $T=\frac{3}{2}$, $S_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ states therefore need no modification in the approximation in which the self-energy graphs (a) and (b) are neglected. The uncrossed graph differs again only in the (−) term, the energy denominator $1/(E+M)$ replacing the previous more complicated denominator $1/(E-E_p-\omega_p-M-E_k-\omega_k)$; this term is effective only for the $T=\frac{1}{2}$, $S_{\frac{1}{2}}$ state, the modification being sufficient to permit carrying through all renormalization necessary for this state.

For the wave function $\psi(k)$ of the system we define

$$\psi(k)=(\omega_k)^{\frac{1}{2}}\sum_{u+}u\langle b_{-ku}a_{k\alpha}\rangle. \tag{36}$$

Suppressing the isotopic spin suffices, Eq. (31) now becomes

$$(\epsilon-\omega_p-E_p)\psi(p)=\left\{(T_N+T_m)\psi(p)+\Omega^+(-p)\frac{G^2}{16\pi^3}\times\int\frac{d_3k}{\omega_k}[Q'C(p,k)+QU(p,k)]\psi(k)\right\}. \tag{37}$$

It is now required to find a solution $\psi(k)$ of the form

$$\psi(k)=\psi_0(k)+P\frac{1}{\epsilon-\omega_k-E_k}f(k), \tag{38}$$

where $\psi_0(k)=\delta(\epsilon-\omega_k-E_k)f_0(k)$. If $f_0(k)$ is an eigenfunction of angular momentum j, parity w, and isotopic spin T, then the phase shift δ_{jwT} for the meson-nucleon scattering in this state is given by

$$\tan\delta_{jwT}=-\pi f(l)/f_0(l), \tag{39}$$

where $\epsilon-\omega_l-E_l=0$.

III. SELF-ENERGY GRAPHS

The contributions of the meson and nucleon self-energy graphs of Fig. 1 to the integral equation for meson-nucleon scattering have been given in Eqs. (32) and (33). Consider first the nucleon self-energy S_N. The expression (32) may be rewritten in the following form:

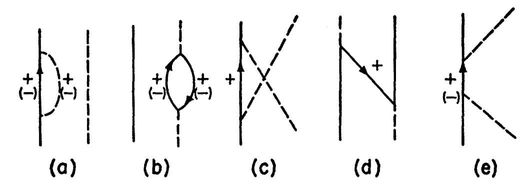

FIG. 1. The graphs responsible for meson-nucleon scattering in the new Tamm-Dancoff theory.

$$T_N(p)=-\frac{M}{E_p}\left\{\bar{v}(p)\left(\frac{3G^2}{16\pi^3}\int\frac{Md_3k}{E_{p+k}\omega_k}\left[\frac{\gamma_5\Lambda^+(-p-k)\gamma_5}{\epsilon-\omega_p-\omega_k-E_{p+k}}-\frac{\gamma_5\Lambda^-(p+k)\gamma_5}{\epsilon-\omega_p+\omega_k+E_{p+k}}\right]\right)v(p)\right\}, \tag{40}$$

where $\Lambda^\pm$ are the covariant projection operators, related to the $\Omega^\pm$ by

$$\Lambda^\pm(\pm p)=\pm\frac{E_p}{M}\Omega^\pm(p)\beta, \tag{41}$$

and $\bar{v}(p)=(M/E_p)^{\frac{1}{2}}u^*(p)\beta$, $v(p)=(M/E_p)^{\frac{1}{2}}u(p)$.

It was pointed out in D1 that this second-order self-energy expression, obtained with the N.T.D. theory, is related to the self-energy Σ_2 of the covariant theory in the following way:

$$T_N(p)=\frac{M}{E_p}\bar{v}(p)\Sigma_2(P)v(p), \tag{42}$$

where

$$\Sigma_2(P)=-\frac{3G^2}{(2\pi)^4}\int\frac{d_4k}{k^2-\mu^2}\left(\gamma_5\frac{1}{\mathbf{P}-\mathbf{k}-M}\gamma_5\right), \tag{43}$$

and P is the four-vector $(\Delta+E_p, -p)$, $\Delta=\epsilon-\omega_p-E_p$. For a momentum p on the energy shell $\Delta=0$, (43) represents the self-energy of a free nucleon. This relationship (42) has been demonstrated by Visscher[5] and in D1, by using an adaptation of Cini's method[8] (old Tamm-Dancoff theory) to the N.T.D. theory. It may also be verified directly by integration of (43) over the variable k_0, the result (40) being obtained after a little rearrangement.

The covariant self-energy Σ_2 has the following form

$$\Sigma_2(P)=\delta M+A(\mathbf{P}-M)+\Sigma_c(\mathbf{P}), \tag{44}$$

where the finite part Σ_c is given by the integral

$$\Sigma_c(\mathbf{P})=-\frac{3G^2}{16\pi^2}\int_0^1dx\left[[M-(1-x)\mathbf{P}]\ln\frac{\phi(P^2)}{\phi(M^2)}+(\mathbf{P}-M)\frac{2M^2x^2(1-x)}{\phi(M^2)}\right], \tag{45}$$

[8] M. Cini, Nuovo cimento 10, 526 and 614 (1953).

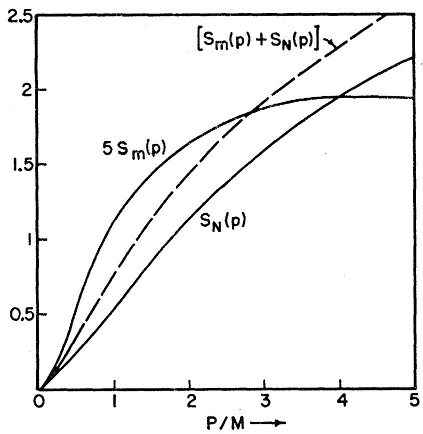

FIG. 2. The self-energy functions $S_N(p)=G^2R_N(p)$ and $S_m(p)=G^2R_m(p)$, plotted for $G^2/4\pi=13$ and $\epsilon=M$.

with $\phi(P^2)=(1-x)\mu^2+xM^2-x(1-x)P^2$. In the expression (40) for T_N, only the expectation value of Σ_2 in the state $v(p)$ is needed. This is obtained by replacing $\mathbf{P}-M$ by $\Delta E_p/M$ and P^2 by $M^2+\Delta(\Delta+2E_p)$ in (44) and (45). If we take $\mu=0$ in the integral (45), then

$$\frac{M}{E_p}\bar{v}(p)\Sigma_2(P)v(p)=\frac{M\delta M}{E_p}+A\Delta+G^2R_N\Delta, \tag{46}$$

where

$$R_N=-\frac{3}{32\pi^2}\frac{\Delta}{E_p}\frac{M^2-E_p(\Delta+2E_p)}{M^2+\Delta(\Delta+2E_p)} \times\left[\frac{\Delta(\Delta+2E_p)}{M^2+\Delta(\Delta+2E_p)}\ln\left(\frac{-\Delta(\Delta+2E_p)}{M^2}\right)-1\right]. \tag{47}$$

According to its definition, $R_N(p)$ vanishes on the energy shell $\Delta=0$. $R_N(p)$ is plotted in Fig. 2 as function of p for $\epsilon=M$.

The first term of (46) may be brought to the left of Eq. (37) to be absorbed as a correction to the nucleon mass. If one neglects $T_m(p)$ for the present, the infinite constant A gives rise to a coupling constant renormalization when the remainder of $T_N(p)$ is brought to the left of (37), this equation then taking the form

$$(\epsilon-\omega_p-E_p)(1-G_1^2R_N(p))\psi(p) =\frac{G_1^2}{16\pi^3}\Lambda^+(-p)\int\frac{d_3k}{\omega_k}(Q'C(p,k)+QU(p,k))\psi(k), \tag{48}$$

where the renormalized coupling constant is $G_1^2=G^2/(1-A)$. The nucleon self-energy divergence which appears explicitly in the integral equation (37) may therefore be renormalized satisfactorily, the integral equation being brought to the finite form (48), where the propagator for the system $(\epsilon-E_p-\omega_p)^{-1}$ is now replaced by a modified propagator $[(\epsilon-\omega_p-E_p)\times(1-G_1^2R_N(p))]^{-1}$.

However it is also necessary to consider the meson self-energy $T_m(p)$. This is also closely related with the corresponding self-energy expression in the covariant theory, in fact

$$T_m(p)=-\frac{1}{2\omega_p}\Omega(Q^2) \tag{49}$$

where Ω is the meson self-energy in lowest approximation,

$$\Omega(Q^2)=\frac{2G^2}{(2\pi)^4}\int d_4k \times\mathrm{Sp}[\gamma_5(Q+\tfrac{1}{2}k-M)^{-1}\gamma_5(Q-\tfrac{1}{2}k-M)^{-1}], \tag{50}$$

and Q is the four-vector $(\Delta+\omega_p, p)$. This relationship has been established by Visscher[5] using the Cini method for the N.T.D. theory, and may readily be verified by integration of (50) over k_0. $\Omega(Q^2)$ is then written in the form

$$\Omega(Q^2)=\delta(\mu^2)+B(Q^2-\mu^2)+\Omega_C(Q^2). \tag{51}$$

The function $\Omega_C(Q^2)$ is well known, and for the value of Q appropriate we will write it in the form

$$\Omega_C(Q^2)=-2\omega_pG^2R_m(p)\Delta, \tag{52}$$

where, neglecting terms of order $(\mu/M)^2$ in $R_m(p)$,

$$R_m(p)=-\frac{1}{2\pi^2}\frac{\Delta+2\omega_p}{2\omega_p}\left\{1-\left(\frac{\Delta(\Delta+2\omega_p)-4M^2}{\Delta(\Delta+2\omega_p)}\right)^{\frac{1}{2}} \times ar\sinh\left(\frac{(-\Delta(\Delta+2\omega_p))^{\frac{1}{2}}}{2M}\right)\right\}. \tag{53}$$

The function $R_m(p)$ is plotted in Fig. 2 for $\epsilon=M$. It also vanishes on the energy shell $\Delta=0$ (by definition) and is generally small compared with the nucleon term $R_N(p)$.

It must be remarked here that the second (infinite) term of (51) is not proportional to Δ as is appropriate for a coupling constant renormalization in the Tamm-Dancoff theory, but is given by $B\Delta(\Delta+2\omega_p)$. Since the meson field satisfies the Klein-Gordon equation rather than a linear equation, such a result is difficult to avoid in any prescription based on the covariant theory. A true mass and coupling-constant renormalization in the Tamm-Dancoff theory requires the subtraction of a term $C+D\Delta$ from the expression $T_m(p)$ of (33); however no such subtraction can succeed in making $T_m(p)$ finite. The N.T.D. method has led to a well-defined calculation of the finite part of the meson self-energy graph, closely related to that of the covariant theory, but it fails in so far as the interpretation of the infinite parts is concerned.

If the infinite parts of $R_m(p)$ are simply omitted, the integral equation takes the form (48) with $\{(\epsilon-\omega_p-E_p)\times[1-G_1^2(R_N(p)+R_m(p))]\}^{-1}$ as the propagator for the system. In view of the difficulty just mentioned, this procedure is not as convincing as it might have been, but there is still one further point to be made. According to their definition, both $R_m(p)$ and $R_N(p)$ must vanish on the energy shell $E_p+\omega_p=\epsilon$. For very large p, they have the following asymptotic forms:

$$R_N(p)\sim\frac{3}{32\pi^2}\left(\ln\left(\frac{2p\epsilon}{M^2}\right)-1\right),$$
$$R_m(p)\sim\frac{\epsilon}{8\pi^2 p}\left(\ln\left(\frac{2p\epsilon}{M^2}\right)-2\right). \quad (54)$$

The second factor of the modified propagator therefore must have a singularity at some high momentum p, for all G^2, since $[R_N(p)+R_m(p)]$ ranges from 0 to $+\infty$. For $G^2/4\pi=13$ and $\epsilon=M$, this singularity lies at $p=1.3M$.

Such a singularity is not physically reasonable. Its existence would imply the presence of a corresponding singularity in the wave function of the meson-nucleon state, whose interpretation would require the presence of waves of very high momentum at infinity, although the ingoing meson-nucleon waves have energy ϵ. This would be possible only if some bound system[9] could be formed with rest mass far below that of the nucleon. This is clearly not the case, and this singularity must be regarded only as a consequence of the approximations made in setting up the present N.T.D. theory. Despite the success of the N.T.D. method in relating the self-energy graphs of the integral equations to self-energy expressions of the covariant theory, the presence of this singularity prevents the use of this renormalization method for calculating self-energy corrections to the calculations of D3 for the $T=\frac{3}{2}$ states.

IV. RENORMALIZATION PROBLEM FOR THE STATES $T=1/2$, $j=1/2$

A further renormalization problem arises for the $S_{\frac{1}{2}}$ and $P_{\frac{1}{2}}$ states of isotopic spin $T=\frac{1}{2}$, since they respectively have the same spin, parity, and isotopic spin as the minus-one-anti-nucleon state and the one nucleon state. In Eq. (37), $Q=3$ for the $T=\frac{1}{2}$ states and the term $U(p,k)$ becomes effective. The integral equation now has no finite solution: this may be recognized most readily by attempting to solve the integral equation as a power series in G^2, by iteration from $\psi_0(k)$. The successive terms correspond to graphs built up from the basis graphs of Fig. 1, and two typical graphs

[9] T. D. Lee [Phys. Rev. **95**, 1329 (1954)] has recently studied a special meson theory in which the Tamm-Dancoff approximation used here is actually exact. Complete solutions are obtained, and it is found that the modified propagator has a pole of this kind, when there exists a bound state lying below the energies of the quanta of the individual fields.

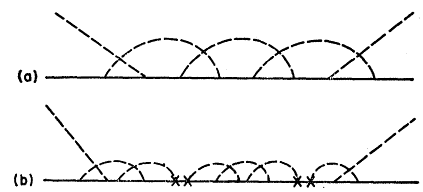

FIG. 3. Typical graphs generated by the kernels $C(p,k)$ and $U(p,k)$ of Eq. (55).

are shown in Fig. 3. Graphs in which the uncrossed graph $U(p,k)$ has operated at least once have a structure of the type (b), and the vertices and self-energy parts thus generated require renormalization.

Consider then the integral equation (38). It will be convenient to use the covariant projection operators (41) and the equation then takes the form

$$(\epsilon-\omega_p-E_p)\psi(p) =G_\epsilon^2(p)\Lambda^+(-p)\frac{M}{E_p}\left\{\int\frac{d_3k}{\omega_k}Q'H(p,k)\psi(k) +\gamma_5\frac{Q\beta}{\epsilon-\beta M}\int\gamma_5\psi(k)\frac{d_3k}{\omega_k}\right\}, \quad (55)$$

where

$$H(p,k)=-\frac{M}{E_{p+k}}\left\{\frac{\gamma_5\Lambda^+(-p-k)\gamma_5}{\epsilon-\omega_p-\omega_k-E_{p+k}} +\frac{\gamma_5\Lambda^-(p+k)\gamma_5}{\epsilon-E_p-E_k-E_{p+k}}\right\}, \quad (56)$$

and we have written $G_\epsilon^2(p)$ in place of $F_\epsilon(p)G^2/16\pi^3$, where the factor $F_\epsilon(p)$ arises from any (nonsingular) modification of the propagator $(\epsilon-\omega_p-E_p)^{-1}$ of the system due to particle self-energy effects. If these are neglected, $F_\epsilon(p)$ is to be replaced by unity. Since the kernel $H(p,k)$ lies between positive energy projection operators, its form may be simplified to

$$H(p,k)=H_-(p,k)(1+\beta)/2+H_+(p,k)(1-\beta)/2, \quad (57)$$

where

$$H_\pm(p,k)=\frac{1}{2E_{p+k}}\left\{\frac{\mp(E_p+E_k-E_{p+k})+M}{\epsilon-\omega_p-\omega_k-E_{p+k}} +\frac{\mp(E_p+E_k+E_{p+k})+M}{\epsilon-E_p-E_k-E_{p+k}}\right\}. \quad (58)$$

For the $T=\frac{1}{2}$ states, Q and Q' take the values 3, -1.

A solution of the form (39),

$$\psi(p)=\psi_0(p)+\frac{P}{\epsilon-\omega_p-E_p}f(p) \quad (59)$$

is now required. Consider first the function $g(p)$ which is the solution of

$$g(p)=G_\epsilon^2(p)\left[B_\epsilon(p)+\Lambda^+(-p)\frac{M}{E_p}\int\frac{d_3k}{\omega_k}Q'H(p,k)\times\frac{P}{\epsilon-\omega_k-E_k}g(k)\right],\quad (60)$$

where $B_\epsilon(p)$ is the Born approximation amplitude

$$B_\epsilon(p)=\Lambda^+(-p)\frac{M}{E_p}\int\frac{d_3k}{\omega_k}Q'H(p,k)\psi_0(k).\quad (61)$$

Procedures for the approximate solution of this Eq. (60) have been discussed in D3 and we shall not consider this problem further here.

By substitution of (59) in the Eq. (55) and use of the Eq. (60), it will be found that the complete function $f(p)$ is now given by

$$f(p)=g(p)+G_\epsilon^2(p)\frac{M}{E_p}\Lambda^+(-p)\Gamma_\epsilon(p)\chi,\quad (62)$$

where $\Gamma_\epsilon(p)$ is the solution of the equation

$$\Gamma_\epsilon(p)=\gamma_5+\frac{G^2}{16\pi^3}\int\frac{d_3k}{\omega_k}Q'H(p,k)P_\epsilon(k)\frac{M}{E_k}\times\Lambda^+(-k)\Gamma_\epsilon(k),\quad (63)$$

and

$$\chi=\frac{Q\beta}{\epsilon-\beta M}\int\gamma_5\psi(k)\frac{d_3k}{\omega_k}.\quad (64)$$

In (63) the notation

$$P_\epsilon(k)=F_\epsilon(k)\frac{P}{\epsilon-\omega_k-E_k}\quad (65)$$

has been used, so that $P_\epsilon(k)$ now represents the modified propagator of the theory. It is useful also to define the adjoint $\Gamma_\epsilon^\dagger(p)$ of $\Gamma_\epsilon(p)$, satisfying the equation

$$\Gamma_\epsilon^\dagger(p)=\gamma_5+\frac{G^2}{16\pi^3}Q'\int\frac{d_3k}{\omega_k}\Gamma_\epsilon^\dagger(k)\Lambda^+(-k)\frac{M}{E_k}\times P_\epsilon(k)H(k,p).\quad (66)$$

After substitution of (59) (62) in Eq. (64), the following equation is obtained for the spinor:

$$\chi=\frac{Q\beta}{\epsilon-\beta M}\int\frac{d_3k}{\omega_k}\gamma_5\times\left(\phi(k)+\frac{G^2}{16\pi^3}P_\epsilon(k)\frac{M}{E_k}\Lambda^+(-k)\Gamma_\epsilon(k)\chi\right),\quad (67)$$

where

$$\phi(k)=\psi_0(k)+\frac{P}{\epsilon-\omega_k-E_k}g(k).\quad (68)$$

With use of (66), the first integral on the right of (67) becomes

$$\begin{aligned}\int\frac{d_3k}{\omega_k}\gamma_5\phi(k)&=\int\frac{d_3q}{\omega_q}\Gamma_\epsilon^\dagger(q)\phi(q)-\frac{G^2}{16\pi^3}\int\int\frac{d_3q}{\omega_q}\Gamma_\epsilon^\dagger(q)\Lambda^+(-q)\times\frac{M}{E_q}P_\epsilon(q)Q'H(q,k)\phi(k)\frac{d_3k}{\omega_k}\\&=\int\frac{d_3q}{\omega_q}\Gamma_\epsilon^\dagger(q)\phi(q)-\int\frac{d_3q}{\omega_q}\Gamma_\epsilon^\dagger(q)\frac{P}{\epsilon-\omega_q-E_q}g(q)\\&=\int\frac{d_3q}{\omega_q}\Gamma_\epsilon^\dagger(q)\psi_0(q),\end{aligned}\quad (69)$$

and the solution χ of Eq. (67) may be written

$$\chi=\frac{Q\beta}{\epsilon-\beta M-Q\beta S(\epsilon)}\int\frac{d_3q}{\omega_q}\Gamma_\epsilon^\dagger(q)\psi_0(q),\quad (70)$$

where

$$S(\epsilon)=\int\gamma_5\frac{md_3k}{E_k\omega_k}P_\epsilon(k)\Lambda^+(-k)\Gamma_\epsilon(k).\quad (71)$$

The function $\Gamma_\epsilon(p)$ is the vertex operator of the present theory and corresponds to the sum of all vertex graphs of the type shown in Fig. 3(b). Its definition by Eq. (63) is purely formal, as $\Gamma_\epsilon(p)$ is an infinite quantity. The function $S(\epsilon)$ is the self-energy expression resulting from joining the outgoing meson and nucleon lines of the vertex $\Gamma_\epsilon(p)$, i.e., it corresponds to the sum of all simple self-energy graphs of the type shown in Fig. 3(b). It is now necessary to discuss the renormalization of $\Gamma_\epsilon(p)$ and $S(\epsilon)$ and the calculation of their finite parts.

(a) Vertex Operator

Only matrix elements of $\Gamma_\epsilon(p)$ leading to a positive energy spinor are effective in Eq. (63) and in the expression (62) owing to the presence of the projection operator $\Lambda^+(-p)$. It is therefore appropriate to reduce (63) to two-component form. We now define the two-component function $\Delta_\epsilon(p)$ by the relation

$$\Delta_\epsilon(p)=\tfrac{1}{2}(1+\beta)\Lambda^+(-p)\Gamma_\epsilon(p).\quad (72)$$

The matrix element appearing in (62) may be expressed in terms of this function by

$$\Lambda^+(-p)\Gamma_\epsilon(p)=\left(1-\frac{\gamma_5\sigma\cdot p}{E_p+M}\right)\Delta_\epsilon(p).\quad (73)$$

Now the spinor χ on which $\Delta_\epsilon(p)$ will operate in (62) corresponds to a state of zero momentum (since the system is being treated in c.m. system). From parity conservation, χ is a positive frequency spinor if the incident system is in a $p_{\frac{1}{2}}$ state, a negative frequency spinor if in an $S_{\frac{1}{2}}$ state. In zero approximation, then, $\Delta_\epsilon(p)$ may be reduced to

$$\Delta_\epsilon^{(1)}(p)=\frac{1+\beta}{2}\left[\frac{E_p+M}{2M}\gamma_5+\frac{\sigma\cdot p}{2M}\right]. \tag{74}$$

Clearly only the first of these terms is effective for the $S_{\frac{1}{2}}$ state, in which case the spinor χ standing to the right of $\Delta_\epsilon(p)$ has negative frequency, and only the second for the $p_{\frac{1}{2}}$ state, χ then being a positive frequency spinor. From considerations of rotational invariance and of parity, the function $\Delta_\epsilon(p)$ will have the general form

$$\Delta_\epsilon(p)=\frac{1+\beta}{2}\left[\frac{E_p+M}{2M}V_S(p)\gamma_5+V_P(p)\frac{\sigma\cdot p}{2M}\right], \tag{75}$$

where $V_S(p)$ and $V_P(p)$ are scalar functions. Corresponding to the adjoint operator $\Gamma_\epsilon^\dagger(p)$, an operator $\Delta_\epsilon^\dagger(p)$ is defined

$$\Delta_\epsilon^\dagger(p)=\Gamma_\epsilon^\dagger(p)\Lambda^+(-p)\frac{1+\beta}{2}$$
$$=\left[\frac{E_p+M}{2M}V^\dagger{}_S(p)\gamma_5+V^\dagger{}_P(p)\frac{\sigma\cdot p}{2M}\right]\frac{1+\beta}{2}. \tag{76}$$

To obtain equations separately for $V_S(p)$ and $V_P(p)$, we proceed from Eqs. (63) and (73), whence

$$\Delta_\epsilon(p)=\Delta_\epsilon^{(1)}(p)+\frac{G^2}{16\pi^3}Q'\int\frac{d_3q}{E_q\omega_q}\Delta_\epsilon^{(1)}(p)\times\left(H_+(p,q)\frac{1-\beta}{2}+H_-(p,q)\frac{1+\beta}{2}\right)P_\epsilon(q)\times\left(1-\frac{\gamma_5\sigma\cdot q}{E_q+M}\right)\Delta_\epsilon(q). \tag{77}$$

The integral on the right of (77) may now be simplified to

$$\frac{1+\beta}{2}\frac{G^2}{16\pi^3}Q'\int\frac{d_3q}{E_q\omega_q}\left(\frac{E_p+M}{2M}\right)\times\left(H_-(p,q)-\frac{\sigma\cdot p\sigma\cdot q}{(E_p+M)(E_q+M)}H_+(p,q)\right)P_\epsilon(q)\Delta_\epsilon(q).$$

Substitution of (75) for $\Delta_\epsilon(p)$, and comparison of terms on either side gives equations for V_S and V_P separately:

$$V_S(p)=1+\frac{G^2}{8\pi^2}Q'\int\frac{q^2dq}{E_q\omega_q}\frac{d\Omega_q}{4\pi}\times\left[H_-(p,q)-\frac{p\cdot q}{(E_p+M)(E_q+M)}H_+(p,q)\right]\times P_\epsilon(q)(E_q+M)V_S(q), \tag{78a}$$

$$V_P(p)=1+\frac{G^2}{8\pi^2}Q'\int\frac{q^2dq}{E_q\omega_q}\frac{d\Omega_q}{4\pi}\times\left[\frac{p\cdot q}{pq}H_-(p,q)-\frac{pq}{(E_p+M)(E_q+M)}H_+(p,q)\right]\times P_\epsilon(q)\frac{q(E_p+M)}{p}V_P(q). \tag{78b}$$

We will write these equations in the form

$$V_\alpha(p)=1+\frac{G^2}{16\pi^2}Q'\int dqL_\alpha(\epsilon,p,q)V_\alpha(q), \tag{79}$$

the kernels $L_\alpha(\epsilon,p,q)$ being obtained by integrating (78) over the angles of q. In terms of the symmetric function $K_\alpha(\epsilon,p,q)$ obtained by averaging the square bracket of (78) over the angle between p and q they are given by

$$L_S(\epsilon,p,q)=\frac{q^2}{E_q\omega_q}(E_q+M)U_S(\epsilon,p,q)P_\epsilon(q),$$
$$L_P(\epsilon,p,q)=\frac{q^3}{E_q\omega_q}\frac{E_p+M}{p}U_P(\epsilon,p,q)P_\epsilon(q). \tag{80}$$

The function $U_\alpha(\epsilon,p,q)$ has been calculated in D3 with the result

$$U_S(\epsilon,p,q)=(A-2M)K_0(C)+(\epsilon-M)K_0(B)+\frac{pq}{(E_p+M)(E_q+M)}(AK_1(C)+(\epsilon+M)K_1(B)),$$
$$U_P(\epsilon,p,q)=(A-2M)K_1(C)+(\epsilon-M)K_1(B)+\frac{pq}{(E_p+M)(E_q+M)}(AK_0(C)+(\epsilon+M)K_0(B)),$$

in which

$$A=E_p+\omega_p+E_q+\omega_q+M-\epsilon,$$
$$B=E_p+E_q-\epsilon,$$
$$C=\omega_p+\omega_q-\epsilon.$$

The K_0 and K_1 functions are explicitly:

$$K_0(x)=\frac{1}{2pq}\ln\left(\frac{\bar{E}+x+R}{\bar{E}+x-R}\right),$$

$$K_1(x)=-\frac{1}{2pq}(M^2+p^2+q^2-x^2)K_0(x)+\frac{1}{2pq}\left(1-\frac{x}{\bar{E}}\right),$$

with $\bar{E}=(E_{p+q}+E_{p-q})/2$, $R=pq/\bar{E}$.

It is now necessary to find an equation for the direct calculation of the finite part of V_S and of V_P. The renormalized function $V_\alpha{}^R(\epsilon,p)$ is related to $V_\alpha(\epsilon,p)$ by

$$V_\alpha{}^R(\epsilon,p)=Z_\alpha{}^{-1}V_\alpha(\epsilon,p). \tag{81}$$

Equation (79) then becomes

$$V_\alpha{}^R(\epsilon,p)=1+\left[\frac{G^2}{16\pi^2}Q'\int dqL_\alpha(\epsilon,p,q)V_\alpha{}^R(\epsilon,q)-\left(1-\frac{1}{Z_\alpha}\right)\right]. \quad (82)$$

The renormalized $V_\alpha{}^R(\epsilon,p)$ may be defined by the condition

$$V_\alpha{}^R(M,0)=1 \quad (83)$$

from which it follows that Z_α is to be chosen

$$\frac{1}{Z_\alpha}=1-\frac{G^2}{16\pi^2}Q'\int dqL_\alpha(\epsilon,0,q)V_\alpha{}^R(M,q). \quad (84)$$

From Eqs. (82) and (84), we may now construct a finite integral equation for $V_\alpha{}^R(M,p)$:

$$V_\alpha{}^R(M,p)=1+\frac{G^2}{16\pi^2}Q'\int dq \times(L_\alpha(M,p,q)-L_\alpha(M,0,q))V_\alpha{}^R(M,q). \quad (85)$$

This is an integral equation whose solution may be obtained by standard numerical methods. It may readily be verified that the dominant term of $L_\alpha(M,p,q)$ for large q is independent of p, so that the subtraction does lead to cancellation of the divergent terms of the integral. This requirement is, of course, essential for the success of this renormalization scheme, and the fact that it is met here provides a strong check on the appropriateness of the N.T.D. equations for the problem at hand.

When $V_\alpha{}^R(M,p)$ is known, the complete function is $V_\alpha{}^R(\epsilon,p)$ given by

$$V_\alpha{}^R(\epsilon,p)=V_\alpha{}^R(M,p)+\frac{G^2}{16\pi^2}Q'W_\alpha(p), \quad (86)$$

where $W_\alpha(p)$ satisfies the equation

$$W_\alpha(p)=\int dq(L_\alpha(\epsilon,p,q)-L_\alpha(M,p,q)V_\alpha{}^R(M,q)+\frac{G^2}{16\pi^2}Q'\int dqL_\alpha(\epsilon,p,q)W_\alpha(q). \quad (87)$$

In the physical case ($Q'=-1$) these kernels correspond to attractive interactions,[10] and the vertex functions $V_\alpha{}^R(\epsilon,p)$ may be shown (Appendix B) to decrease as a negative power of p for large p. The integral in (84), logarithmically divergent in Born approximation (where $V_\alpha{}^R(\epsilon,p)=1$), is now convergent and it may be shown (Appendix B) that, in fact,[11] $1/Z_\alpha=0$. It should be remarked that, in this theory, the expressions for Z_S and Z_P are quite different; this is a consequence of the asymmetry between the positive and negative frequency states included in the theory, required by the considerations of Sec. II.

In the covariant theory, the renormalized vertex operator $\Gamma_5{}^R(p',p)$ may be defined by the condition

$$\mathrm{Lim}_{p'\to p}[\bar{u}(p')\Gamma_5{}^R(p',p)u(p)]/[\bar{u}(p')\gamma_5u(p)]=1, \quad (88)$$

where p is the four-momentum of a real nucleon. It is to be noted that this condition only makes a direct statement concerning the diagonal elements of $\Gamma_5{}^R(p',p)$—the nondiagonal elements of $\Gamma_5{}^R(p',p)$ between positive and negative frequency states, are, of course, well-determined by this condition and are definite, but unknown, functions of $G^2/4\pi$. In the $p_{\frac{1}{2}}$ state, V_P involves the diagonal element of $\Gamma_\epsilon(q)$ and the renormalization procedure (83) adopted is the direct analog of (88), the four-momentum p being chosen $(M,0)$. However, in the $S_{\frac{1}{2}}$ state, the nondiagonal element of $\Gamma_\epsilon(q)$ is effective and V_S should be renormalized not to unity, but to a value whose magnitude we are at present unable to determine. $V_S{}^R$ and Z_S are therefore each undetermined to a constant factor. This inability to relate V_S to the diagonal elements of $\Gamma_\epsilon(q)$ is again a consequence of the lack of symmetry between positive and negative frequency states in the theory.

(b) Self-Energy Graphs

Consider now the calculation of the finite part of the self-energy $S(\epsilon)$ of (70). As Salam[12] has emphasized, there occur overlap divergences in these self energy parts, which require careful consideration. However Ward[13] has shown how these difficulties may be avoided by a special method which we shall adapt to the present case.

Consider then the derivative[14] of $S(\epsilon)$,

$$\frac{d}{d\epsilon}S(\epsilon)=\frac{G^2}{16\pi^3}\int\frac{Md_3k}{E_k\omega_k}\gamma_5\left(\frac{dP_\epsilon(k)}{d\epsilon}\Lambda^+(-k)\Gamma_\epsilon(k)+P_\epsilon(k)\Lambda^+(-k)\frac{d\Gamma_\epsilon(k)}{d\epsilon}\right). \quad (89)$$

[10] $L_S(\epsilon,p,q)$ is $(-\frac{1}{2})$ times the interaction kernel for the $T=\frac{3}{2}$, $S_{\frac{1}{2}}$ state, known to correspond to a strong repulsion. Similarly for the kernel $L_P(\epsilon,p,q)$.

[11] This result implies that, for the case of physical interest, $V_\alpha{}^R(\epsilon,p)$ satisfies the homogeneous part of Eq. (79). Edwards (Phys. Rev. **90**, 284 (1953)) has also pointed this out in his study of a covariant vertex function, based on a different approximation. The solution of this homogeneous equation is uncertain to a factor $A(\epsilon)$. Only $A(M)$ is specified by the boundary condition (83), and $A(\epsilon)$ must be obtained by comparison of the asymptotic forms obtained for energies ϵ and M. The numerical solution of a singular homogeneous integral equation requires great care in the region of high momenta, whereas the Eqs. (85), (87) are well-behaved and stable in this respect. It should also be noted that, for positive Q' (i.e., a repulsive interaction), the renormalized vertex function is still well-defined by Eqs. (85), (87), but does not satisfy the homogeneous equation.

[12] A. Salam, Phys. Rev. **82**, 217 (1951).

[13] J. C. Ward, Proc. Phys. Soc. (London) **A64**, 54 (1951).

[14] The derivative of a principal value integral $\int\phi(p)P_\epsilon(p)dp$ with respect to the position of the singularity will be denoted symbolically by $\int\phi(p)(dP_\epsilon(p)/d\epsilon)dp$. Its evaluation will be discussed in Appendix A.

Substituting for γ_5 the expression given in Eq. (66) and using the derivative of (63), namely

$$\frac{d\Gamma_\epsilon(q)}{d\epsilon}=\frac{G^2}{16\pi^3}Q'\int\frac{d_3k}{E_k\omega_k}\times\left(\left[\frac{dH(\epsilon,q,k)}{d\epsilon}P_\epsilon(k)+H(\epsilon,q,k)\frac{dP_\epsilon(k)}{d\epsilon}\right]\Lambda^+(-k)\Gamma_\epsilon(k)+H(\epsilon,q,k)P_\epsilon(k)\Lambda^+(-k)\frac{d\Gamma_\epsilon(k)}{d\epsilon}\right), \quad (90)$$

it will be found that

$$\frac{dS(\epsilon)}{d\epsilon}=\frac{G^2}{16\pi^3}\int\frac{Md_3k}{E_k\omega_k}\Gamma_\epsilon^\dagger(k)\Lambda^+(-k)\frac{dP_\epsilon(k)}{d\epsilon}\Lambda^+(-k)\Gamma_\epsilon(k)+\left(\frac{G^2}{16\pi^3}\right)^2Q'\int\int\frac{M^2d_3pd_3q}{E_p\omega_pE_q\omega_q}\left[\Gamma_\epsilon^\dagger(p)\Lambda^+(-p)P_\epsilon(p)\times\frac{dH(\epsilon,p,q)}{d\epsilon}P_\epsilon(q)\Lambda^+(-q)\Gamma_\epsilon(q)\right]. \quad (91)$$

This may now be expressed in terms of V_S^R and V_S^P by substituting (73), (75), and (81). After some reduction, this becomes

$$\frac{dS(\epsilon)}{d\epsilon}=\frac{1-\beta}{2}Z_S^2\Lambda_S+\frac{1+\beta}{2}Z_P^2\Lambda_P, \quad (92)$$

where Λ_S and Λ_P are given by

$$\Lambda_\alpha(\epsilon)=\frac{G^2}{8\pi^2}\int\frac{k^2dk}{M\omega_k(E_k+M)}\frac{dP_\epsilon(k)}{d\epsilon}|v_\alpha(k)|^2+\left(\frac{G^2}{8\pi^2}\right)^2Q'\int\int\frac{q^2dq}{E_q\omega_q}\frac{p^2dp}{E_p\omega_p}\times\left(\bar{v}_\alpha(q)P_\epsilon(q)\frac{dK_\alpha(\epsilon,q,p)}{d\epsilon}P_\epsilon(p)v_\alpha(p)\right), \quad (93)$$

and β may be taken as $+1$ for the $p_{\frac{1}{2}}$ state ($\alpha=P$), -1 for the $S_{\frac{1}{2}}$ state ($\alpha=S$). The functions $v_\alpha(p)$ are given by $v_S(p)=(E_p+M)V_S^R(p)$ and $v_P(p)=pV_P^R(p)$. The quantities Λ defined by Eq. (93) each have the structure of a vertex. The first term of Λ diverges less than linearly. The second term is convergent over p and q integrations separately, and diverges less than linearly for the joint integration. The quantity $\Lambda(\epsilon)-\Lambda(\beta M)$ is therefore a convergent integral if the integrands are subtracted before the integration. The self-energy $S_\alpha(\epsilon)$ is then, on integration of (92)

$$S_\alpha(\epsilon)=S_\alpha(\beta M)+(\epsilon-\beta M)Z_\alpha^2\Lambda_\alpha(\beta M)+Z_\alpha^2\int_{\beta M}^{\epsilon}(\Lambda_\alpha(\epsilon')-\Lambda_\alpha(\beta M))d\epsilon'. \quad (94)$$

The last integral of (94) represents the finite part of the self energy $S_\alpha(\epsilon)$ after renormalization—we shall denote this finite part by $(G^2/16\pi^3)S_\alpha^R(\epsilon)$. Of the other terms, the first represents a correction to the nucleon mass and the second produces a coupling-constant renormalization at certain points.

The final expression for $f(p)$ is now

$$f(p)=g(p)+\frac{G_2^2}{16\pi^3}\frac{M}{E_p}F_\epsilon(p)\Lambda^+(-p)\Gamma_\epsilon(p)\times\frac{Q\beta}{\epsilon-\beta M-(G_2^2/16\pi^3)Q\beta S_\alpha^R(\epsilon)}\times\int\frac{d_3k}{\omega_k}\Gamma_\epsilon^{R\dagger}(k)\psi_0(k), \quad (95)$$

where G_2^2 is the renormalized coupling constant $G^2Z_\alpha^2/(1-Z_\alpha^2\Lambda_\alpha(\beta M))$. At all other places in this expression, where the coupling constant occurs implicitly, the unrenormalized coupling constant G^2 is effective. This corresponds to the appearance of G_2^2 only at the points marked with a cross on the graph (b) of Fig. 1. For an approximate theory, such as the present one, it is not to be expected that a renormalization scheme can be obtained which is complete and consistent at every point. What is significant is that, with the N.T.D. method, the finite parts of all divergent expressions have been identified uniquely and it has been shown that the infinite parts occurring in self-energy and vertex terms may be interpreted as mass and coupling constant renormalizations. For the $p_{\frac{1}{2}}$ state, it would be natural to identify G_2 with G, which corresponds simply to dropping all infinite terms. However, for the $S_{\frac{1}{2}}$ state, the renormalized vertex function is uncertain to a constant factor—if a finite vertex operator is defined corresponding to some standard condition such as (83), this uncertainty is transferred to an uncertainty in G_2. For the $S_{\frac{1}{2}}$ state, therefore, the two coupling constants G and G_2 are to be considered as quantities to be determined independently from comparison with experiment.

To obtain the phase shift, only the large components of $f(p)$ are needed. For the $S_{\frac{1}{2}}$ state, taking $\psi_0(p)=\delta(\epsilon-\omega_p-E_p)$,

$$f_S^+(p)=g_S^+(p)-\frac{G_2^2}{4\pi^2}\frac{E_p+2M}{3E_p}V_S^R(p)\times\frac{3}{\epsilon+M+(3G_2^2/16\pi^3)S_S^R(\epsilon)}\frac{l(E_l+M)}{2M\epsilon}V^\dagger{}_S^R(l). \quad (96a)$$

For the $p_{\frac{1}{2}}$ state, $\psi_0(q)$ is $\delta(\epsilon-\omega_p-E_p)\sigma\cdot p/p$ and

$$f_P{}^+(p)=g_P{}^+(p)+\frac{G_2{}^2}{4\pi^2}\frac{1}{2E_p}V_P{}^R(p)\times\frac{3}{\epsilon-M-(3G_2{}^2/16\pi^3)S_P{}^R(\epsilon)}\frac{l^2}{2M\epsilon}V\dagger_P{}^R(l). \quad (96b)$$

The phase shift is then given by expression (39).

V. DISCUSSION

It is of interest to summarize in this section the various difficulties which have appeared in the course of this work, and to compare the situation with that for the corresponding covariant Bethe-Salpeter equation. In lowest approximation for meson-nucleon scattering, this equation may be written

$$(\mathbf{p}-M-\Sigma_2(\mathbf{p}))(q^2-\mu^2-\Omega_2(q^2))\phi(p,q)=\frac{G^2}{(2\pi)^4}\int d_4k\left(\gamma_5\frac{Q'}{\mathbf{p}-\mathbf{k}-M}\gamma_5+\gamma_5\frac{Q}{\mathbf{p}+\mathbf{q}-M}\gamma_5\right)\phi(p+q-k,k). \quad (97)$$

This equation goes a little beyond the lowest approximation in that the use of the modified propagators corresponds to the inclusion of some terms of order G^4. For Eq. (97) a consistent coupling constant renormalization is not strictly possible since a product of propagators occurs, but it is reasonable to simply drop the infinite parts of Σ_2 and Ω_2. However, as Feldman[15] has pointed out, the modified nucleon propagator $[\mathbf{p}-M-\Sigma_2(\mathbf{p})]^{-1}$ has a pole in the complex p_0-plane, which probably has the consequence that (99) has no finite solution in its present form. In Sec. III, it has been remarked that the modified propagator in the Tamm-Dancoff theory also has a nonphysical pole, which prevents its use in the scattering calculations. The physical interpretation of the pole is quite different in the two cases and it appears most probable that there is no simple relationship between them. Each of these situations illustrates that approximations to the modified propagators may be used only with caution, and only when they contain no singularities beyond those required by the physical processes possible. Both in (37) and (97) then, the presence of a spurious pole prevents the use of the appropriate modified propagators in the theory.

For the $T=\frac{1}{2}$ states, the calculation of a covariant renormalized vertex may be reduced to the solution of a finite integral equation in a manner similar to that developed in Sec. IIIa, and the self-energy renormalization procedure may be followed through according to the method of Ward.[13] However the difficulty in the covariant theory is that reliable solutions cannot be obtained for these four-dimensional integral equations at present, especially as numerical techniques are inadequate to deal with integral equations involving several variables. The advantage of the N.T.D. theory is that all these operations involve the solution of one-dimensional integral equations by standard techniques. The advantage of the covariant theory is that it includes both positive and negative frequencies; the vertex renormalization is then well-determined in that the relation between the renormalized vertices effective for the $S_{\frac{1}{2}}$ state and for the $p_{\frac{1}{2}}$ state is known explicitly. For the graphs generated from (97), only an incomplete renormalization is possible, of course, in that the renormalization of each coupling constant is not independent of its position in the graphical structure. This situation also holds in the N.T.D. schemes; it is not a serous matter since the important thing is to calculate the finite parts corresponding to a given graph. The difficulty for the N.T.D. scheme is that, in the $S_{\frac{1}{2}}$ state, the uncertainty in the vertex renormalization involves essentially the introduction of a new parameter in the theory.

Both the B-S and the N.T.D. theories have the defect of not satisfying the symmetry principle of Goldberger and Gell-Mann,[16] that to every uncrossed graph generated in the theory a corresponding crossed graph should be included. One consequence of this symmetry principle is that the difference between the $T=\frac{3}{2}$ and $T=\frac{1}{2}$ scattering lengths at zero momentum must approach zero in the limit of vanishing meson mass. If this feature of the complete γ_5 theory is not present in the approximate calculations of S-state scattering, it may well be that the difference in slope calculated may reflect the approximations of the method rather than the content of the complete theory. This defect will exist in any strict Tamm-Dancoff theory, and this suggests the direction in which any further modifications of the noncovariant theory should tend.

In a more complete theory, one may expect that equations of the type (37) may be obtained, but with a far more complicated kernel. The modification to the S-state kernel used here may be expected to be quite different from that for P-state kernels, so that a comparison of (37) with experimental results may well lead to effective coupling constants different for S- and P-states. A practical difficulty (see Appendix A) in the use of the considerations of Sec. IV is that for the $T=\frac{1}{2}$, $j=\frac{1}{2}$ states the integral equations (60) and (85) have satisfactory solutions only if $G^2/4\pi$ is less than 6.7. This is certainly not a deep difficulty since for reasonable coupling strengths the unsatisfactory character of the solution only shows up for very large momenta (say $p>10M$) where the Tamm-Dancoff kernel could not be regarded as a reasonable approximation.

[15] G. Feldman, Proc. Roy. Soc. (London) **A223**, 112 (1954).

[16] M. Goldberger and M. Gell-Mann, Proceedings of the Rochester Conference (University of Rochester, Rochester, 1954).

Several more general criticisms of the N.T.D. method, made recently, should also be mentioned here. Symanzig[17] has considered the soluble case of the anharmonic oscillator and has shown that the N.T.D. amplitudes do not diminish with increasing complexity of the amplitude considered but may even increase exponentially, although the old Tamm-Dancoff amplitudes describing an excited state of this system do diminish satisfactorily. It seems possible that this lack of convergence may be a general feature of the N.T.D. method and that the equations considered here could not be regarded as the first approximation in a converging sequence of equations. Renormalization of explicit self-energy expressions occurring in higher approximations of the theory have been considered by Taylor[18] who concludes that these expressions do not have the structure necessary for the success of the renormalization procedure. The effect of these objections on the present theory is not yet clear however, but it seems probable that the equations studied here should still provide a first approximation to the more extended theory.

VI. ACKNOWLEDGMENTS

In conclusion, we wish to express our appreciation to Professor H. A. Bethe for discussions of the problems raised in this work.

APPENDIX A. CALCULATION OF DERIVATIVES OF PRINCIPAL VALUE INTEGRALS

The derivative of the following integral $I_\epsilon(k)$,

$$I_\epsilon(k)=\int_0^\infty P_\epsilon(k)\phi(k)dk \tag{A1}$$

with respect to energy ϵ, is to be calculated. $P_\epsilon(k)$ is singular at $k=l$, where $\epsilon=\omega_l+E_l$, so that (A1) is to be understood as a principal value integral. We may write

$$P_\epsilon(k)=A_\epsilon(k)/(k-l). \tag{A2}$$

The differentiation of $I_\epsilon(k)$ then leads to

$$\frac{dI_\epsilon(k)}{d\epsilon}=\int_0^\infty \frac{1}{(k-l)}\frac{dA_\epsilon(k)}{d\epsilon}\phi(k) +\frac{dl}{d\epsilon}\left[\frac{d}{dl}\int_0^\infty \frac{1}{k-l}A_\epsilon(k)\phi(k)dk\right]. \tag{A3}$$

The first term of (A3) is a principal value integral again, so that only the square bracket need be considered. The principal value integral appearing there may be given by the average of the integrals along two contours from 0 to $+\infty$ in the complex plane, are passing above $k=l$, the other below. Then for these contours C, the square bracket becomes simply

$$\int_C \frac{1}{(k-l)^2}B(k)dk, \tag{A4}$$

where $B(k)=\phi(k)A_\epsilon(k)$. Making use of the integral

$$\int_{-\infty}^{\infty} dk/(k-l)^2=0,$$

(A4) may be rearranged to give the following explicitly finite expression

$$\int_0^l \frac{B(l+x)-2B(l)+B(l-x)}{x^2}dx +\int_{2l}^\infty \frac{B(k)}{(k-l)^2}dk+\frac{2B(l)}{l}, \tag{A5}$$

suitable for numerical computations.

APPENDIX B. BEHAVIOR OF THE FUNCTIONS $g(p)$ AND $V_R(\epsilon,p)$ FOR LARGE p

Consider first the function $g(p)$ satisfying Eq. (60). The states $S_{\frac{1}{2}}$ and $P_{\frac{1}{2}}$ are of special interest in the present work, and our detailed remarks will be confined to this case. Only the large components $g_\alpha^+(p)$ of $g_\alpha(p)$ need be considered, and the integral equation for $g_\alpha^+(p)$ has been given in D3. With the notation of the present paper,

$$g_\alpha^+(p)=\frac{G^2}{8\pi^2}Q'B_\alpha(p)+\frac{G^2}{8\pi^2}Q'\int dkL_\alpha(\epsilon,p,k)g_\alpha^+(k), \tag{B1}$$

where $L_\alpha(\epsilon,p,k)$ is given by Eq. (80) and $B_\alpha(p)$ is obtained from the integral of (B1) by replacing $P_\epsilon(k)$ of L_α by $\delta(\epsilon-\omega_k-E_k)$ and $g_\alpha^+(k)$ by 1.

Now for $p\gg k$,

$$\begin{aligned} L_\alpha(\epsilon,p,k)&\sim\frac{k^2}{E_k\omega_k}\frac{E_k\pm M}{\epsilon-\omega_k-E_k}\frac{1}{p}+O\left(\frac{1}{p^2}\right), \\ B_\alpha(p)&\sim\frac{l}{\epsilon}\frac{E_l\pm M}{p}+\cdots, \end{aligned} \tag{B2}$$

where the + sign refers to the S-state, the − sign to the P-state. If p and k are comparable, and large compared with M, $L_\alpha(\epsilon,p,k)$ is given by the following expression. With $a>b\gg M$,

$$L_\alpha(\epsilon,a,b)\simeq L_\alpha(\epsilon,b,a) \simeq\left[-\frac{a+b}{ab}\ln\frac{a+b}{a}+\frac{1}{2}\frac{a+b}{a^2}\right]. \tag{B3}$$

In the integral equation (B1), consider the ranges of integration $k\gtrless X$, where $X\gg M$. Then for $p\gg X$, the

[17] K. Symanzig, Gottingen Dissertation, March, 1954 (unpublished).
[18] J. Taylor, Phys. Rev. 95, 1313 (1954).

integral up to $k=X$ and the term $B_\alpha(p)$ are each of form $1/p$, while the kernel is homogeneous for $k>X$. This integral equation is therefore singular at infinity and not of the Fredholm type. Such equations, homogeneous for large momenta, characteristically have solutions with an asymptotic form $\sim p^\lambda$. If $\lambda>-1$, the integration for $X<k<\infty$ may be extended down to $k=0$ using the asymptotic form of the kernel, with error of lower order, and this term (being homogeneous) reproduces the form p^λ with a coefficient depending on λ. This term is dominant on the right of (B1), and equating this with $g_\alpha^+(p)$ gives the following equation for λ:

$$\frac{16\pi^2}{G^2}=-Q'D(\lambda), \tag{B4}$$

where

$$D(\lambda)=\frac{\pi\,\mathrm{cosec}\pi\lambda}{\lambda(1+\lambda)}-\frac{1}{\lambda^2}-\frac{1}{(1+\lambda)^2}+\frac{1}{2\lambda(1+\lambda)}+\frac{3}{2(\lambda-1)(2+\lambda)}.$$

This derivation of (B4) is valid only for $-1<\lambda<0$, $D(\lambda)$ is singular at $\lambda=0$ and $\lambda=-1$. In Born approximation, the asymptotic form of $g_\alpha^+(p)$ corresponds to $\lambda=-1$, and, for $T=\frac{1}{2}$, $j=\frac{1}{2}$ states ($Q'=-1$), the value of λ increases with increasing G^2 until it reaches $\lambda=-\frac{1}{2}$ at $G^2/4\pi=3\pi/(3\pi-8)\sim 6\cdot 7$. At this critical value of G^2, the function $g(p)$ is no longer normalizable for large p. For G^2 beyond this critical value, the value of λ is complex, of the form $\lambda=-\frac{1}{2}\pm iu$—the equation no longer has a uniquely defined solution, and no solutions are normalizable. Such a critical value $G_c^2(j)$ exists for every attractive meson-nucleon state, but owing to the centrifugal repulsion, $G_c^2(j)$ increases with increasing j, being about[19] 26.9 for $j=\frac{3}{2}$.

For the $T=\frac{3}{2}$, $j=\frac{1}{2}$ states, Eq. (B4) has no solution in the range $-1<\lambda<0$. However it may be shown that the asymptotic form of the solution is still p^λ, where $\lambda<-1$ and is given by the same Eq. (B4). For these states, as G^2 is increased, the value of λ decreases from $\lambda=-1$, and the solution decreases faster than Born approximation for large p.

The function $V_\alpha^R(M,p)$ is defined by the integral Eq. (85). In Born approximation $V_\alpha^R(M,p)$ is asymptotically constant. For definite G^2, $V_\alpha^R(M,p)$ may be expected to have asymptotic form p^λ, where λ may be either $\lambda\lesseqgtr 0$.

If $\lambda<0$, then the integrals on the right of (86) are separately convergent, and in the limit $p\rightarrow\infty$, the equation becomes

$$1-\frac{G^2}{8\pi^2}Q'\int dkL_\alpha(M,0,k)V_\alpha^R(M,k)=0. \tag{B5}$$

This implies that $1/Z_\alpha=0$, and that $V_\alpha^R(M,p)$ satisfies the homogeneous part of integral equation (79). For this equation, it is easily shown the dominant part of the integral comes from the asymptotic region, so the value of λ is again given by Eq. (B4). This equation does in fact give a negative λ for the physical case $Q'=-1$. As G^2 increases from zero, λ decreases from zero to negative values, reaching $\lambda=-\frac{1}{2}$ for $G^2=G_c^2$. For this attractive interaction, then, $V_\alpha^R(\epsilon,p)$ decreases more and more rapidly with p, as the interaction becomes stronger.

For $\lambda>0$, more care would be needed. With Eq. (85), the dominant terms on the right come from large momenta k and the asymptotic form of $L_\alpha(M,0,k)$ must now be included. When this is done, the calculation may be carried through as above, with the result that Eq. (B4) is valid again for positive λ. Hence, for a repulsive interaction, the value of λ runs from zero along the right-hand branch of the function $D(\lambda)$, and $V_\alpha^R(M,p)$ increases as p increases. For this case integral (B5) diverges and $1/Z_\alpha=\infty$. The function $V_\alpha^R(M,p)$ now does not satisfy the homogeneous part of the integral equation (80)—in fact, this homogeneous integral equation now has no solution, in general.

For this case, it is necessary to examine also Eq. (87). Since $L_\alpha(\epsilon,q,k)$ is independent of ϵ for large k, the dominant part of the inhomogeneous term in (87) has the asymptotic form $\sim X(\epsilon)p^{\nu-1}$, corresponding to asymptotic form p^ν for $V_\alpha^R(M,p)$. If the function $W_\alpha(k)$ is $\sim k^\mu$ for large k, the form of (87) requires $\mu<0$ and a comparison of dominant terms requires $\mu=\nu-1$. $W_\alpha(k)$ then has asymptotic form $A(\epsilon)q^{\nu-1}$, where

$$A(\epsilon)\left(1+\frac{G^2}{16\pi^2}D(\nu-1)\right)=X(\epsilon). \tag{B6}$$

The difference between $V_\alpha^R(\epsilon,p)$ and $V_\alpha^R(M,p)$ is therefore of one order lower than the separate vertex-functions. This is of importance since the renormalization procedure adopted would not otherwise succeed.

[19] H. A. Bethe and F. J. Dyson, Phys. Rev. **90**, 372 (1953). See also H. A. Bethe and F. de Hoffmann, *Mesons and Fields* (Row, Peterson, and Company, Evanston, 1955), Vol. 2.