Conservation of Charge in Einstein's Generalization of Gravitation Theory*

JAMES L. ANDERSON AND PARVIZ MERAT
Department of Physics, University of Maryland, College Park, Maryland

(Received April 14, 1955)

The consequences of λ invariance in Einstein's new modified field theory are investigated. It is shown that, as a consequence of this proposed invariance, it is possible to replace the antisymmetric part of the metric tensor with a four-potential. By further postulating invariance of the theory under a gauge transformation of the theory, it is possible to arrive at a quantity which can be interpreted as a current density four-vector.

INTRODUCTION

FROM our previous experiences, we have come to expect that theories which are invariant with respect to one or more continuous groups of transformations have associated with them a like number of conservation laws.[1] For instance, rotational invariance leads to the conservation law of angular momentum while gauge invariance of electromagnetic theory leads to the conservation of charge. It is important to note that the charge conservation law differs from the angular momentum conservation law in that the charge law is valid at each space-time point independent of whether the field equations are satisfied at this point while the angular momentum law depends upon the satisfaction of the field equations. This difference is intimately related to the nature of the related invariance group. The rotation group is a six-parameter Lie group while the gauge group is an infinite parameter Lie group; it is necessary to specify a function of the four-coordinates in order to specify a member of the group. The same situation pertains in general relativity. There it is necessary to specify four functions of the coordinates, the relations between the old and the new coordinates, in order to specify an element of the group. The corresponding conservation laws are those of energy and momentum.

In a revised version of his generalization of gravitation theory,[2] Einstein has proposed that his theory be invariant under a new type of invariance, the λ invariance. Under a λ transformation the Christoffel symbol $\Gamma^{\lambda}{}_{\iota\kappa}$ transforms according to

$$\bar{\Gamma}^{\lambda}{}_{\iota\kappa}=\Gamma^{\lambda}{}_{\iota\kappa}+\delta^{\lambda}{}_{\iota}\lambda_{\kappa}, \tag{1}$$

where $\delta^{\lambda}{}_{\iota}$ is the Kronecker delta tensor and λ_{κ} is a set of four arbitrary space-time functions. Under this transformation, the Lagrangian

$$\mathfrak{L}=\mathfrak{g}^{\iota\kappa}R_{\iota\kappa}, \tag{2}$$

where

$$R_{\iota\kappa}=[\Gamma^{\sigma}{}_{\iota\kappa,\sigma}-\Gamma^{\lambda}{}_{\iota\sigma}\Gamma^{\sigma}{}_{\lambda\kappa}]-[\Gamma^{\sigma}{}_{\iota\sigma,\kappa}-\Gamma^{\lambda}{}_{\iota\kappa}\Gamma^{\sigma}{}_{\lambda\sigma}] \tag{3}$$

changes by an amount

$$\begin{aligned}\delta\mathfrak{L}&=\mathfrak{g}^{\iota\kappa}(\lambda_{\iota,\kappa}-\lambda_{\kappa,\iota})\\&=2\mathfrak{g}^{\underset{\vee}{\iota\kappa}}\lambda_{\iota,\kappa}.\end{aligned} \tag{4}$$

Here $\mathfrak{g}^{\underset{\vee}{\iota\kappa}}$ represents the antisymmetric part of $\mathfrak{g}^{\iota\kappa}$. The change in the integrated Lagrangian is thus

$$\begin{aligned}\delta\int\mathfrak{L}&=2\int\mathfrak{g}^{\underset{\vee}{\iota\kappa}}\lambda_{\iota,\kappa}d^4x\\&=-2\int\mathfrak{g}^{\underset{\vee}{\iota\kappa}}{}_{,\kappa}\lambda_{\iota}d^4x,\end{aligned} \tag{5}$$

provided of course that λ_{ι} vanish on the surface bounding the region of integration. Now in order that the theory be invariant under arbitrary λ transformations, the change in the integrated Lagrangian must vanish identically. Since the λ_{ι} are completely arbitrary we can conclude from Eq. (5) that

$$\mathfrak{g}^{\underset{\vee}{\iota\kappa}}{}_{,\kappa}\equiv0. \tag{6}$$

These are the "Bianchi" identities associated with λ invariance. While it is true that they are already in the form of conservation laws, we cannot interpret them without further work.

FOUR-POTENTIALS

The Eqs. (6) are identities; they are valid regardless of any particular dependence of the field variables on their arguments. However, it is clear from the form of Eqs. (6) that we would encounter an inconsistency in the requirement of λ invariance if we interpreted the $\mathfrak{g}^{\underset{\vee}{\iota\kappa}}$ as the fundamental field variables. The Eqs. (6) can only be satisfied by a very restricted set of field variables. On the other hand, if the $\mathfrak{g}^{\underset{\vee}{\iota\kappa}}$ were themselves to depend upon other field variables, then it should be possible to arrange this dependence in such a manner that Eqs. (6) are really identities in these new field variables.

In order to find these new field variables and the dependence of $\mathfrak{g}^{\underset{\vee}{\iota\kappa}}$ on them, we define the dual $\tilde{\mathfrak{g}}_{\underset{\vee}{\iota\kappa}}$ to $\mathfrak{g}^{\underset{\vee}{\iota\kappa}}$ by the equations

$$\tilde{\mathfrak{g}}_{\underset{\wedge}{\mu\nu}}=\tfrac{1}{2}\eta_{\mu\nu\iota\kappa}\mathfrak{g}^{\underset{\vee}{\iota\kappa}}, \tag{7}$$

where $\eta_{\mu\nu\iota\kappa}$ is the Levi-Civita tensor density. Written

* This work was supported by the Office of Naval Research and the National Science Foundation.

[1] See E. L. Hill, Revs. Modern Phys. **23**, 253 (1951) for a discussion of the relation between invariance and conservation laws. For a treatment of the types of invariance and their conservation laws treated in this paper, see J. N. Goldberg, Phys. Rev. **89**, 263 (1953).

[2] A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, 1953), fourth edition, Appendix II.

in terms of $\tilde{\mathfrak{g}}_{\mu\nu}$, (6) becomes

$$(\tilde{\mathfrak{g}}_{\underset{\wedge}{\kappa}\underset{\vee}{\lambda},\mu}+\tilde{\mathfrak{g}}_{\underset{\wedge}{\mu}\kappa,\lambda}+\tilde{\mathfrak{g}}_{\lambda\underset{\vee}{\mu},\kappa})\equiv 0, \tag{8}$$

from which it follows that $\tilde{\mathfrak{g}}_{\underset{\vee}{\mu\nu}}$ can be written as

$$\tilde{\mathfrak{g}}_{\underset{\vee}{\mu\nu}}=(\varphi_{\mu,\nu}-\varphi_{\nu,\mu}), \tag{9}$$

and thus we can write

$$\begin{aligned}\mathfrak{g}^{\underset{\vee}{\iota\kappa}}&=\tfrac{1}{2}\eta^{\iota\kappa\mu\nu}\tilde{\mathfrak{g}}_{\mu\nu}=\tfrac{1}{2}\eta^{\iota\kappa\mu\nu}(\varphi_{\mu,\nu}-\varphi_{\nu,\mu})\\&=\eta^{\iota\kappa\lambda\mu}\varphi_{\lambda,\mu}.\end{aligned} \tag{10}$$

It is therefore reasonable to take as our fundamental field variables the "potentials" φ_λ. In terms of them Eqs. (6) are indeed identities.

With these new variables the Lagrangian of the theory becomes

$$\mathfrak{L}=(\mathfrak{g}^{\underline{\iota\kappa}}+\eta^{\iota\kappa\lambda\mu}\varphi_{\lambda,\mu})R_{\iota\kappa}, \tag{11}$$

and thus, considering the $\mathfrak{g}^{\underline{\iota\kappa}}$, φ_λ, and $\Gamma^\lambda{}_{\iota\kappa}$ as independent field variables, our variational principal becomes

$$\int\{B^{\lambda\kappa}{}_\iota\delta\Gamma^\iota{}_{\lambda\kappa}+R_{\underline{\iota\kappa}}\delta\mathfrak{g}^{\underline{\iota\kappa}}+\eta^{\iota\kappa\lambda\mu}R_{\underset{\vee}{\iota\kappa},\mu}\delta\varphi_\lambda\}d^4x=0. \tag{12}$$

Our field equations are then

$$B^{\iota\kappa}{}_\lambda=-\mathfrak{g}^{\underset{+}{\iota}\underset{-}{\kappa}}{}_{;\lambda}+\mathfrak{g}^{\underset{+}{\iota}\underset{-}{\sigma}}{}_{;\sigma}\delta^\kappa{}_\lambda+\mathfrak{g}^{\iota\kappa}\Gamma_\lambda+\mathfrak{g}^{\iota\sigma}\Gamma_\sigma\delta^\kappa{}_\lambda=0, \tag{13a}$$

$$R_{\underline{\iota\kappa}}=0, \tag{13b}$$

$$\eta^{\iota\kappa\lambda\mu}R_{\underset{\vee}{\iota\kappa},\mu}=0. \tag{13c}$$

(The notation is that of Einstein.) These equations are equivalent to those obtained by Einstein provided we choose λ_κ so that $\Gamma_\kappa=0$ with the exception that now φ_κ has replaced $\mathfrak{g}^{\underset{\vee}{\iota\kappa}}$ as a fundamental variable.

CURRENT DENSITY

As we mentioned in the Introduction, we can expect that a law of charge conservation will be associated with some invariance property of the theory akin to gauge invariance. The simplest way to introduce this invariance is to postulate that the theory is invariant under the potential transformation

$$\varphi_\lambda\rightarrow\varphi_\lambda+\Lambda_{,\lambda}, \tag{14}$$

where Λ is an arbitrary function of the space-time coordinates.

In order that the Lagrangian (11) be invariant under this transformation, it is sufficient to require that

$$\eta^{\iota\kappa\lambda\mu}R_{\underset{\vee}{\iota\kappa},\mu\lambda}\equiv 0 \tag{15}$$

as can be seen from Eq. (12). Because of the symmetry properties of $\eta^{\iota\kappa\lambda\mu}$ it is evident that the left-hand-side of (15) is indeed identically zero. It does not seem unreasonable therefore, to take as our current-density four-vector

$$\mathfrak{J}^\lambda=\eta^{\iota\kappa\lambda\mu}R_{\underset{\vee}{\iota\kappa},\mu}, \tag{16}$$

with the associated conservation law

$$\mathfrak{J}^\lambda{}_{,\lambda}\equiv 0. \tag{17}$$

The total charge enclosed in a region Ω is given by

$$Q=\int_\Omega\eta^{\iota\kappa 4\mu}R_{\underset{\vee}{\iota\kappa},\mu}d^4x, \tag{18}$$

which, by Gauss' theorem can be written as

$$Q=\int_\Sigma\eta^{\iota\kappa 4r}R_{\underset{\vee}{\iota\kappa}}d\sigma_r, \tag{18}$$

where $d\sigma_r$ is a surface element with normal along the r coordinate axis. It is evident from Eq. (17) that, for a charge concentrated within a finite volume, the integral is independent of the surface Σ provided only that it completely enclosed the charge. We should require this property if the integral is to represent the charge enclosed within a given region of space.

DISCUSSION

In order to obtain any information at all from a theory like the one we have been considering, it is necessary to construct quantities like the energy-momentum tensor and the current-density vector in order that a connection can be made with experiment. The field equations by themselves tell us nothing since in general we do not know how to interpret the field variables which appear therein. In order that quantities which represent the physical observables of the theory are not introduced in a completely ad hoc manner, we look for some guiding principle to aid us in our choice. At present that principle seems to lie in the close connection between invariances of the theory and the corresponding conservation laws. We have used this principle in postulating that the right-hand side of Eq. (16) really does represent the current density of the theory.

Our assertion differs from that made by Einstein. He postulates that

$$\mathfrak{J}^\mu=\eta^{\iota\kappa\lambda\mu}g_{\underset{\vee}{\iota\kappa},\lambda}, \tag{20}$$

and indeed it is true that

$$\mathfrak{J}^\mu{}_{,\mu}=\eta^{\iota\kappa\lambda\mu}g_{\underset{\vee}{\iota\kappa},\lambda\mu}\equiv 0, \tag{21}$$

so that the quantity appearing on the right-hand side of (20) is conserved. However, it is not at all evident that there are not many other quantities which possess this property and indeed we have found one such additional quantity. We believe that one must have a choice based on a general principle applicable to all cases, such as the aforementioned relation between invariance and conservation laws.

Our choice (16) however is not completely free from objection. Einstein has pointed out[3] that our current-

[3] A. Einstein (private communication).

density vanishes everywhere that the field equations are satisfied and hence is unsatisfactory since the only acceptable solutions to the field equations are those which satisfy them everywhere. Thus, for this solution, our current density would be everywhere zero. We believe, however, that it is necessary to consider solutions of the field equations which do not satisfy them everywhere. In certain small regions of space it might well be that the field equations are not even valid. One can take the position that the field equations describe correctly the interaction of elementary particles but are insufficient to describe the particles themselves and hence, at small distances from these particles, must be replaced by some other kind of mathematical construction.[4] If such were the case we would no longer need to restrict our solutions and the current density of Eq. (16) could be nonzero for certain regions of space. In fact, if one takes seriously the relation between invariance and conservation laws, one is almost forced into this latter position. Regardless of the form of the Lagrangian, the current density which follows from gauge invariance will vanish whenever the field equations are satisfied.

The only way out seems to be to construct a theory in which it is impossible to introduce potentials. While there would still be an invariance associated with the conservation of charge, it may be of such a nature that the current-density need not vanish when the field equations are satisfied. For instance, the energy-momentum tensor associated with arbitrary coordinate invariance does not vanish when the field equations are satisfied. It is evident that a theory for which it is impossible to introduce potentials will differ from the present theory and hence lies outside the scope of our investigation.

[4] This view is diametrically opposite to that held by Einstein. He contends that his field equations are everywhere valid, i.e., that they do correctly describe the elementary particles.

PHYSICAL REVIEW VOLUME 99, NUMBER 6 SEPTEMBER 15, 1955

Gravitational Radiation

JOSHUA N. GOLDBERG
Armour Research Foundation, Illinois Institute of Technology, Chicago, Illinois
(Received October 19, 1953; revised manuscript received May 27, 1955)

In this paper, we investigate the existence of gravitational radiation within the framework of the EIH approximation method. Following a prescription introduced by Infeld, the radiation terms of the EIH expansion are begun with functions of the time alone. We find that these terms do not have physical significance if they are introduced in the scalar or longitudinal components of the gravitational potentials. However, if the radiation terms are introduced in the fifth order of the transverse-transverse components, one finds a contribution to the curvature tensor in the seventh order, a contribution to the equations of motion in the ninth order, and radiation in the tenth order. The existence of radiation is determined by calculating the energy passing through a spherical surface which is an infinite distance from all source points. This definition of radiation agrees with that used in the theory of electromagnetism.

I. INTRODUCTION

IN recent years there has been some controversy concerning gravitational radiation. Infeld and Scheidegger have maintained that the possibility of radiation does not exist within the framework of the EIH (Einstein, Infeld, and Hoffman) approximation method.[1–4] This conclusion has been accepted for the scalar and longitudinal components. Indeed, in his book Bergmann has shown[5] that in the linearized gravitational equations these terms do not contribute to radiation; however, the transverse-transverse components do make a contribution. It was on this basis that the proof set forth by Infeld and Scheidegger was first criticized.[6] This paper will attempt to clarify the situation by proposing an unambiguous definition of radiation.

II. EIH APPROXIMATION METHOD

In a previous paper[7] the surface integrals leading to the equations of motion were found without recourse to an approximation method. The possibility of doing so depended on the existence of superpotentials for the components of the energy-momentum psuedo-tensor:*

$$T_\mu{}^\nu = U_\mu{}^{[\nu\sigma]}{}_{,\sigma}. \qquad (1)$$

$T_\mu{}^\nu$ is the energy-momentum psuedo-tensor and $U_\mu{}^{[\nu\sigma]}$

[1] L. Infeld and A. E. Scheidegger, Can. J. Math. **3**, 195 (1951).
[2] A. E. Scheidegger, Phys. Rev. **82**, 883 (1951).
[3] L. Infeld, Can. J. Math. **5**, 17 (1953).
[4] A. E. Scheidegger, Revs. Modern Phys. **25**, 451 (1953).
[5] P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall Publications, Inc., New York, 1947), pp. 187–189.
[6] P. G. Bergmann (private communication).
[7] J. N. Goldberg, Phys. Rev. **89**, 263 (1953).
* The energy-momentum pseudo-tensor does not have simple geometrical transformation properties (see reference 5, page 196). Hence, the word *pseudo-tensor* should not be confused by the current use of the prefix *pseudo-* to describe a density of weight one.