Traveling Density Waves in Positive Columns

SATOSI WATANABE AND NORMAN L. OLESON
United States Naval Postgraduate School, Monterey, California
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The continuity equations for positive ions and for electrons coupled to each other by the Coulomb force and the effect of ionizing collisions have a solution representing traveling density waves whose frequencies are widely different from the usual plasma oscillations.

THE aim of this paper is to show that there can exist, in the positive column, traveling waves of ion density and electron density whose physical nature is different from the so-called plasma or plasma-ion oscillations.[1] It is premature to identify these waves with the moving striations, but a further elaboration of the theory accompanied by an adequate consideration of modes of excitation of the waves may perhaps provide a theoretical basis for explaining moving striations. The present paper differs from the previous papers by various authors which tried to explain positive striations with the help of diffusion equations, for they considered only the electron density or assumed equality of the electron and ion densities.[2]

The starting equations are diffusion equations for positive and negative ions. Since the observed wavelengths of striations are larger than the mean free paths and since the observed frequencies are smaller than the reciprocal of the time necessary to establish local equilibrium, the use of the diffusion equations is justifiable. These two diffusion equations are coupled to each other through two physical phenomena. First, the electrostatic field generated by positive space charges reacts on negative charges, and vice versa. Second, both positive and negative ions are created by collisions of electrons with neutral molecules, the density of which is much larger than the ion and electron densities. The coupling between the two ion densities under an external electrostatic field gives rise to the possibility of traveling density waves.

In the following, more emphasis should be placed on the possibility of the existence of such waves than on the numerical results. However, a rough approximation gives as the wavelength something of the order of magnitude of the radius of the tube, which is comparable with the observed data. The propagation velocity of waves in the first approximation is of the order of magnitude of the positive ion mobility times the impressed electric field. This is apparently smaller than the observed velocity of moving striations.[3] However, a qualitative discussion at the end of the paper will show that a further refinement of the theory may probably bring the propagation velocity to the neighborhood of the sound velocity of the neutral gas in the tube.

The basic equations are:

$$\begin{aligned}(\partial n^-/\partial t)+\mathrm{div}(n^-\mathbf{v}^-)-zn^-=0,\\ (\partial n^+/\partial t)+\mathrm{div}(n^+\mathbf{v}^+)-zn^-=0,\end{aligned} \tag{1}$$

with

$$\begin{aligned}\mathbf{v}^-=-K^-\mathbf{E}-(D^-/n^-)\,\mathrm{grad}n^-,\\ \mathbf{v}^+=+K^+\mathbf{E}-(D^+/n^+)\,\mathrm{grad}n^+,\end{aligned} \tag{2}$$

where the symbols have the following meanings:

n^+, n^-: positive and negative ion densities, respectively,
$\mathbf{v}^+$, $\mathbf{v}^-$: positive and negative drift velocities,
K^+, K^-: positive and negative mobilities,
D^+, D^-: positive and negative diffusion coefficients,
z: number of ion pairs created per unit time per one electron.

The assumption that z is constant may be justified if the temperature variations can be considered very small. The electric field $\mathbf{E}$ consists of two parts: $\mathbf{E}_0$ and $\mathbf{E}'$, where $\mathbf{E}_0$ is the impressed electric field and $\mathbf{E}'$ the electric field generated by space charges.

$$\begin{aligned}\mathbf{E}=\mathbf{E}_0+\mathbf{E}',\\ \mathrm{div}\mathbf{E}=\mathrm{div}\mathbf{E}'=4\pi e(n^+-n^-),\end{aligned} \tag{3}$$

where e is the electronic charge. We consider only singly charged ions and assume that the tube has a circular cross section and ignore the effect of the both ends of the positive column.

We propose now to solve these equations by assuming that the actual solution is a superposition of a steady,

[1] This can be seen in Eq. (19) which gives the frequency depending on the external electric field.

[2] M. T. Druyvesteyn, Physica **1**, 273 and 1003 (1934); M. F. Shirokov, Doklady Akad. Nauk S.S.S.R. **89**, 837 (1953). See also Armstrong, Emeleus, and Neill, Proc. Roy. Irish Acad. **54-A**, 291 (1951). Our thanks are due to Professor Emeleus who informed us of Dr. V. D. Farris' yet unpublished theory on moving striations. [Proc. Phys. Soc. (London) (to be published).] His work, however, also assumes approximate equality of positive and negative densities. Yoshimoto *et al.*, treat the same problem without this assumption, but their results are essentially different from ours since they do not take creation of ions into consideration. Our results, as can be seen from our Eq. (26), depends crucially on the rate of production of ions. See Yoshimoto, Sato, and Nakao, Repts. Phys. Dept., Okayama Univ. **10**, 15 (1954).

[3] In a tube approximately one centimeter in diameter and containing argon at 12 mm Hg pressure, we have observed positive striations in the glow discharge moving with velocities of 25 meters per second or higher. Using the data of J. A. Hornbeck, Phys. Rev. **84**, 615 (1951), on drift velocities of A^+ in argon, one finds that an average field intensity of 2600 volts per meter would be required to account for the observed velocity. The velocity of sound in argon under the above conditions is 320 meters per second.

uniform solution $n_0^{\pm}$ plus a small nonsteady, nonuniform disturbance $\nu^{\pm}$. The assumption that $\nu^{\pm}$ is small compared with $n_0^{\pm}$ may not be realistic if the theory is to be applied to the actual striations. This assumption thus is a mathematical expediency to linearize the equations and to help discover certain aspects of the complicated physical situation.

$$\begin{aligned} n^+&=n_0^+ + \nu^+, \quad |\nu^+| \ll n_0^+, \\ n^-&=n_0^- + \nu^-, \quad |\nu^-| \ll n_0^-. \end{aligned} \tag{4}$$

We shall write $\mathcal{E}$ for the electric field generated by the steady, uniform charge densities $n_0^{\pm}$. The word "uniform" here means that the quantities do not depend on the x-coordinate taken along the axis of the tube.

$$\begin{aligned} \partial n_0^+/\partial t = \partial n_0^-/\partial t = \partial \mathcal{E}/\partial t = 0, \\ \partial n_0^+/\partial x = \partial n_0^-/\partial x = \partial \mathcal{E}/\partial x = 0, \\ \mathcal{E}_x = 0. \end{aligned} \tag{5}$$

These n_0^+, n_0^- and $\mathcal{E}=\mathbf{E}'$ must satisfy Eqs. (1), (2) and (3).

The well-known ambipolar diffusion solution[4] may be considered as an approximate solution satisfying these conditions, i.e., Eqs. (1), (2), (3), and (5). This ambipolar diffusion solution is not self-consistent in the sense that the assumption $n_0^-=n_0^+$ leads to $\mathbf{E}'=\mathcal{E}=0$ according to Eq. (3), while the other assumption which is also used that $\mathbf{v}_r^+=\mathbf{v}_r^-$ (r meaning radial component) leads, on account of Eq. (2), to the electric field given by

$$\mathcal{E}_r = \frac{D^+ - D^-}{K^+ + K^-} \cdot \frac{1}{n_0} \cdot \frac{dn_0}{dr}, \tag{6}$$

where r is the radial distance from the tube axis.

However, we can expect that a rigorous solution of Eqs. (1), (2), (3), and (5) under adequate boundary conditions is not very different from the ambipolar diffusion solution, in particular in the region not too close to the wall of the tube. We assume specifically that the rigorous solution has, in common with this approximation, the following features. First, in the region which is not very far from the tube axis the densities n_0^+ and n_0^- are almost constant and approximately equal to each other. Second, the rate z of production of ion pairs is roughly given by

$$z = (2.405/R)^2 D_a, \tag{7}$$

where the ambipolar diffusion coefficient D_a is defined by

$$D_a = (D^+K^- + D^-K^+)/(K^+ + K^-) \doteq K^+ \kappa T_e/e, \tag{8}$$

where κ is the Boltzmann constant and T_e is the electron temperature.[4] R in Eq. (7) is the radius of the tube.

Now we substitute Eq. (4) in Eqs. (1) and (2), and simplify the equations with the help of the fact that $n_0^{\pm}$ is the solution of Eqs. (1), (2), and (3), satisfying Eq. (5). The second order term in the disturbance is to be neglected. For instance, in the term $\operatorname{div}(n^{\pm}\mathbf{v}^{\pm})$ of Eq. (1), there will appear a term like

$$\operatorname{div}(\nu^{\pm} K^{\pm} \mathbf{e}), \tag{9}$$

where $\mathbf{e}$ is the electric field generated by the disturbance $\nu^{\pm}$, i.e.,

$$\operatorname{div}\mathbf{e} = 4\pi e(\nu^+ - \nu^-). \tag{10}$$

The term (9) is neglected. Then Eq. (1) becomes a set of linear equations for ν^+ and ν^-, which are

$$\begin{aligned} (\partial\nu^-/\partial t) + \operatorname{div}[-K^-(n_0^-\mathbf{e} + \nu^-\mathbf{E}_0 + \nu^-\mathcal{E}) \\ - D^- \operatorname{grad}\nu^-] - z\nu^- = 0. \\ (\partial\nu^+/\partial t) + \operatorname{div}[+K^+(n_0^+\mathbf{e} + \nu^+\mathbf{E}_0 + \nu^+\mathcal{E}) \\ - D^+ \operatorname{grad}\nu^+] - z\nu^- = 0. \end{aligned} \tag{11}$$

where $\mathcal{E}$ and $\mathbf{e}$ are the fields generated respectively by $n_0^{\pm}$ and $\nu^{\pm}$.

As we are interested in the region not too close to the wall, we assume

$$\begin{aligned} \partial\nu^-/\partial r = \partial\nu^+/\partial r = 0, \\ \mathbf{e}_r = 0. \end{aligned} \tag{12}$$

Remembering that $\mathcal{E}_x=0$, Eq. (5), and that $n_0^{\pm}$ is almost constant in the central region, we can simplify Eq. (11) to the form

$$\begin{aligned} (\partial\nu^-/\partial t) - K^-E_0(\partial\nu^-/\partial x) - D^-(\partial^2\nu^-/\partial x^2) \\ + \gamma_1\nu^- - \gamma_2\nu^+ = 0, \\ (\partial\nu^+/\partial t) + K^+E_0(\partial\nu^+/\partial x) - D^+(\partial^2\nu^+/\partial x^2) \\ + \gamma_3\nu^+ - \gamma_4\nu^- = 0, \end{aligned} \tag{13}$$

with

$$\begin{aligned} \gamma_1 &= 4\pi e K^-(2n_0^- - n_0^+) - z. \\ \gamma_2 &= 4\pi e K^- n_0^-, \\ \gamma_3 &= 4\pi e K^+(2n_0^+ - n_0^-), \\ \gamma_4 &= 4\pi e K^+ n_0^+ + z, \end{aligned} \tag{14}$$

where n_0^+ and n_0^- are approximately equal to each other.

$$n_0^+ = n_0^- (\equiv n_0). \tag{15}$$

The coefficients γ_2 and γ_4 represent the coupling between the two densities.

We now proceed to show that Eq. (13) has a solution which represents traveling waves of ν^+ and ν^- both having the same space-time dependence. Let us insert in Eq. (13) the expression:

$$\begin{aligned} \nu^- &= a^- e^{ikx - i\omega t}, \\ \nu^+ &= a^+ e^{ikx - i\omega t}, \end{aligned} \tag{16}$$

where a^- and a^+ are complex amplitudes. We then obtain

$$\begin{aligned} (-i\omega - iK^-E_0k + D^-k^2 + \gamma_1)a^- - \gamma_2 a^+ = 0. \\ -\gamma_4 a^- + (-i\omega + iK^+E_0k + D^+k^2 + \gamma_3)a^+ = 0. \end{aligned} \tag{17}$$

[4] See for instance J. D. Cobine, *Gaseous Conductors* (McGraw-Hill Book Company, Inc., New York, 1941), p. 236 ff.

Compatibility of these two homogeneous linear equations leads to two real equations:

$$(D^-k^2+\gamma_1)(D^+k^2+\gamma_3)-\gamma_2\gamma_4 +k^2E_0^2(K^++K^-)^2(D^+k^2+\gamma_3)(D^-k^2+\gamma_1)/[(D^+k^2+\gamma_3)+(D^-k^2+\gamma_1)]^2=0, \quad (18)$$

and

$$\omega=kE_0[(K^+D^--K^-D^+)k^2-(K^-\gamma_3-K^+\gamma_1)]/[(D^++D^-)k^2+(\gamma_1+\gamma_3)]. \quad (19)$$

The fact that these Eqs. (18) and (19) have real solutions for k and ω shows the possibility of traveling waves. We can use Eq. (18) to determine k, and then use Eq. (19) to determine the propagation velocity

$$V=\omega/k. \quad (20)$$

In order to obtain an approximate solution of Eq. (18), we introduce two variables

$$\xi=D^-k^2+\gamma_1, \quad \eta=D^+k^2+\gamma_3, \quad (21)$$

and observe the intersection of two straight lines:

$$\eta=\gamma_3+(D^+/D^-)(\xi-\gamma_0), \qquad \eta=\gamma_3-\{(1+g)^2/[g(1+g)^2+gh/\gamma_3]\}(\xi-g\gamma_3), \quad (22)$$

with

$$g=\gamma_2\gamma_4/\gamma_3^2, \quad h=(K^++K^-)^2E_0^2/D^+. \quad (23)$$

The first equation in Eq. (22) is a direct consequence of Eq. (21). The second equation in Eq. (22) is the tangent at $\eta=\gamma_3$ to the curve representing Eq. (18) in the $\xi-\eta$ plane. The substitution of this tangent for the curve is justified by the fact that $D^-\gg D^+$, which means that the straight line: $\eta=\gamma_3+(D^+/D^-)(\xi-\gamma_1)$ cuts the said curve at a point very close to $\xi=\gamma_2\gamma_4/\gamma_3$, $\eta=\gamma_3$. This approximation yields

$$k^2=(\gamma_3g-\gamma_1)/\{D^-+D^+[g+gh/\gamma_3(1+g)^2]\}. \quad (24)$$

Making use of Eqs. (14) and (15) and noticing that

$$z\ll 4\pi eK^+n, \quad (25)$$

we obtain from Eq. (24):

$$k^2=(K^++K^-)z/(K^+D^-+K^-D^++K^+K^-E_0^2/4\pi ne), \quad (26)$$

or by the help of Eqs. (7) and (8),

$$\lambda=2\pi/k=(2\pi R/2.405)[1+\tfrac{1}{2}a^2/(eE_0/\kappa T_e)^2], \quad (27)$$

where a is the Debye-Hückel radius of the electron:

$$a=(\kappa T_e/4\pi n_0e)^{\frac{1}{2}}.$$

To determine the sign of the propagation velocity V, Eq. (20), let us take the case $E_0>0$. Then, Eq. (19) shows that V is positive (i.e., traveling from anode to cathode) for sufficiently large values of k and negative (i.e., traveling from cathode to anode) for sufficiently small values of k. When k is very large, we obtain from Eq. (19), with the help of the usually satisfied conditions: $(K^+/D^+)\gg(K^-/D^-)$ and $D^-\gg D^+$,

$$V_{k\to\infty}=K^+E_0. \quad (28)$$

V vanishes, when

$$k^2=z/D^-\ll 1 \text{ cm}^{-2}. \quad (29)$$

The amplitude ratio and phase relation between the ion density wave a^+ and the electron density wave a^- can be easily calculated from Eq. (17). Putting

$$a^+/a^-=Ae^{-i\phi}, \quad (A,\phi \text{ real}), \quad (30)$$

we obtain

$$\tan\phi=(K^++K^-)kE_0/[(D^++D^-)k^2+(\gamma_1+\gamma_3)], \quad (31)$$

and

$$A=(1/\gamma_2)(D^-k^2+\gamma_1)(1+\tan^2\phi)^{\frac{1}{2}}. \quad (32)$$

The order of magnitude of the factor $(D^-k^2+\gamma_1)/\gamma_2$ in Eq. (32) is unity, meaning that the positive amplitude and the negative amplitude are of the same order of magnitude unless $\tan\phi$ becomes very large compared with unity.

If we assume $A>0$, $k>0$, $E_0>0$, then we see from Eq. (17) that

$$0<\phi<\pi/2. \quad (33)$$

This means that for a wave propagating from anode to cathode, the phase of the positive density wave is leading that of the negative density wave by an angle between 0 and $\pi/2$. For very large values of k and for very small values of k, this phase difference vanishes, as can be seen from Eq. (31). Between these two limits $\tan\phi$ takes one maximum which is given by

$$(\tan\phi)_{\max}^2=(K^++K^-)^2E_0/[4(D^++D^-)(\gamma_1+\gamma_3)]. \quad (34)$$

Under the usual discharge conditions, the order of magnitude of Eq. (34) cannot be said to be either extremely large or extremely small.

It should be noted that the actual drift current of ions (as well as of electrons) is composed of three parts: (1) the part due to the external field $n_0^+K^+\mathbf{E}_0$ in the axial direction, (2) the part constituting the ambipolar diffusion in the radial direction, $n_0^+K^+\mathcal{E}-D^+\,\text{grad}n_0^+$, and (3) the part due to the disturbance. Except for the negligible term given in formula (9), this last part is given by the expression in the brackets following div in Eq. (11). As the radial field $\mathcal{E}$ is negligible in the central region, this part of drift current is in the axial direction, i.e., in the direction of propagation of the wave represented by Eq. (16). Furthermore, the time average of this drift current vanishes since each term in this quantity is oscillating in time. In other words, the traveling wave in question is accompanied by a longitudinal drift current whose time average is zero. In this respect, this wave is a kind of "sound wave," only its mechanism is electrical instead of being dynamical.

The ions, as far as their dynamic properties are concerned, are almost identical with the neutral molecules. Therefore, once the ion density wave described in the

foregoing occurs, a genuine sound wave of neutral molecules will be excited on account of the collisions between ions and molecules. If such a genuine sound wave is excited, the ions will "ride on" this wave, i.e., they will participate on this wave.

The drift velocity determined in Eq. (2) is essentially the terminal velocity in the presence of friction between the neutral molecules and ions due to the relative drift velocity. The calculation given in this paper is based on the assumption that the drift velocity of the neutral molecules is zero. If the neutral molecules perform sound motion, then the friction will be proportional to the difference between the ion drift velocity and the molecule drift velocity accompanying the sound. When these two drift velocities coincide, the friction will disappear and a "resonance" will set in. This situation, which is outside the scope of Eqs. (1) and (2), may explain the fact that the actual propagation velocity of positive striations is found to lie between the value as given by Eq. (28) and the velocity of sound in the gas. We are now undertaking an investigation in this direction.

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Helium Film Transfer Rate from 0.14 to 2.19°K*

R. K. WARING, JR.†
Sloane Physics Laboratory, Yale University, New Haven, Connecticut
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The transfer rate of the mobile helium film, over Pyrex glass, was measured as a function of temperature in the range from 0.14°K to the lambda point. The bulk of the observations was concerned with film transport due to a thermal gradient but a measurement, using the same apparatus, was made at 1.1°K under a gravitational potential as a comparison. At this temperature it was found that the transport rates, in the two cases, differ by a factor of nearly two, and below 1°K the rate was found to increase with decreasing temperature by about 10%.

INTRODUCTION

AMBLER and Kürti[1] have recently measured the rate of transfer by the helium film, under a gravitational potential difference, at temperatures below 1°K. They observed a 30% increase in the rate as the temperature was reduced from 0.85°K to 0.15°K. The only other data, that of Lesinsky and Boorse,[2] whose lowest temperature point was 0.75°K, lend some support to this result. Since all but about one percent of the liquid is already superfluid at 1°K, these observations seem to negate the hypothesis that the temperature dependence of the transfer rate is given by

$$R=(\rho_s/\rho)v_c d,$$

where the product $v_c d$ is a constant approximately equal to $\hbar/2m$. Here R is the flow rate in $cm^3\ cm^{-1}\ sec^{-1}$, v_c is the critical velocity of the superfluid, d is the film thickness, and ρ_s/ρ is the fraction of the liquid which is superfluid.

Liquid helium may also be transferred, by the mobile film, from a position at low temperature to a position at a higher temperature providing that no portion of the surface over which the helium must flow is above the lambda point. Thus thermal film transfer rates may be measured by allowing the film to flow into a region where it evaporates, the rate of vapor efflux then being determined. In view of the results of Ambler and Kürti it appeared desirable to investigate the thermal transfer rates at temperatures below 1°K, and this method offered a convenient means for observing these as well as rates above 1°K.

THE APPARATUS

The portion of the apparatus contained in the helium cryostat is schematically represented in Fig. 1. The bulk liquid helium II, which supported the film flow, was contained in a small Pyrex reservoir (*a*), the flow being constricted by a 211-micron i.d. Pyrex capillary (*b*). Except for a baking process to be described, this capillary was untreated. The surface of the liquid in reservoir (*a*) was always farther from the lower end of the Pyrex capillary than the surface tension rise appropriate to a 211 micron tube (~5 mm). This prevented the Pyrex capillary from filling with bulk liquid via the film.[3]

Above this constriction was a glass to Kovar seal (*c*) leading to a 0.014-inch i.d. stainless steel tube (*d*) with a 0.003-inch wall. A tube this fine was used in order to inhibit recondensation, into the reservoir, of helium evaporated from the film. It was necessary, since excessive recondensation would have caused the salt to warm up too rapidly after a demagnetization,

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