density vanishes everywhere that the field equations are satisfied and hence is unsatisfactory since the only acceptable solutions to the field equations are those which satisfy them everywhere. Thus, for this solution, our current density would be everywhere zero. We believe, however, that it is necessary to consider solutions of the field equations which do not satisfy them everywhere. In certain small regions of space it might well be that the field equations are not even valid. One can take the position that the field equations describe correctly the interaction of elementary particles but are insufficient to describe the particles themselves and hence, at small distances from these particles, must be replaced by some other kind of mathematical construction.[4] If such were the case we would no longer need to restrict our solutions and the current density of Eq. (16) could be nonzero for certain regions of space. In fact, if one takes seriously the relation between invariance and conservation laws, one is almost forced into this latter position. Regardless of the form of the Lagrangian, the current density which follows from gauge invariance will vanish whenever the field equations are satisfied.

The only way out seems to be to construct a theory in which it is impossible to introduce potentials. While there would still be an invariance associated with the conservation of charge, it may be of such a nature that the current-density need not vanish when the field equations are satisfied. For instance, the energy-momentum tensor associated with arbitrary coordinate invariance does not vanish when the field equations are satisfied. It is evident that a theory for which it is impossible to introduce potentials will differ from the present theory and hence lies outside the scope of our investigation.

[4] This view is diametrically opposite to that held by Einstein. He contends that his field equations are everywhere valid, i.e., that they do correctly describe the elementary particles.

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Gravitational Radiation

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In this paper, we investigate the existence of gravitational radiation within the framework of the EIH approximation method. Following a prescription introduced by Infeld, the radiation terms of the EIH expansion are begun with functions of the time alone. We find that these terms do not have physical significance if they are introduced in the scalar or longitudinal components of the gravitational potentials. However, if the radiation terms are introduced in the fifth order of the transverse-transverse components, one finds a contribution to the curvature tensor in the seventh order, a contribution to the equations of motion in the ninth order, and radiation in the tenth order. The existence of radiation is determined by calculating the energy passing through a spherical surface which is an infinite distance from all source points. This definition of radiation agrees with that used in the theory of electromagnetism.

I. INTRODUCTION

In recent years there has been some controversy concerning gravitational radiation. Infeld and Scheidegger have maintained that the possibility of radiation does not exist within the framework of the EIH (Einstein, Infeld, and Hoffman) approximation method.[1–4] This conclusion has been accepted for the scalar and longitudinal components. Indeed, in his book Bergmann has shown[5] that in the linearized gravitational equations these terms do not contribute to radiation; however, the transverse-transverse components do make a contribution. It was on this basis that the proof set forth by Infeld and Scheidegger was first criticized.[6] This paper will attempt to clarify the situation by proposing an unambiguous definition of radiation.

II. EIH APPROXIMATION METHOD

In a previous paper[7] the surface integrals leading to the equations of motion were found without recourse to an approximation method. The possibility of doing so depended on the existence of superpotentials for the components of the energy-momentum psuedo-tensor:*

$$T_\mu{}^\nu = U_\mu{}^{[\nu\sigma]}{}_{,\sigma}. \tag{1}$$

$T_\mu{}^\nu$ is the energy-momentum psuedo-tensor and $U_\mu{}^{[\nu\sigma]}$

[1] L. Infeld and A. E. Scheidegger, Can. J. Math. **3**, 195 (1951).
[2] A. E. Scheidegger, Phys. Rev. **82**, 883 (1951).
[3] L. Infeld, Can. J. Math. **5**, 17 (1953).
[4] A. E. Scheidegger, Revs. Modern Phys. **25**, 451 (1953).
[5] P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall Publications, Inc., New York, 1947), pp. 187–189.
[6] P. G. Bergmann (private communication).
[7] J. N. Goldberg, Phys. Rev. **89**, 263 (1953).
* The energy-momentum pseudo-tensor does not have simple geometrical transformation properties (see reference 5, page 196). Hence, the word *pseudo-tensor* should not be confused by the current use of the prefix *pseudo-* to describe a density of weight one.

is the antisymmetric superpotential. As a result, it was shown that whenever the field equations of a covariant field theory are satisfied on a two-dimensional closed surface, that is,

$$L^A=0 \tag{2}$$

on the two-dimensional surface, the following integrals taken over that surface vanish:

$$\Sigma_\mu \equiv \oint (U_\mu{}^{[4s]}{}_{,4}+t_\mu{}^s)n_s dS=0. \tag{3}$$

In the above equations, L^A are the field equations and $t_\mu{}^\nu$ is equal to the energy-momentum psuedo-tensor when the field equations are satisfied:

$$t_\mu{}^\nu=-\delta_\mu{}^\nu L+y_{A,\mu}\partial^{A\nu}L. \tag{4}$$

The above surface integrals, Eq. (3), were shown to be equivalent to those used by EIH in their solution of the problem of motion in the general theory of relativity. They are independent of the surface chosen in the sense that the restrictions placed on the coordinates of any singularities enclosed by the surface are the same regardless of the choice of surface. For our purposes it will be convenient to discuss the EIH approximation method using the notation of this paragraph.

The method of EIH[8–10] assumes that the particles generating the gravitational field are represented by singularities in the field and that the velocities of these particles are small compared with that of light ($v/c\ll 1$). It follows that the derivative of the field variables (gravitational potentials) with respect to time will be of a higher order in v/c than the derivatives with respect to the space coordinates. One introduces a parameter τ such that $\tau=\lambda x^4 (\lambda=v/c)$. Derivatives taken with respect to τ are assumed to be of the same order as those with respect to the space coordinates. τ differentiation will be represented by a zero following a comma; i.e., $y_{A,4}=\lambda y_{A,0}$. For the sake of consistency, in the following Greek indices will run from 0 to 3. This notation has the advantage of constantly reminding us of the nature of the approximation method. As usual, Latin indices run from 1 to 3. Finally, the field variables, y_A, are expanded into a power series in λ:

$$y_A=\sum_{n=0}^{\infty}\lambda^n {}_ny_A. \tag{5}$$

As a result, the field equations and the surface integrals will also be power series in λ and the coefficients of the various powers of λ may be equated to zero separately:

$${}_nL^A=0, \tag{6}$$

$${}_n\Sigma_\mu=0. \tag{7}$$

[8] Einstein, Infeld, and Hoffman, Ann. Math **39**, 66 (1938).
[9] A. Einstein and L. Infeld, Ann. Math **41**, 455 (1940).
[10] A. Einstein and L. Infeld, Can. J. Math **1**, 209 (1949).

Let us assume that the field equations have been solved and the surface integrals satisfied through the nth order. Therefore we know ${}_0y_A, \cdots, {}_ny_A$. The superpotential ${}_nU_\mu{}^{[0s]}$ contains terms which are linear in ${}_ny_A$ as well as the lower order field variables in combinations of higher degree. In the surface integrals, however, this term is differentiated with respect to τ and therefore is of order $n+1$. The components of the weak energy-momentum psuedo-tensor $t_\mu{}^\nu$ are homogeneous quadratic in the first derivatives of the field variables and therefore knowledge of the field variables to the nth order gives us $t_\mu{}^\nu$ at least to the $(n+1)$th order. Therefore, if the solution of the field equations up to the nth order is to be consistent, the surface integrals in the $(n+1)$th order must vanish. In general one cannot expect these surface integrals to be satisfied when the surface encloses a singularity. However, Einstein and Infeld have shown[10] that in the theory of gravitation ${}_{n+1}\Sigma_4$ may be satisfied by the addition of poles in the nth order and ${}_{n+1}\Sigma_s$ by the addition of dipoles in the $(n-1)$th order. When the solution has been carried as far as desired, the sum of all the dipoles added is set equal to zero. That this process yields the equations of motion for the singularities follows from the fact that the surface integrals are merely conservation laws for linear momentum. For arbitrary particle motions, linear momentum can be conserved by the addition of appropriate time dependent dipoles. The ultimate prohibition of such dipoles forces the particles to move in such a manner as to conserve linear momentum.

There is still a certain amount of arbitrariness in the solution and hence in the equations of motion which has not been discussed. In the nth order the field equations will consist of linear terms involving the field variables of the nth order and linear as well as nonlinear terms involving the field variables of lower orders. Thus in each order, except for the lowest, one has to solve inhomogeneous linear partial differential equations. Therefore, to the solution of the inhomogeneous equations one can add an arbitrary solution of the homogeneous equations. One must decide in advance what singularities are to be permitted if unique equations of motion are to be obtained. Having chosen a particular solution in the lowest order, EIH prohibit the addition of any other arbitrary solutions of the homogeneous equations.

Apart from the above, there is an additional arbitrariness which results from the question of radiation: Should one choose a retarded, advanced, or a linear combination of retarded and advanced solutions? EIH avoid the question of radiation by choosing the standing wave solution (retarded plus advanced).

In the theory of gravitation it is convenient to introduce linear combinations of the deviation from flat space as the field variables in the approximation method:

$$\gamma_{\mu\nu}=h_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}h_{\rho\sigma}, \tag{8}$$

where

$$h_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu} \tag{9}$$

is the deviation from flat space. $\eta_{\mu\nu}$ $(-1, -1, -1, +1)$ is the Minkowski metric. The $\gamma_{\mu\nu}$ are then expanded into a power series in λ beginning with λ^2:

$$\begin{aligned}\gamma_{00}&=\sum_{n=1}^{\infty}\lambda^{2n}\,{}_{2n}\gamma_{00}+\sum_{n=1}^{\infty}\lambda^{2n+1}\,{}_{2n+1}\gamma_{00},\\ \gamma_{0r}&=\sum_{n=1}^{\infty}\lambda^{2n+1}\,{}_{2n+1}\gamma_{0r}+\sum_{n=1}^{\infty}\lambda^{2n+2}\,{}_{2n+2}\gamma_{0r},\\ \gamma_{rs}&=\sum_{n=1}^{\infty}\lambda^{2n+2}\,{}_{2n+2}\gamma_{rs}+\sum_{n=1}^{\infty}\lambda^{2n+3}\,{}_{2n+3}\gamma_{rs}.\end{aligned} \tag{10}$$

As indicated above, the expansions of the $\gamma_{\mu\nu}$ separate into two independent sets. The field equations have the property that each set must be introduced separately. By comparing the above expansion with the corresponding expansion of the electromagnetic theory, Infeld[11] concluded that the first set represents an advanced plus retarded field (standing waves) while the second set represents a retarded minus advanced field. Therefore, he called the second group of terms the "radiation terms." The solution of EIH considers only the standing waves.

In the theory of electromagnetism, the radiation terms start off with a function of time alone. Arguing by analogy, Infeld assumed that this situation should also be true in the theory of gravitation. Together with Scheidegger, he has shown that any function of time introduced into the radiation terms can be eliminated by means of a coordinate transformation.[1] Therefore, they concluded that gravitational radiation does not exist in the EIH framework.

The fallacy of their argument can be shown at this point. Let us consider the curvature tensor (see Appendix 2). The linear terms of this tensor contain only second derivatives of the field variables while higher degree terms contain a part which is homogeneous-linear in the second derivatives and a part which is homogeneous-quadratic in the first derivatives of the field variables. It is clear then that the introduction of an arbitrary function of time into the nth order of the radiation terms may make a contribution to the curvature tensor not in the nth order, but in the $(n+2)$th order. Thus, to say that the arbitrary function of time which has been introduced may be transformed away in the nth order is trivial, for it does not contribute to anything physical in that order. It follows, then, that one must investigate the higher order terms in order to determine whether anything physical results from the introduction of a function of time into the radiation terms.

This discussion may be clarified by examining the corresponding situation in the theory of electromagnetism where we know what the physically meaningful variables are. In this case the field equations for the radiation terms of the vector potential are

$${}_{2n+1}\varphi_{0,ss}={}_{2n-1}\varphi_{0,00},\quad {}_{2n+2}\varphi_{r,ss}={}_{2n}\varphi_{r,00}, \tag{11}$$

with the gauge condition

$${}_{2n+1}\varphi_{0,0}-{}_{2n+2}\varphi_{r,r}=0. \tag{12}$$

If we choose to begin the radiation terms with

$${}_{2n}\varphi_r=f_r(\tau), \tag{13}$$

this term may be removed by the gauge transformation generated by

$${}_{2n}\psi=-x^n f_n. \tag{14}$$

However, ${}_{2n}\varphi_r$ does not contribute to any physical quantity in the $(2n)$th order, but rather in the $(2n+1)$th order. Choosing

$${}_{2n+1}\varphi_0=0, \tag{15}$$

we find for the electric field strength

$${}_{2n+1}E_r={}_{2n+1}\varphi_{0,r}-{}_{2n}\varphi_{r,0}=-f_{r,0}. \tag{16}$$

Clearly the gauge transformation of Eq. (14) which removes ${}_{2n}\varphi_r$ introduces

$${}_{2n+1}\varphi_0=-x^n f_{n,0}, \tag{17}$$

so that Eq. (16) is maintained as it should be. Had the higher order effect of the gauge transformation been neglected, we would no longer have been considering the same physical problem we started with.

In our investigation, the coordinate transformations will be used to simplify the necessary calculations.

III. DEFINITION OF RADIATION

Thus far nothing has been said about how one is to know whether or not radiation exists. Certainly the existence of so-called "radiation" terms does not prove that radiation occurs. In the theory of electromagnetism one calculates the energy flux through a closed two-dimensional surface. If there is a finite flux through the surface when the surface is infinitely far from all source points, one says that radiation occurs. The existence of the strong conservations laws in the general theory of relativity allows the same definition to be applied in this case. The energy and momentum density of the gravitational field, including the source points, is given by $(1/16\pi\kappa)T_\mu{}^0$ where κ is the gravitational constant. Thus, the total energy and momentum, P_μ, contained in a volume V with a surface S is given by

$$P_\mu=\frac{1}{16\pi\kappa}\int_V T_\mu{}^0 dV. \tag{18}$$

The negative time derivative of P_μ yields the flow of

[11] L. Infeld, Phys. Rev. 53, 836 (1938).

energy and momentum, W_μ, out of the volume V:

$$W_\mu = -\frac{1}{16\pi\kappa}\int_V T_\mu{}^0{}_{,0}dV = \frac{1}{16\pi\kappa}\oint T_\mu{}^s\, n_s dS. \tag{19}$$

In the last step, we have made use of the strong conservation laws

$$T_\mu{}^\nu{}_{,\nu} \equiv 0,$$

and then applied Gauss' theorem. If the field equations are satisfied on the surface S, the weak energy-momentum tensor, Eq. (4) may be substituted into Eq. (19):

$$W_\mu = \frac{1}{16\pi\kappa}\oint t_\mu{}^s\, n_s dS. \tag{20}$$

We shall say that gravitational radiation exists if the surface integral of Eq. (20) with $\mu=0$ yields a finite result when the surface is infinitely far from all source points. The total flow of momentum may be similarly defined with $\mu=1, 2, 3$.

By means of the surface integrals of Eq. (3), or directly from the definition of the superpotentials in Eq. (1), the radiation may be defined in terms of the superpotentials. However, as pointed out previously, the superpotentials contain linear terms whereas the $t_\mu{}^\nu$ do not. Therefore, once the field equations have been solved up to a given order, Eq. (20) permits calculation of the radiation to a higher order than does the corresponding surface integral involving the superpotentials.

One can easily show, though we shall not do so here, that with the above definition of radiation the solution obtained by EIH does not contain radiation and hence corresponds to a standing wave.

IV. TRANSFORMATION EQUATIONS

We shall be concerned with coordinate transformations which change the coordinate values by an amount proportional to a given power of λ:

$$x^\rho = x'^\rho + \lambda^m\, {}_m v^\rho(x'). \tag{21}$$

In order to preserve the slow time variation of the field variables, the transformation functions should also possess this character. From the transformation properties of the gravitational potentials, the $g_\mu{}^\nu$, one can establish that the $\gamma_{\mu\nu}$ transform as follows:

$$\begin{aligned}
&\text{(a)} && {}_m\gamma'_{00} = {}_m\gamma_{00} - {}_m v^n{}_{,n},\\
&\text{(b)} && {}_m\gamma'_{0r} = {}_m\gamma_{0r} + {}_m v^0{}_{,r},\\
&\text{(c)} && {}_m\gamma'_{rs} = {}_m\gamma_{rs} - {}_m v^r{}_{,s} - {}_m v^s{}_{,r} + \delta_{rs}\, {}_m v^n{}_{,n},\\
&\text{(d)} && {}_{m+1}\gamma'_{00} = {}_{m+1}\gamma_{00} + {}_m v^0{}_{,0},\\
&\text{(e)} && {}_{m+1}\gamma'_{0r} = {}_{m+1}\gamma_{0r} - {}_m v^r{}_{,0},\\
&\text{(f)} && {}_{m+1}\gamma'_{rs} = {}_{m+1}\gamma_{rs} + \delta_{rs}\, {}_m v^0{}_{,0}.
\end{aligned} \tag{22}$$

In the above equations all functions are considered to be functions of the new coordinates and all derivatives are taken with respect to the new coordinates.

There are other higher order effects of the transformation which we shall not consider. These higher order terms will combine with the untransformed solution to produce solutions either of the homogeneous or inhomogeneous equations in the higher orders. Since the solution of the inhomogeneous equations will be generated as we proceed with the approximation method, we lose only solutions of the homogeneous equations by disregarding higher order terms in the transformation equations. In view of the previously discussed ambiguity of the solution all such solutions of the homogeneous equations may be discarded. It is clear from the above discussion that the prohibition of solutions of the homogeneous equations is not a covariant requirement. Beyond the lowest nonvanishing order we shall always prohibit the appearance of arbitrary solutions of the homogeneous equations in that coordinate system in which we are working. Whether or not there exist other coordinate systems, in which this condition is also satisfied, will not concern us.

We are now prepared to solve the field equations for the radiation terms. The fundamental problem to be considered is that of two point masses interacting. EIH have already solved the stationary wave part of this problem. We shall look for a nonsingular solution of the field equations (radiation terms) which cannot be removed by a coordinate transformation. Having found such a solution, we must then investigate, in the manner of Sec. III, whether or not radiation of energy occurs. The introduction of radiation terms will not alter the EIH solution. Fortunately, for our purposes we shall only require ${}_2\gamma_{00}$. As has been mentioned previously, we do not question Infeld and Scheidegger's result if the radiation terms are begun in the scalar or longitudinal components. However, for the sake of completeness we shall consider these cases as well as that of the transverse-transverse components.

In order not to obscure the principal argument with long cumbersome equations, we shall put the gravitational field equations into an appendix and bring forward only those terms required for our purposes. Equations in the appendices will be referred to as Eq. (A7), etc.

V. SCALAR AND LONGITUDINAL COMPONENTS

Although the radiation terms may be introduced in any order, we shall do so as early as possible, i.e., into ${}_3\gamma_{00}$, ${}_4\gamma_{0r}$, and ${}_5\gamma_{rs}$. The argument is not confined to this choice, for the same types of terms are combined in the field equations and the surface integrals regardless of the order. The field equations in any order may be found from Appendix 1. In all our considerations we shall begin by assuming the standard coordinate conditions introduced by EIH:

$$_m\gamma_{00,0}-{}_{m+1}\gamma_{0r,r}=0, \tag{23}$$

$$_m\gamma_{rs,s}=0. \tag{24}$$

However, we shall not restrict the coordinate transformations to those which preserve these conditions.

The field equations for the scalar and longitudinal components, in the third and fourth orders respectively, are

$$_3\gamma_{00,ss}=0, \tag{25}$$

$$_4\gamma_{0r,ss}=0. \tag{26}$$

In the above equations, the coordinate conditions have already been introduced. Following the prescription given by Infeld,[11] we start the scalar term with the solution

$$_3\gamma_{00}=f(\tau). \tag{27}$$

Before proceeding to the next stage of the approximation, the surface integral $_4\sum_0$ [Eq. (3)] should be evaluated to insure the consistency of the solution $_3\gamma_{00}$. Since the value of the surface integral is independent of the specific surface of integration, the only terms of the integrand which can lead to a nonvanishing integral are those which have an r^{-2} behavior. Clearly, such terms cannot appear in $_4\sum_0$ and as a result the consistency of the solution in the third order is assured. This argument will be valid throughout our discussion as we proceed from the solution in one order to that in the next. Therefore, in all such cases we need not be concerned with the surface integrals.

The solution for $_4\gamma_{0r}$ which is consistent with the coordinate conditions is given by [Eqs. (26) and (23)]

$$_4\gamma_{0r}=\tfrac{1}{3}x^r f_{,0}. \tag{28}$$

Now another question arises in relation to the surface integrals: Is it necessary to examine $_3\gamma_{00}$ and $_4\gamma_{0r}$ for consistency before carrying out a coordinate transformation to remove $_3\gamma_{00}$? More generally the question is whether it is necessary to examine the consistency of a solution before carrying out a coordinate transformation which affects that solution. If the solution is not consistent, then appropriate poles and dipoles must be added to it. After carrying out the transformation, we must examine the transformed solutions for consistency. The net result will be to alter the poles and dipoles which had been added. Therefore, we can delay investigating the surface integrals in a given order until a coordinate transformation has been carried out which affects the solution in a lower order only. As a result of this and the previous argument we shall not need to consider the surface integrals until the solutions have been carried as far as desired.

A coordinate transformation may now be carried out to remove $_3\gamma_{00}$. From Eq. (22), with

$$_3v^r=\tfrac{1}{4}x^r f(\tau), \tag{29}$$

and

$$_2v^0=-\tfrac{1}{4}\int f(\tau)d\tau,$$

we find

$$_3\gamma'_{00}=0, \tag{30a}$$

and

$$_4\gamma'_{0r}=\tfrac{1}{12}x^r f_{,00}. \tag{30b}$$

Clearly the coordinate conditions in the fourth order have been altered by the transformation. In the fifth order, the field equations are

$$_5\gamma_{rs,nn}-{}_5\gamma_{rn,sn}-{}_5\gamma_{sn,rn}+\delta_{rs}\,{}_5\gamma_{nm,mn} = -{}_4\gamma_{0r,0s}-{}_4\gamma_{0s,0r}+2\delta_{rs}\,{}_4\gamma_{0n,0n}-\delta_{rs}\,{}_3\gamma_{00,00} \tag{31a}$$

$$_5\gamma_{00,nn}={}_5\gamma_{mn,mn}, \tag{31b}$$

and the coordinate conditions are

$$_5\gamma_{rs,s}=0. \tag{31c}$$

Hence,

$$_5\gamma'_{rs,nn}=\tfrac{1}{3}\delta_{rs}f_{,00}, \tag{32a}$$

$$_5\gamma'_{00,nn}=0. \tag{32b}$$

The solutions of these equations may be found in a straightforward manner:

$$_5\gamma'_{rs}=(1/15)\delta_{rs}r^2 f_{,00}-(1/30)x^r x^s f_{,00}, \tag{33a}$$

$$_5\gamma'_{00}=0. \tag{33b}$$

Another coordinate transformation may be carried out to eliminate the longitudinal component in the fourth order. Consider

$$_4v^0=-(1/24)r^2 f_{,0}. \tag{34}$$

From Eqs. (22), we find that

$$_5\gamma''_{rs}=[(1/40)\delta_{rs}r^2-(1/30)x^r x^s]f_{,00}, \tag{35a}$$

$$_5\gamma''_{00}=-(1/24)r^2 f_{,00}. \tag{35b}$$

The field equations in the sixth order are

$$_6\gamma_{0s,nn}-{}_6\gamma_{0n,ns}={}_5\gamma_{rs,0r}-{}_5\gamma_{00,0s}, \tag{36a}$$

with the coordinate condition

$$_6\gamma_{0s,s}-{}_5\gamma_{00,0}=0. \tag{36b}$$

From Eqs. (35), we obtain

$$_6\gamma''_{0s,nn}=-(1/12)x^s f_{,000}, \tag{37a}$$

$$_6\gamma''_{0s,s}=-(1/24)r^2 f_{,000}. \tag{37b}$$

The solution of these equations is

$$_6\gamma''_{0s}=-(1/120)x^s r^2 f_{,000}. \tag{38}$$

Now, one final transformation may be carried out which wipes out the entire solution. An examination of the transformation equations shows that

$$_5v^s = -(1/120)x^s r^2 f_{,00} \qquad (39)$$

is the desired transformation function.

Now let us examine what happens if the radiations terms are begun in the longitudinal components with the solution

$$_4\gamma_{0r} = f_r(\tau). \qquad (40)$$

In this case the proof is much simpler for we need not consider the higher order solutions. Let

$$_3v^r = \int f_r(\tau)d\tau. \qquad (41)$$

An examination of Eqs. (22) shows that $_4\gamma'_{0r}$ vanishes and that no other term is affected. Thus, the entire solution is wiped out by the coordinate transformation.

One may argue that if higher order effects of the coordinate transformations had not been neglected, the solution as a whole could not have been removed by the transformations considered. Furthermore, by our stated intent to keep solutions of the homogeneous equations in the lowest nonvanishing order, we should have been obliged to consider the higher order effects. In the lowest orders the solutions introduced by these higher order effects could have been of two types: nonsingular functions of time alone and singular functions with time dependent coefficients. The purely time dependent functions could be removed by coordinate transformations and hence are of no consequence. The singular terms, on the other hand, would correspond to time-dependent multipoles. Such terms cannot be of physical importance because we always have the possibility of altering the equations of motion by the addition of multipoles.

VI. TRANSVERSE-TRANSVERSE COMPONENTS

The field equations for γ_{rs} in the fifth order are

$$_5\gamma_{rs,\,nn} = 0, \qquad (42)$$

if the coordinate conditions given in Eqs. (23) and (24) are used. Therefore, in this case the solutions may also be started off with an arbitrary function of time:

$$_5\gamma_{rs} = f_{rs}(\tau). \qquad (43)$$

The transformations to be considered in this case are handled in the same manner as those in the previous section. Therefore, in order to avoid the monotony of repetition, we shall merely state the results of the transformations and relegate the explicit treatment of the coordinate transformations to the Appendix (Appendices 4 and 5).

Starting from an arbitrary, though symmetric, set of functions $f_{rs}{}^*(\tau)$ one can always transform the set to another, $f_{rs}(\tau)$, with vanishing trace (Appendix 4); i.e., such that

$$f_{ss}(\tau) = 0. \qquad (44)$$

This result is in agreement with the discussion of gravitational radiation given in Bergmann's book.[5] Continuing the solution with the above condition, we find, after the appropriate transformations, that the lowest nonvanishing order is the seventh (Appendix 5). In the seventh and eighth orders the solutions are

$$_7\gamma_{rs} = (12/77)r^2 f_{rs,\,00} - (18/77)x^n(x^r f_{ns} + x^s f_{nr})_{,\,00} + (3/7)\delta_{ns}x^m x^n f_{mn,\,00}, \qquad (45a)$$

$$_7\gamma_{00} = 0, \qquad (45b)$$

$$_8\gamma_{0r} = 0. \qquad (45c)$$

The standard coordinate conditions are satisfied by these solutions.

The question still remains whether the above solution leads to anything physical. An examination of the curvature tensor is the simplest test of this point. From Eq. (A15) we find that the $(s00,r)$ component of the curvature tensor in the seventh order is

$$_7R_{s00}{}^r = -\tfrac{1}{4}({}_7\gamma_{00,\,rs} + {}_7\gamma_{nn,\,rs}). \qquad (46)$$

Substituting Eqs. (35), we have

$$_7R_{s00}{}^r = -(9/22)f_{rs,\,00}. \qquad (47)$$

Thus, it is clear that introducing the radiation terms through the transverse-transverse components has a physical significance.

Before computing the radiation in the tenth order, we shall examine the surface integrals in the ninth order, $_9\sum_m$, to establish the consistency of our solution and also because there is a contribution to the equations of motion in this order. From Eqs. (A9) and (A10), we find

$$_9\sum{}_m = \oint [-{}_7\gamma_{ms,\,00} - \tfrac{1}{4}\delta_{sm}\,{}_2\gamma_{00,\,r}\,{}_7\gamma_{nn,\,r} + \tfrac{1}{4}{}_2\gamma_{00,\,n}\,{}_7\gamma_{nn,\,s} + \tfrac{1}{4}{}_2\gamma_{00,\,s}\,{}_7\gamma_{nn,\,m}]n_s dS. \qquad (48)$$

Previously, we stated that the problem we were considering involved two particles interacting. Up to this time we have not required this information. The solution of $_2\gamma_{00}$ involving two particles is[4,10]

$$\begin{aligned} {}_2\gamma_{00} &= -4(m_1/r_1 + m_2/r_2), \\ (r_1)^2 &= (x^s - \xi^s)(x^s - \xi^s), \\ (r_2)^2 &= (x^s - \eta^s)(x^s - \eta^s), \end{aligned} \qquad (49)$$

ξ^s and η^s specify the locations of the particles. Inserting Eqs. (45a) and (49) into (48) and taking a surface which encloses only the first particle, we find the following contribution to the equations of motion:

$$(1/4\pi)\,{}_9\sum{}_m = (18/11)m_1\xi^n f_{nm,\,00}. \qquad (50)$$

The surface integrals have been computed with the aid of Appendix 3. One cannot say more about this contribution to the equations of motion until information is available as to the form of f_{rs}.

If the momentum passing through a surface at infinity which encloses both particles is now computed, [Eq. (20)], one obtains

$$ {}_9W_m = (9/22\kappa)(m_1\xi^n + m_2\eta^n) f_{nm,00}. \quad (51) $$

This momentum transfer vanishes if the origin is taken at the center of mass of the system. Since the momentum transfer is of a higher order in v/c than the energy transfer, there can be no passage of momentum through a surface at infinity in this order. This result implies that the radiation appears to come from the center of mass of the system. Indeed, if in the solution for ${}_7\gamma_{rs}$, $\mathbf{x}$ is replaced by $\mathbf{x}-\mathbf{x}'$, where $\mathbf{x}'$ is the center of mass, the momentum transfer vanishes regardless of the location of the origin. For the sake of simplicity we shall assume the origin to be at the center of mass.

Because of the quadratic nature of the $t_\mu{}^\nu$, solutions of higher order than the eighth cannot contribute to the radiation in the tenth order. Also, we shall not add the dipole to ${}_7\gamma_{00}$ as this term cannot contribute to the radiation because it falls off too rapidly as a function of r. Therefore, the only terms we need be concerned with are ${}_7\gamma_{rs}$ [Eq. (45a)] and ${}_2\gamma_{00}$ [Eq. (49)]. We find from Eq. (A9) that these terms make the following contribution to $t_0{}^s$:

$$ {}_{10}t_0{}^s = -\tfrac{1}{4}({}_2\gamma_{00,0}\,{}_7\gamma_{nn,s} + {}_2\gamma_{00,s}\,{}_7\gamma_{nn,0}). \quad (52) $$

If the above expression is now integrated over an infinite spherical surface centered at the origin, we obtain, with the aid of Appendix 3,

$$ {}_{10}W_0 = (9/55\kappa)[(m_1\xi^r\xi^s{}_{,0} + m_2\eta^r\eta^s{}_{,0}) f_{rs,00} - \tfrac{3}{4} f_{rs,000}(m_1\xi^r\xi^s + m_2\eta^r\eta^s)]. \quad (53) $$

Thus, the introduction of the radiation terms through the transverse-transverse components in the fifth order lead to gravitational radiation in the tenth order and thereby justifies the nomenclature.

There is one other point which requires discussion. In a recent paper,[3] Infeld has shown that one can eliminate the contribution to the equations of motion in all orders higher than the sixth by an appropriate choice of coordinate conditions. The equations of motion through the sixth order are independent of the coordinate conditions.[10] However, in his proof he does not consider the radiation terms. By the same method, one can prove for the radiation terms that the contribution to the equations of motion through the ninth order does not depend on the coordinate conditions in the seventh order. Indeed, one can go further and prove that the radiation in the tenth order also does not depend on the coordinate conditions in the seventh order. Therefore, gravitational radiation has an invariant significance.

VII. CONCLUSION

We have examined the effects of introducing functions of time into the fifth order of the radiation terms and have found that they lead to a finite curvature tensor in the seventh order, a contribution to the equations of motion in the ninth order, and gravitational radiation in the tenth order. However, these results in no way constitute a proof of the existence of gravitational radiation for a detailed discussion of these effects cannot be given without knowing the specific form of the functions f_{rs} which were introduced in the fifth order. Clearly the choice of the f_{rs} will depend on what radiation effects are sought—those resulting from an externally applied field or from the interaction of two or more mass points. If we restrict our attention to the latter case, which is the case of principal interest, the f_{rs} are no longer arbitrary. Once we decide to consider a retarded (or advanced) solution, the starting functions in the radiation terms are related to the standing wave solutions in a precise manner. For example, consider the retarded and advanced potentials

$$ \varphi_- = \frac{f(t-r/c)}{r}, \quad \varphi_+ = \frac{f(t+r/c)}{r}. $$

If these functions are expanded into a power series in $1/c$, we obtain for the standing wave solution

$$ \tfrac{1}{2}(\varphi_- + \varphi_+) = \frac{1}{r}\left[f(t) + \frac{1}{c^2}\frac{r^2}{2!}\frac{d^2}{dt^2} f(t) + \cdots \right]. $$

Similarly, we obtain for the radiation terms

$$ \tfrac{1}{2}(\varphi_- - \varphi_+) = -\frac{1}{c}\left[\frac{d}{dt} f(t) + \frac{1}{c^2}\frac{r^2}{3!}\frac{d^3}{dt^3} f(t) + \cdots \right]. $$

Thus, the starting term for the radiation terms is just the time derivative of the starting term for the standing wave solution. In general, however, the relationship is not as simple. Consider a function $g(r,t,\lambda)$ where $\lambda = v/c$. By generalizing a calculation by Page for the Liénard-Wiechert potentials[12] we find

$$ g_\mp(r,t,\pm\lambda) = g(r,t,0) \pm \frac{1}{c}\left[\frac{dr}{dt}\left(\frac{d}{d\lambda} g(r,t,\lambda) \right)_{\lambda=0} - r\frac{d}{dt} g(r,t,0) \right] + \cdots, $$

where the subscripts $-$ and $+$ mean that the function is to be evaluated at the retarded and advanced times respectively. Clearly, in order to determine the relationship between the radiation terms and the standing waves, the explicit dependence of g on λ must be known. The EIH approximation method cannot reveal this

[12] L. Page, Phys. Rev. 24, 296 (1924).

dependence. Hence, it is not the appropriate formalism to use for an investigation of gravitational radiation and an approximation method which does not restrict the particle velocities must be found. The only purpose of the present paper is to show, contrary to previous results, that the EIH does not exclude the possibility of radiation.

In an accompanying paper,[13] Dr. Scheidegger remarks that our treatment of the transformation equations alters the physical situation because we neglect higher order solutions of the homogeneous field equations. On the other hand, keeping these terms violates the EIH method of approximation.[10] Therefore, he concludes that the starting function for the radiation terms must be rejected as a possible solution for freely interacting particles. It would appear, however, that a more reasonable solution of the problem, if the physical situation is altered by the transformations, would be to reject the transformations themselves. Indeed, Infeld[3] rejects a coordinate transformation which eliminates the perihelion precession of a double star system precisely because the transformation violates the EIH prescriptions. Since these prescriptions are rather stringent, it is hardly surprising that there exist some coordinate transformations which are not allowed. However, our purpose was to show that there exists a nonsingular solution for the radiation terms which cannot be removed by a coordinate transformation and which leads to the radiation of energy. Inasmuch as we know nothing about the form of the gravitational radiation terms, if any, the existence of a nonsingular solution which leads to radiative effects shows that radiation of energy by freely interacting masses cannot be ruled out. Only by finding another approximate solution for the gravitational field equations—one which does not depend on small particle velocities—can we determine whether or not freely interacting bodies radiate energy.

However, one can say that if gravitational radiation exists, it plays a very small role in the problem of motion. The EIH solution has been carried out to yield the equations of motion through the sixth order in v/c. This solution is sufficient to give the perihelion precession of a double star system.[14] On the basis of the present calculation, radiation effects should first appear in the eleventh order (one order above that in which the radiation of energy occurs). In addition Eq. (53) implies that, to be observable, the radiating system should not only have a large moment of inertia, but also a large rate of change for its moment of inertia. If these conditions are satisfied, the EIH approximation method may be poor and other methods will have to be found.

I should like to express my appreciation to Professor Bergmann for discussions on the significance of the EIH approximation method as well as for a critical reading of the manuscript.

[13] A. Scheidegger, following paper [Phys. Rev. **99**, 1883 (1955)].

[14] H. P. Robertson, Ann. Math **39**, 101 (1938).

APPENDIX 1. FIELD EQUATIONS

In all covariant field theories whose field equations are derivable from a variational principle, the strong energy-momentum psuedo-tensor has the form[15]

$$T_\mu{}^\nu = -F_{A\mu}{}^{B\nu} y_B L^A + t_\mu{}^\nu. \qquad \text{(A1)}$$

The $F_{A\mu}{}^{B\nu}$ are certain constants, y_A are the field variables, L^A the field equations, and $t_\mu{}^\nu$ the weak energy-momentum psuedo-tensor defined by Eq. (4). Since the $T_\mu{}^\nu$ may be derived from the superpotentials,

$$T_\mu{}^\nu = U_\mu{}^{[\nu\sigma]}{}_{,\sigma}, \qquad \text{(A2)}$$

it follows that linear combinations of the field equations may be expressed in terms of the superpotentials:

$$-F_{A\mu}{}^{B\nu} y_B L^A = U_\mu{}^{[\nu\sigma]}{}_{,\sigma} - t_\mu{}^\nu. \qquad \text{(A3)}$$

We shall find it convenient to use the field equations in this form. If, as in the general theory of relativity, the Lagrangian density is homogeneous quadratic in the first derivatives of the field variables,

$$L = \Lambda^{A\rho B\sigma} y_{A,\rho} y_{B,\sigma}, \qquad \text{(A4)}$$

the weak energy-momentum psuedo-tensor takes the simple form:

$$t_\mu{}^\nu = -\delta_\mu{}^\nu \Lambda^{A\rho B\sigma} y_{A,\rho} y_{B,\sigma} + 2\Lambda^{A\nu B\sigma} y_{A,\mu} y_{B,\sigma}. \qquad \text{(A5)}$$

Specializing the above relations to the theory of gravitation, we find

$$-F_{(\alpha\beta)\mu}{}^{(\gamma\delta)\nu} g_{\gamma\delta} \sqrt{-g} G^{\alpha\beta} = 2\sqrt{-g} G_\mu{}^\nu \qquad \text{(A6)}$$

$$\Lambda^{(\alpha\beta)\rho(\gamma\delta)\sigma} = \frac{\sqrt{-g}}{8} \{ g^{\alpha\beta}(g^{\gamma\rho}g^{\delta\sigma} + g^{\gamma\sigma}g^{\delta\rho}) + g^{\gamma\delta}(g^{\alpha\rho}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\rho}) + g^{\rho\sigma}(g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma} - 2g^{\alpha\beta}g^{\gamma\delta}) - [g^{\alpha\sigma}(g^{\beta\gamma}g^{\rho\delta} + g^{\beta\delta}g^{\rho\gamma}) + g^{\beta\sigma}(g^{\alpha\gamma}g^{\rho\delta} + g^{\alpha\delta}g^{\rho\gamma})] \}, \qquad \text{(A7)}$$

and[16]

$$U_\mu{}^{[\nu\sigma]} = \sqrt{-g} \{ \delta_\mu{}^\nu (g^{\sigma\tau}g^{\rho\lambda} - g^{\lambda\sigma}g^{\rho\tau}) - \delta_\mu{}^\sigma (g^{\nu\tau}g^{\rho\lambda} - g^{\nu\lambda}g^{\rho\tau}) + \delta_\mu{}^\rho (g^{\nu\tau}g^{\lambda\sigma} - g^{\sigma\tau}g^{\lambda\nu}) \} g_{\rho\lambda,\tau}. \qquad \text{(A8)}$$

In terms of the $\gamma_\mu{}^\nu$, (A5) and (A8) become:

$$t_\mu{}^\nu = -\tfrac{1}{4}\delta_\mu{}^\nu (\gamma_{\alpha\beta,\rho}\gamma_{\alpha\beta,\rho} - 2\gamma_{\alpha\beta,\rho}\gamma_{\rho\alpha,\beta} - \tfrac{1}{2}\gamma_{,\rho}\gamma_{,\rho}) + \tfrac{1}{2}\eta^{\rho\nu}(\gamma_{\alpha\beta,\mu}\gamma_{\alpha\beta,\rho} - 2\gamma_{\alpha\beta,\mu}\gamma_{\rho\alpha,\beta} - \tfrac{1}{2}\gamma_{,\mu}\gamma_{,\rho}) + \text{terms of higher degree}, \qquad \text{(A9)}$$

[15] P. G. Bergmann, Phys. Rev. **75**, 680 (1949).

[16] P. Freud, Ann. Math **40**, 417 (1939). The definition of $T_\mu{}^\nu$ given here differs from that of Freud by a factor of 2. Therefore, in reference 7 Eq. (2.23) should be multiplied by 2.

$$U_\mu{}^{[\nu\sigma]}=(\delta_\mu{}^\sigma\,\eta^{\nu\lambda}-\delta_\mu{}^\nu\,\eta^{\lambda\sigma})\gamma_{\rho\lambda,\,\rho}+(\eta^{\nu\tau}\,\eta^{\lambda\sigma}-\eta^{\nu\lambda}\,\eta^{\sigma\tau})\gamma_{\mu\lambda,\,\tau}$$
$$+(\delta_\mu{}^\nu\,\eta^{\sigma\tau}-\delta_\mu{}^\sigma\,\eta^{\nu\tau})(-\gamma_{\rho\lambda}\,\gamma_{\rho\lambda,\,\tau}+\gamma_{\tau\lambda}\gamma_{\rho\lambda,\,\rho}$$
$$-\tfrac{1}{2}\gamma\gamma_{\rho\tau,\,\rho}+\gamma_{\rho\lambda}\gamma_{\rho\tau,\,\lambda}+\tfrac{1}{2}\gamma\gamma_{,\,\tau}-\tfrac{1}{2}\gamma_{\rho\tau}\gamma_{,\,\rho})$$
$$+(\eta^{\sigma\lambda}\eta^{\nu\tau}-\eta^{\nu\lambda}\eta^{\sigma\tau})(\tfrac{1}{2}\gamma\gamma_{\mu\lambda,\,\tau}-\gamma_{\rho\tau}\gamma_{\mu\lambda,\,\rho}$$
$$-\gamma_{\lambda\rho}\gamma_{\mu\rho,\,\tau}+\tfrac{1}{2}\gamma_{\mu\lambda}\gamma_{,\,\tau})$$
$$+\text{terms of higher degree.}\quad\text{(A10)}$$

In order to simplify the appearance of the above equations we have used the following notation:

$$\eta^{\rho\sigma}\gamma_{\mu\rho}\gamma_{\nu\sigma}=\gamma_{\mu\rho}\gamma_{\nu\rho}=\gamma_{\mu4}\gamma_{\nu4}-\gamma_{\mu s}\gamma_{\nu s},\quad\text{(A11)}$$
$$\gamma=\gamma_{\rho\rho}.\quad\text{(A12)}$$

The expansions in Eqs. (A9) and (A10) are sufficient to give us the field equations and surface integrals to all orders of interest. The field equations are obtained by substituting Eqs. (A6), (A9), and (A10) into (A3); the surface integrals, by substituting (A9) and (A10) into Eq. (3); and the radiation, by substituting (A9) into Eq. (20).

APPENDIX 2. CURVATURE TENSOR

In terms of the Christoffel symbols, the curvature tensor is

$$R_{\iota\kappa\lambda}{}^\nu=\begin{Bmatrix}\nu\\ \lambda\iota\end{Bmatrix}_{,\kappa}-\begin{Bmatrix}\nu\\ \lambda\kappa\end{Bmatrix}_{,\iota}-\begin{Bmatrix}\nu\\ \sigma\iota\end{Bmatrix}\begin{Bmatrix}\sigma\\ \lambda\kappa\end{Bmatrix}+\begin{Bmatrix}\nu\\ \sigma\kappa\end{Bmatrix}\begin{Bmatrix}\sigma\\ \lambda\iota\end{Bmatrix}.\quad\text{(A13)}$$

Introducing the $\gamma_{\mu\nu}$, we have

$$R_{\iota\kappa\lambda}{}^\nu=\tfrac{1}{2}\eta^{\nu\rho}[(\gamma_{\iota\rho,\,\lambda}-\gamma_{\iota\lambda,\,\rho}-\tfrac{1}{2}\eta_{\iota\rho}\gamma_{,\,\lambda}+\tfrac{1}{2}\eta_{\iota\lambda}\gamma_{,\,\rho})_{,\,\kappa}$$
$$-(\gamma_{\kappa\rho,\,\lambda}-\gamma_{\lambda\kappa,\,\rho}-\tfrac{1}{2}\eta_{\kappa\rho}\gamma_{,\,\lambda}+\tfrac{1}{2}\eta_{\lambda\kappa}\gamma_{,\,\rho})_{,\,\iota}]$$
$$+\text{terms of higher degree.}\quad\text{(A14)}$$

In particular, we find for the $(s00,r)$ component:

$$R_{s00}{}^r=-\tfrac{1}{2}(\gamma_{rs,\,00}-\gamma_{s0,\,0r}-\gamma_{r0,\,0s}-\tfrac{1}{2}\eta_{rs}\gamma_{,\,00}$$
$$-\tfrac{1}{2}\gamma_{,\,rs}+\gamma_{00,\,rs})+\cdots.\quad\text{(A15)}$$

APPENDIX 3. TABLE OF SURFACE INTEGRALS

The following table is helpful in evaluating the surface integral:

I. $$\frac{1}{4\pi r^2}\oint n_s dS=0,$$

II. $$\frac{1}{4\pi r^2}\oint n_r n_s dS=\frac{1}{3}\delta_{rs},$$

III. $$\frac{1}{4\pi r^2}\oint n_n n_r n_s dS=0,$$

IV. $$\frac{1}{4\pi r^2}\oint n_m n_n n_r n_s dS=\frac{1}{15}(\delta_{mn}\delta_{rs}+\delta_{mr}\delta_{ns}+\delta_{ms}\delta_{nr}).$$

In the above, the n_s are the components of the outward normal to the surface:

$$n_s=x^s/r$$

for a spherical surface.

APPENDIX 4. PROOF THAT ONE MAY CHOOSE $f_{ss}=0$

We start with the following solutions in the fifth and sixth orders:

$$_5\gamma_{rs}=f_{rs}{}^*(\tau),\quad\text{(A16a)}$$
$$_5\gamma_{00}=0,\quad\text{(A16b)}$$
$$_6\gamma_{0r}=0,\quad\text{(A16c)}$$

where $f_{rs}{}^*$ is a symmetric set of arbitrary functions of time. Consider the coordinate transformation generated by

$$_4v^0(\tau)=-\tfrac{1}{4}\int f_{nn}{}^*(\tau)d\tau,\quad\text{(A17a)}$$
$$_5v^r(\tau)=-\tfrac{1}{4}x^n f_{rn}{}^*(\tau).\quad\text{(A17b)}$$

From the transformation equations Eqs. (22), we find

$$_5\gamma'_{rs}=\tfrac{3}{2}f_{rs}{}^*-\tfrac{1}{2}\delta_{rs}f_{nn}{}^*\equiv f_{rs}(\tau),\quad\text{(A18a)}$$
$$_5\gamma'_{00}=0,\quad\text{(A18b)}$$
$$_6\gamma'_{0r}=\tfrac{1}{4}x^n f_{rn}{}^*{}_{,\,0}.\quad\text{(A18c)}$$

Clearly, $f_{nn}=0$; however, $f_{rs}{}^*$ still contributes to the solution through $_6\gamma'_{0r}$. The remainder of the discussion will show that solutions of the higher order field equations can be determined so that they depend only on f_{rs}.

In the seventh order the field equations are

$$_7\gamma_{rs,\,nn}-{}_7\gamma_{rn,\,sn}-{}_7\gamma_{sn,\,rn}+\delta_{rs}\,{}_7\gamma_{mn,\,mn}$$
$$=-{}_6\gamma_{0r,\,0s}-{}_6\gamma_{0s,\,0r}+2\delta_{rs}\,{}_6\gamma_{0n,\,0n}$$
$$+{}_5\gamma_{rs,\,00}-\delta_{rs}\,{}_5\gamma_{00,\,00}+\text{N.L.}\quad\text{(A19a)}$$
$$_7\gamma_{00,\,nn}={}_7\gamma_{mn,\,mn}+\text{N.L.}\quad\text{(A19b)}$$

with the coordinate conditions

$$_7\gamma_{rs,\,s}=0.\quad\text{(A19c)}$$

The nonlinear terms, N.L., contain products of $_2\gamma_{00}$ and $_5\gamma_{rs}$ only. Hence, they do not interest us because they depend on f_{rs} and not on $f_{rs}{}^*$. Therefore, after the substitution of Eqs. (18) the above equations become

$$_7\gamma'_{rs,\,nn}=f_{rs,\,00}{}^*+\text{N.L.},\quad\text{(A20a)}$$
$$_7\gamma'_{00,\,nn}=\text{N.L.}\quad\text{(A20b)}$$

The solutions of these equations consistent with the coordinate conditions [Eq. (A19c)]

$$_7\gamma'_{rs}=\frac{5}{22}r^2 f_{rs,\,00}{}^*-\frac{1}{11}x^n(x^r f_{sn}{}^*+x^s f_{rn}{}^*)_{,\,00}$$
$$+\frac{3}{110}x^r x^s f_{nn}{}^*-\frac{1}{110}\delta_{rs}r^2 f_{nn}{}^*+\text{N.L.},\quad\text{(A21a)}$$
$$_7\gamma'_{00}=\text{N.L.}\quad\text{(A21b)}$$

The following transformation in the sixth order removes ${}_6\gamma'_{0r}$:

$$ {}_6v^0 = \tfrac{1}{8} x^m x^n f_{mn,0}^* . \qquad \text{(A22)} $$

In the seventh order, we find [Eq. (22)]

$$ {}_7\gamma''_{rs} = {}_7\gamma'_{rs} - \tfrac{1}{8}\delta_{rs} x^m x^n f_{mn,00}^*, \qquad \text{(A23a)} $$

$$ {}_7\gamma''_{00} = {}_7\gamma'_{00} - \tfrac{1}{8} x^m x^n f_{mn,00}^*, \qquad \text{(A23b)} $$

and the coordinate conditions are

$$ {}_7\gamma''_{rs,s} = -\tfrac{1}{4} x^n f_{nr,00}^* . \qquad \text{(A23c)} $$

The field equations in the eighth order are

$$ {}_8\gamma_{0r,nn} - {}_8\gamma_{0n,rn} = {}_7\gamma_{rn,n0} - {}_7\gamma_{00,0r} + \text{N.L.}, \qquad \text{(A24)} $$

with the coordinate condition

$$ {}_8\gamma_{0r,r} = {}_7\gamma_{00,0} . \qquad \text{(A25)} $$

From Eqs. (A23), we find for these equations

$$ {}_8\gamma''_{0r,nn} = -\tfrac{1}{4} x^n f_{rn,000}^* + \text{N.L.}, \qquad \text{(A26)} $$

$$ {}_8\gamma''_{0r,r} = -\tfrac{1}{8} x^m x^n f_{mn,000}^* + \text{N.L.}, \qquad \text{(A27)} $$

and the solutions

$$ {}_8\gamma''_{0r} = -\frac{1}{56} x^r x^m x^n f_{mn,000}^* - \frac{1}{56} x^n r^2 f_{nr,000}^* + \frac{1}{280} x^r r^2 f_{nn,000}^* + \text{N.L.} \qquad \text{(A28)} $$

A final transformation eliminates the "linear" terms in ${}_7\gamma''_{00}$ and ${}_8\gamma''_{0r}$ and brings back the standard coordinate conditions in the seventh order:

$$ {}_7v^r = -\frac{1}{56} x^r x^m x^n f_{mn,00}^* - \frac{1}{56} x^n r^2 f_{nr,000}^* + \frac{1}{280} x^r r^2 f_{nn,00}^* . \qquad \text{(A29)} $$

As a result of the above transformation, the solutions in the seventh and eighth orders are

$$ {}_7\gamma'''_{rs} = \frac{81}{308} r^2 f_{rs,00}^* - \frac{3}{154} x^n (x^s f_{rn}^* + x^n f_{sn}^*)_{,00} + \frac{1}{77} x^r x^s f_{nn,00}^* - \frac{5}{308} \delta_{rs} r^2 f_{nn,00}^* - \frac{3}{14} \delta_{rs} x^m x^n f_{mn,00}^* + \text{N.L.}, \qquad \text{(A30a)} $$

$$ {}_7\gamma'''_{00} = \text{N.L.}, \qquad \text{(A30b)} $$

$$ {}_8\gamma'''_{0r} = \text{N.L.} \qquad \text{(A30c)} $$

A little arithmetic shows that Eq. (A30a) may be written as

$$ {}_7\gamma'''_{rs} = \frac{27}{154} r^2 f_{rs,00} - \frac{1}{77} x^n (x^r f_{sn} + x^s f_{rn})_{,00} - \frac{1}{7} \delta_{rs} x^m x^n f_{mn,00}, \qquad \text{(A31)} $$

where f_{rs} is defined by Eq. (A18a). Clearly, the solution no longer depends on f_{rs}^*.

APPENDIX 5. SOLUTION FOR THE TRANSVERSE-TRANSVERSE COMPONENTS

In this case the initial solutions in the fifth and sixth orders are

$$ {}_5\gamma_{rs} = f_{rs}(\tau), \qquad \text{(A32a)} $$

$$ {}_5\gamma_{00} = 0, \qquad \text{(A32b)} $$

$$ {}_6\gamma_{0r} = 0, \qquad \text{(A32c)} $$

with (see Appendix 4)

$$ f_{ss}(\tau) = 0. \qquad \text{(A33)} $$

The coordinate transformation generated by the following functions eliminates ${}_5\gamma_{rs}$:

$$ {}_5v^r = \tfrac{1}{2} x^n f_{rn}(\tau). \qquad \text{(A34)} $$

From the transformation equations, Eq. (22), we find

$$ {}_5\gamma'_{rs} = 0, \qquad \text{(A35a)} $$

$$ {}_5\gamma'_{00} = 0, \qquad \text{(A35b)} $$

$$ {}_6\gamma'_{0r} = -\tfrac{1}{2} x^n f_{rn,0} . \qquad \text{(A35c)} $$

Comparing Eq. (A35c) with (A18c) we observe that the solutions differ only by a factor of two. Therefore, the remaining transformations required for this discussion will follow the pattern of the previous Appendix.

In the seventh order the field equations are [Eqs. (A19) and (A35)]

$$ {}_7\gamma'_{rs,nn} = f_{rs,00}, \qquad \text{(A36a)} $$

$$ {}_7\gamma'_{00,nn} = 0, \qquad \text{(A36b)} $$

and the coordinate conditions are

$$ {}_7\gamma'_{rs,s} = 0. \qquad \text{(A37)} $$

If no additional harmonic functions are added to our solution, we obtain in a straightforward manner

$$ {}_7\gamma'_{rs} = \frac{5}{22} r^2 f_{rs,00} - \frac{1}{11} x^n (x^2 f_{ns} + x^s f_{nr})_{,00}, \qquad \text{(A38a)} $$

$$ {}_7\gamma'_{00} = 0. \qquad \text{(A38b)} $$

This solution differs from that of Eq. (A31) only by a harmonic function.

With the choice of transformation function

$$ {}_6v^0=\tfrac{1}{4}x^mx^nf_{mn,\,0}, \tag{A39}$$

the longitudinal component ${}_6\gamma'_{0r}$ can be eliminated. In addition, in the seventh order we find

$$ {}_7\gamma''_{rs}={}_7\gamma'_{rs}+\tfrac{1}{4}\delta_{rs}x^mx^nf_{mn,\,00}, \tag{A40a}$$

$$ {}_7\gamma''_{00}=\tfrac{1}{4}x^mx^nf_{mn,\,00}, \tag{A40b}$$

and the coordinate conditions

$$ {}_7\gamma''_{rs}=\tfrac{1}{2}x^nf_{nr,\,00}. \tag{A40c}$$

An examination of the transformation equations, Eq. (22), shows that these solutions cannot be removed by any coordinate transformation in the seventh order.

From Eq. (A9) it follows that the solutions in the seventh order cannot give rise to radiation effects until the tenth order. Therefore, we require the solution in the eighth order. The field equations in the eighth order are [Eqs. (A24) and (A40)]

$$ {}_8\gamma''_{0r,\,nn}=\tfrac{1}{2}x^nf_{nr,\,000}, \tag{A41a}$$

and the coordinate condition is

$$ {}_8\gamma''_{0r,\,r}={}_7\gamma''_{00,\,0}=\tfrac{1}{4}x^mx^nf_{mn,\,000}. \tag{A41b}$$

The solution of the above equations is easily found to be

$$ {}_8\gamma''_{0r}=(1/28)x^nx^m(x^mf_{nr}+x^rf_{nm}),_{\,000}. \tag{A42}$$

It is now possible to carry out another coordinate transformation which not only will eliminate ${}_7\gamma''_{00}$ and ${}_8\gamma''_{0r}$, but also returns the standard coordinate conditions in the seventh order. This transformation is characterized by

$$ {}_7v^r=(1/28)x^nx^m(x^mf_{nr}+x^rf_{nm}),_{\,00}. \tag{A43}$$

By means of Eq. (22), we find

$$ {}_7\gamma'''_{rs}=(12/77)r^2f_{rs,\,00}-(18/77)x^n(x^rf_{ns}+x^sf_{nr}),_{\,00} +(3/7)\delta_{rs}x^mx^nf_{mn,\,00}, \tag{A44a}$$

$$ {}_7\gamma'''_{00}=0, \tag{A44b}$$

$$ {}_8\gamma'''_{0r}=0, \tag{A44c}$$

and the coordinate condition

$$ {}_7\gamma'''_{rs,\,s}=0. \tag{A45}$$

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Gravitational Radiation

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Recently, several solutions of the field equations of general relativity theory have been published which are based upon the Einstein-Infeld-Hoffmann approximation method and claim to represent radiative effects in the motion of freely gravitating particles. These solutions are analyzed and it is shown that none produced to date represents a satisfactory discussion of the possibility of radiation by freely gravitating particles.

The problem of the possibility of gravitational radiation by free particles is then investigated upon general grounds. It is shown that the possibility of such radiation depends on the definition of free particles. This definition depends on the (unproven) assumption that the Einstein-Infeld-Hoffmann method is, in its physical outcome, independent of the coordinate system. If this assumption were not true, this would constitute a severe limitation of the method and imply that all results have to be expressed in a standard coordinate system. Under these circumstances, the definition of free particles is obviously equivalent to a postulate of absence of radiation.

IT has been claimed by Infeld and the author[1–4] that gravitational radiation within the framework of the Einstein-Infeld-Hoffmann (EIH) formalism[5] of solving the field equations of general relativity theory does not correspond to radiation from a purely mechanical system. In fact, it was shown that the introduction of a "radiation" term (for the latter's definition see Scheidegger[3]) at any stage of the EIH procedure corresponds to altering the coordinate system at that stage of the procedure, and therefore it was reasoned that the solution could always be continued in such a manner so as not to contain any radiation at all. The last train of thought is dependent upon the assumption that the EIH procedure, including the prescription for the characterization of freely gravitating particles, is a consistent formalism, yielding equations of motion that are physically independent of the formal choice of the coordinate system. As the latter assumption has never been proven, the argument of Infeld and the writer suffers from the same deficiency.

Every once in a while some calculations are published which seem to show explicitly that gravitational

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[3] A. E. Scheidegger, Revs. Modern Phys. **25**, 451 (1953).
[4] L. Infeld, Can. J. Math. **5**, 17 (1953).
[5] See, e.g. A. Einstein and L. Infeld, Can. J. Math. **1**, 209 (1949).