Statistical Mechanics of Liquid He^3

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The partition function which Feynman has proposed for liquid He^4 is extended to the case of liquid He^3, taking into account the Fermi-Dirac statistics and the nuclear spin of $\frac{1}{2}$. The nuclear magnetic susceptibility is calculated and compared with the observation of Fairbank, Ard, and Walters. Theoretical and experimental curves fit when the effective mass of the He^3 atom is taken about four times its true mass. The entropy of the system has not been calculated.

1. INTRODUCTION

FEYNMAN recently developed[1] a new method of treating quantum statistical mechanics of cooperative systems, and applied it to liquid He^4. Though it did not seem completely successful at first because in his paper he obtained a transition of a third order, it has been shown later[2] that when one takes into account more rigorously the geometrical correlation among atoms, Feynman's formulation does give a second-order transition.

This success of Feynman's approach encourages us to go one step further and examine the case of liquid He^3, which obeys Fermi-Dirac statistics, in order to make the discussion about liquid helium complete. The present paper is an attempt in this direction and discusses the problem based on the partition function and the lattice model Feynman introduced in F I.

2. ALL PARALLEL SPINS

The difference between He^4 and He^3 is not only the statistics but that the latter has a nuclear spin of magnitude $\frac{1}{2}$, whereas the former has not. In order to show the difference between the Bose-Einstein and the Fermi-Dirac statistics clearly, in this section we shall treat a case in which the effect of the spin can be neglected, namely the state of liquid He^3 in which all the nuclear spins are aligned in the same direction.

The expression of the partition function for Bose-Einstein particles Feynman derived is shown in Eq. (5) of F I and can be written in the following form:

$$Q_{BE}=N!^{-1}\sum_P\int(P\mathbf{z}_1,P\mathbf{z}_2,\cdots P\mathbf{z}_N \times|e^{-\beta H}|\mathbf{z}_1,\cdots,\mathbf{z}_N)d\mathbf{z}_1\cdots d\mathbf{z}_N, \quad (2.1)$$

in which the function in the integrand is defined by

$$(\mathbf{z}_1',\mathbf{z}_2',\cdots,\mathbf{z}_N'|e^{-\beta H}|\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_N) =\int_{\rm tr}\exp\left\{-\int_0^\beta\left[\frac{m}{2\hbar^2}\sum_i\left(\frac{d\mathbf{x}_i}{du}\right)^2 +\sum_{i,j}V(\mathbf{x}_i-\mathbf{x}_j)\right]du\right\}\mathfrak{D}^N\mathbf{x}_i(u), \quad (2.2)$$

where for each trajectory the following conditions are to be satisfied:

$$\mathbf{x}_i(0)=\mathbf{z}_i \quad \text{and} \quad x_i(\beta)=z_i'. \quad (2.3)$$

P in Eq. (2.1) is a permutation of the coordinates of N particles among themselves. For the derivation of these expressions and the detail of the notations, readers are referred to the original paper, F I.

It can easily be shown that for the partition function Q_{FD} for Fermi-Dirac particles one has to bring in a factor $(-1)^P$ into Eq. (2.1), so that

$$Q_{FD}=N!^{-1}\sum_P(-1)^P\int(P\mathbf{z}_1,P\mathbf{z}_2,\cdots P\mathbf{z}_N \times|e^{-\beta H}|\mathbf{z}_1,\cdots\mathbf{z}_N)d\mathbf{z}_1\cdots d\mathbf{z}_N. \quad (2.4)$$

The integrand of Eq. (2.4) is the same function as the integrand of Eq. (2.1) and is given by Eq. (2.2), independent of the kind of statistics. Therefore we can use the same argument as F I[3,4] in reducing Eq. (2.2) to arrive at

$$(P\mathbf{z}_1,P\mathbf{z}_2,\cdots P\mathbf{z}_N|e^{-\beta H}|\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_N) =K_\beta\left(\frac{m'}{2\pi\beta\hbar^2}\right)^{3N/2}\exp\left[-\frac{m'}{2\beta\hbar^2}\sum_i(\mathbf{z}_i-P\mathbf{z}_i)^2\right] \times\rho(\mathbf{z}_1,\cdots,\mathbf{z}_N). \quad (2.5)$$

This is further simplified if $\rho(\mathbf{z}_1,\cdots,\mathbf{z}_N)$ is assumed to

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[1] R. P. Feynman, Phys. Rev. **91**, 1291 (1953), hereafter called F I.

[2] R. Kikuchi, Phys. Rev. **96**, 563 (1954).

[3] See Eq. (7) of F I.

[4] Professor Feynman in private communication has emphasized the point he raised in footnote 9 of F I and said that, although the effect of a moving atom in permuting other atoms is not important for the Bose-Einstein case where permutations make no difference, this effect must be considered in more detail if we are to apply these ideas to the Fermi-Dirac case. Although it may simply mean a larger effective mass m', it may also be that expression (2.5) is not sufficiently accurate at low temperatures with any m'. In addition, one must be especially careful in Fermi statistics because the plus and minus contributions of even and odd permutations nearly balance out, so that great precision may be needed to keep the balance representative of the truth. Although (2.5) may be correct, he feels that the arguments of F I justify its use only in the E.B. case.

This comment of Professor Feynman warns that one must accept the calculations in the following sections with critical eyes. The author has decided to publish this work, nevertheless, as he believes the expression (2.5) has as much meaning to the problem of He^3 as the Ising model does to that of ferromagnetism.

vanish except for $\mathbf{z}$'s located on a simple cubic lattice,[5] yielding

$$\int (P\mathbf{z}_1, P\mathbf{z}_2, \cdots, P\mathbf{z}_N | e^{-\beta H} | \mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_N) d\mathbf{z}_1 \cdots d\mathbf{z}_N = N! Q_0 \exp[-a \textstyle\sum_i (\mathbf{x}_i - P\mathbf{x}_i)^2], \quad (2.6)$$

where $\mathbf{x}_i$ indicates a lattice point of a simple cubic lattice, a is a constant[6]:

$$a = m'/(2\beta\hbar^2), \quad (2.7)$$

and Q_0 is a factor introduced in Eq. (2.5):

$$Q_0 = K_\beta [m'/(2\pi\beta\hbar^2)]^{3N/2}. \quad (2.8)$$

This is considered a smoothly varying function of temperature,[7] and is independent of permutations pertinent to the present problem.

From Eqs. (2.4) and (2.6), one obtains for the partition function

$$Q_{FD} = Q_0 \textstyle\sum_P (-1)^P \exp[-a \sum_i (\mathbf{x}_i - P\mathbf{x}_i)^2]. \quad (2.9)$$

The technique used by Kac and Ward[8] in the treatment of the two-dimensional Ising model suggests that Eq. (2.9) can be written in a determinantal form as follows[9]:

$$q \equiv Q_{FD}/Q_0 = \det \mathbf{A}, \quad (2.10)$$

where an element of the matrix $\mathbf{A}$ is defined as

$$\mathbf{A}(\mathbf{x},\mathbf{x}') = \exp[-a(\mathbf{x}-\mathbf{x}')^2], \quad (2.11)$$

$\mathbf{x}$ and $\mathbf{x}'$ taking values of lattice points on a simple cubic lattice. When N is the number of lattice points in the volume or the system, $\mathbf{A}$ is an $N \times N$ matrix. Following Newell and Montroll,[8] one transforms the matrix $\mathbf{A}$ with a unitary matrix $\mathbf{U}$ defined by

$$\mathbf{U}(\mathbf{x},\mathbf{k}) = N^{-\frac{1}{2}} e^{2\pi i \mathbf{x}\cdot\mathbf{k}}, \quad (2.12)$$

obtaining another matrix $\mathbf{B}$ such that

$$\mathbf{B} = \mathbf{U}\mathbf{A}\mathbf{U}^{-1}. \quad (2.13)$$

Then

$$\det \mathbf{A} = \det \mathbf{B}. \quad (2.14)$$

$\mathbf{x}$ and $\mathbf{k}$ in Eq. (2.12), written in components, have the following forms:

$$\mathbf{x} = (k,l,m) \times d, \quad \mathbf{k} = (\kappa,\lambda,\mu) \times (Ld)^{-1}, \quad (2.15)$$

where d is the lattice constant, and $L = N^{\frac{1}{3}}$. k, l, m, κ, λ, and μ in Eq. (2.15) are integers. Because of the lattice structure and the periodic boundary condition with a period of Ld, each of these integers takes one of the values between $-L/2$ and $L/2$.

Combining Eqs. (2.11) and (2.12) and taking account of the periodicity of the lattice, one has

$$\mathbf{B}(\mathbf{k},\mathbf{k}') = \delta(\mathbf{k},\mathbf{k}') \textstyle\sum_{\mathbf{x}} \mathbf{A}(0,\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}, \quad (2.16)$$

where the summation over $\mathbf{x}$ is to be carried over all the lattice points. Putting Eq. (2.11) into Eq. (2.16) and replacing $\mathbf{x}$ and $\mathbf{k}$ by Eq. (2.15), one obtains

$$\mathbf{B}(\mathbf{k},\mathbf{k}) = \sum_{k,l,m=-L/2}^{L/2} \exp[-ad^2(k^2+l^2+m^2) - 2\pi i(\kappa k + \lambda l + \mu m)/K] = \vartheta_3\left(\frac{\kappa}{K}; e^{-\tau}\right) \vartheta_3\left(\frac{\lambda}{K}; e^{-\tau}\right) \vartheta_3\left(\frac{\mu}{K}; e^{-\tau}\right), \quad (2.17)$$

where ϑ_3 is one of the theta functions and is defined as[10]

$$\vartheta_3(x;t) \equiv \sum_{n=-\infty}^{\infty} t^{n^2} e^{2\pi i n x}, \quad (2.18)$$

and

$$\tau = ad^2 = m'd^2kT/(2\hbar^2). \quad (2.19)$$

In arriving at the ϑ_3-function in Eq. (2.17), L is assumed very large, effectively infinite.

One introduces F by

$$q = e^{-F/kT}, \quad (2.20)$$

uses Eqs. (2.10), (2.14), and (2.17), transforms the summation into an integral, and obtains

$$F/kT = -3N \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \ln \vartheta_3(x; e^{-\tau}). \quad (2.21)^{11,12}$$

This is the part of the free energy which is pertinent to the Fermi-Dirac statistics. As τ is proportional to temperature and $\vartheta_3(x; e^{-\tau})$ is a well-behaved function of its arguments, one can conclude from Eq. (2.21) that this substance does not show any phase change.

It should be noticed that after Eq. (2.9) no approximation is made except the reasonable one that $L \gg 1$. Particularly, it seems worth while to mention here clearly that, although in the previous treatment[2] of He^4 only polygons having sides of length d as shown in Fig. 1(a) was taken into account, in the present paper all of the sides of length longer than d such as shown in Fig. 1(b) are also taken account of.

As the form of Eq. (2.12) suggests, what we have

[5] For the sake of simplicity, the explanation and calculation are based on a simple cubic lattice in the first part of the paper. The face-centered cubic lattice is treated in the later part.

[6] This a is different from a used in reference 2 by a factor of d^2/T.

[7] See p. 1296 of F I.

[8] M. Kac and J. C. Ward, Phys. Rev. **88**, 1332 (1952); G. F. Newell and E. W. Montroll, Revs. Modern Phys. **25**, 353 (1953).

[9] For Bose particles, the partition function q is written as a permanent of $\mathbf{A}$ instead of a determinant as in Eq. (2.10). That is why the technique shown below cannot be applied to He^4.

[10] See, for instance, E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge and the McMillan Company, New York, 1946), Chap. 21.

[11] D. Lieberman of California Institute of Technology has derived essentially the same formula independently.

[12] Using relations satisfied by the theta functions, Eq. (2.21) can be reduced to a simpler form:

$$F/NkT = \ln 2 - (\tau/4) - \ln[\vartheta_2(0; e^{-\tau}) \vartheta_3(0; e^{-\tau}) \vartheta_4(0; e^{-\tau})],$$

where ϑ_2, ϑ_3, and ϑ_4 follow the notations of Whittaker and Watson, reference 10.

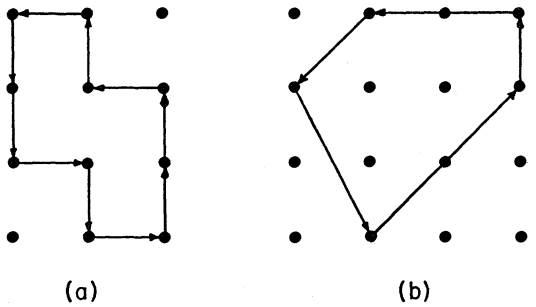


FIG. 1. Examples of polygons on a lattice. Each arrow indicates a side of the polygon.

done in this section was to evaluate the partition function using momentum space representation, although we started from the coordinate representation introduced by Feynman. This situation becomes clearer, when one recalls the alternative way of writing the ϑ_3 function:

$$\vartheta_3(x;e^{-\tau})=\left(\frac{\pi}{\tau}\right)^{\frac{1}{2}}\sum_{n=-\infty}^{\infty}\exp[-\pi^2(x+n)^2/\tau]. \quad (2.22)$$

Inserting this into Eq. (2.17), one obtains

$$\mathbf{B}(\mathbf{k},\mathbf{k})=\left(\frac{h^2}{2\pi m'd^2kT}\right)^{\frac{3}{2}}\times\sum_{\mathbf{n}}\exp[-h^2(\mathbf{k}+\mathbf{n}/d)^2/2m'kT], \quad (2.23)$$

where $\mathbf{n}$ is a vector having integers (plus, minus and zero) as its components. Comparing Eq. (2.23) with the momentum space approach of the ideal Fermi particles, one may tell the similarity and also the difference between the two.

At any rate, Eq. (2.21) does not have much physical meaning, because it is for the case of all parallel spins. We are more interested in the effect of plus and minus spins superposed on the Fermi-Dirac statistics and this will be treated in the following sections.

3. FERMI-DIRAC PARTICLES WITH A SPIN 1/2

In order to treat He^3, one has to take into account the fact that an atom possesses nuclear spin of $\frac{1}{2}$. Then the partition function of the previous section has to be revised so as to include spin functions. Extending Eq. (2.4), one sees that the partition function Q is written as

$$Q=N^{-1}\sum_{\text{distr}}\sum_{P_\alpha}\sum_{P_\beta}(-1)^{P_\alpha+P_\beta}\int(P\mathbf{z}_1,P\mathbf{z}_2\cdots P\mathbf{z}_N\times|e^{-\beta H}|\mathbf{z}_1,\cdots,\mathbf{z}_N)d\mathbf{z}_1\cdots d\mathbf{z}_N, \quad (3.1)$$

where P_α (or P_β) is a permutation among atoms having α (or β) spins and $\sum_{\text{distr}}$ is the summation over different distributions of α and β spins over the lattice. P stands for $P_\alpha P_\beta$. Using Eq. (2.5) and assuming as before that the function ρ has nonvanishing values only when its arguments are on a simple cubic lattice, Q is reduced to

$$Q=Q_0\sum_{\text{distr}}\sum_{P_\alpha}\sum_{P_\beta}(-1)^{P_\alpha+P_\beta}\times\exp[-a\sum_j(\mathbf{x}_j-P_\alpha P_\beta\mathbf{x}_j)^2], \quad (3.2)$$

where Q_0 is the same as Eq. (2.8).

Applying the same technique as in the previous section, Eq. (3.2) can be transformed into a determinantal form as

$$q=Q/Q_0=\sum_{\text{distr}}\det\mathbf{A}, \quad (3.3)$$

where an element of the matrix $\mathbf{A}$ is

$$\mathbf{A}(\mathbf{x},\mathbf{x}')=f(\mathbf{x},\mathbf{x}')\exp[-a(\mathbf{x}-\mathbf{x}')^2], \quad (3.4)$$

with $f(\mathbf{x},\mathbf{x}')$ defined by

$$f(\mathbf{x},\mathbf{x}')=\begin{cases}1, & \text{when atoms on } \mathbf{x} \text{ and } \mathbf{x}' \text{ have the same spin,}\\ 0, & \text{when atoms on } \mathbf{x} \text{ and } \mathbf{x}' \text{ have different spins.}\end{cases} \quad (3.5)$$

As the next step, let us work on one of the terms in the summation of Eq. (3.3). Although it is not indicated explicitly, it should be remembered that a term, $\det\mathbf{A}$, corresponds to a certain distribution of spins over the lattice. To evaluate $\det\mathbf{A}$, one transforms the matrix $\mathbf{A}$ by the unitary matrix $\mathbf{U}$ defined in Eq. (2.12), obtaining another matrix $\mathbf{B}$ of Eq. (2.13), whose element is

$$\mathbf{B}(\mathbf{k}',\mathbf{k})=N^{-1}\sum_{\mathbf{x}}\sum_{\mathbf{x}'}e^{2\pi i\mathbf{x}'\cdot\mathbf{k}'}\mathbf{A}(\mathbf{x}',\mathbf{x})e^{-2\pi i\mathbf{x}\cdot\mathbf{k}}. \quad (3.6)$$

Here $\mathbf{x}$ and $\mathbf{x}'$ run through all the lattice points. Now divide the lattice points into groups so that the lattice points in one group have the same distribution of spins in its neighborhood (in its nearest neighbor points or in its nearest and next nearest points, etc., depending on the approximations one chooses). Then

$$\begin{aligned}\mathbf{B}(\mathbf{k}',\mathbf{k})&=N^{-1}\sum_C\sum_{\mathbf{x}'\epsilon C}\sum_{\mathbf{x}}e^{2\pi i\mathbf{x}'\cdot(\mathbf{k}'-\mathbf{k})}\times\mathbf{A}_C(0,\mathbf{x}-\mathbf{x}')e^{-2\pi i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{k}}\\&=N^{-1}\sum_C[\sum_{\mathbf{x}'\epsilon C}e^{2\pi i\mathbf{x}'\cdot(\mathbf{k}'-\mathbf{k})}]\times\sum_{\mathbf{x}''}\mathbf{A}_C(0,\mathbf{x}'')e^{-2\pi i\mathbf{x}''\cdot\mathbf{k}}, \quad (3.7)\end{aligned}$$

where $\mathbf{x}'\epsilon C$ indicates $\mathbf{x}'$ is a member of a group C and $\sum_C$ the summation over different groups. $\mathbf{A}_C(0,\mathbf{x}-\mathbf{x}')$ denotes $\mathbf{A}(0,\mathbf{x}-\mathbf{x}')$ when the origin 0 is a member of the group C. Now we make use of a relation

$$\sum_{\mathbf{x}\epsilon C}e^{2\pi i\mathbf{x}\cdot\mathbf{k}}=Np(C)\delta(\mathbf{k},0), \quad (3.8)$$

where $p(C)$ is the probability of finding a lattice point belonging to the group C. When the points belonging to the group C are scattered at random all over the

lattice, this relation (3.8) is justified as is shown in Appendix. Equations (3.7), (3.8), and (3.4) give

$$\mathbf{B}(\mathbf{k}',\mathbf{k})=\delta(\mathbf{k}',\mathbf{k})\sum_C p(C) \times\sum_{\mathbf{x}} \exp(-a\mathbf{x}^2-2\pi i\mathbf{x}\cdot\mathbf{k})f_C(0,\mathbf{x}) =\delta(\mathbf{k}',\mathbf{k})\sum_{\mathbf{x}} \exp(-a\mathbf{x}^2-2\pi i\mathbf{x}\cdot\mathbf{k})p(\mathbf{x}), \quad (3.9)$$

where $f_C(0,\mathbf{x})$ means $f(0,\mathbf{x})$ when the origin 0 belongs to a group C, and

$$p(\mathbf{x})=\sum_C p(C)f_C(0,\mathbf{x}) \quad (3.10)$$

is equal to the probability of finding α atoms both at the origin and at the point $\mathbf{x}$, plus the probability of finding β atoms both at the origin and at $\mathbf{x}$.

As is seen in the following sections, a distribution of plus and minus spins over the lattice is specified by a set of parameters. Then, Eq. (3.3) can be written as

$$q=\sum_{\text{distr}} \det\mathbf{B}=\sum_D G[D]\prod_{\mathbf{k}} \mathbf{B}(\mathbf{k},\mathbf{k}), \quad (3.11)$$

where D indicates a set of parameters specifying a distribution and $G[D]$ is a number of different distributions having the same set of values D. In Eq. (3.11), $\mathbf{B}(\mathbf{k},\mathbf{k})$ is a function of D through $p(\mathbf{x})$. If one applies an external field H, the energy of the system increases

TABLE I. Definition of parameters, x_i's, for a lattice point.

Configuration	Probability
α	x_1
β	x_2

by $M[D,H]$ which is a function of D and the field H. When one is interested only in the nuclear magnetic susceptibility, the magnetic energy $M[D,H]$ is due to the nuclear spin magnetic moment, which is assumed to commute with the rest of the Hamiltonian which depends on the coordinates of particles. Therefore the effect of the external field H on q of Eq. (3.11) is multiplication by a factor $\exp(-M[D,H]/kT)$. One evaluates q of Eq. (3.11) by its maximum term, or one maximizes

$$-F/kT=-M[D,H]/kT+\ln G[D] +\sum_{\mathbf{k}} \ln\mathbf{B}(\mathbf{k},\mathbf{k}), \quad (3.12)$$

with respect to the set of parameters D. F in Eq. (3.12) is the part of the free energy we are concerned with.

For the next procedure, it is convenient to rewrite $\mathbf{B}(\mathbf{k},\mathbf{k})$ of Eq. (3.9) using Eq. (2.17) as follows:

$$\mathbf{B}(\mathbf{k},\mathbf{k})=\sum \mathbf{x}\exp(-a\mathbf{x}^2-2\pi i\mathbf{x}\cdot\mathbf{k})[p(\mathbf{x})-p_\infty] +p_\infty\vartheta_3\left(\frac{\kappa}{K};e^{-\tau}\right)\vartheta_3\left(\frac{\lambda}{K};e^{-\tau}\right)\vartheta_3\left(\frac{\mu}{K};e^{-\tau}\right). \quad (3.13)$$

where

$$p_\infty=\lim_{|\mathbf{x}|\to\infty} p(\mathbf{x}). \quad (3.14)$$

τ in Eq. (3.13) is defined in Eq. (2.19) and is proportional to temperature.

TABLE II. Definition of parameters, y_i's, for a bond. ν_i is the number of configurations having the same probability y_i.

Configuration	Probability	ν_i
$\alpha-\alpha$	y_1	1
$\alpha-\beta$	y_2	2
$\beta-\beta$	y_3	1

4. SIMPLE CUBIC LATTICE TREATMENT

In order to proceed further, one chooses a set of parameters to specify distributions. The simplest approximate treatment is explained in this section. The procedure follows closely that derived by the author in the treatment of the Ising model,[13] and readers are referred to the original paper if necessary.

In the first approximation, to be called the "pair approximation," the parameters one uses are probabilities of finding configurations on lattice points and those on bonds as shown in Tables I and II. From geometrical consideration, one derives the relations among the parameters as shown in Table III.

For the weight factor $G[D]$ of Eq. (3.12), one uses the last equation of Sec. B of TCP I, or

$$G=X_N{}^5Y_N{}^{-3}N!^{-2}, \quad (4.1)^{14}$$

where

$$X_N=\prod_{i=1}^{2}(x_iN)!, \quad Y_N=\prod_{i=1}^{3}(y_iN)!^{\nu_i}. \quad (4.2)$$

Hence, using Stirling's formula,

$$\ln G=N\left[5\sum_{i=1}^{2} x_i\ln x_i-3\sum_{i=1}^{3}\nu_i y_i\ln y_i\right]. \quad (4.3)$$

Using the definition of $p(\mathbf{x})$ and Table II, one derives

$$\begin{aligned} &p(\text{origin})=1,\\ &p(\text{nearest neighbor})=y_1+y_3,\\ &p(\text{further neighbor})\doteqdot p_\infty=x_1{}^2+x_2{}^2. \end{aligned} \quad (4.4)$$

The last of these relations is approximate, but seems consistent with the approximation one makes for G in

TABLE III. Relations among parameters for the simple cubic lattice treatment. The meaning of the table is, for instance, $y_1=\frac{1}{2}+\xi_1-y_2$. ξ_1 and y_2 are independent variables.

	1	ξ_1	y_2
x_1	$\frac{1}{2}$	1	
x_2	$\frac{1}{2}$	-1	
y_1	$\frac{1}{2}$	1	-1
y_3	$\frac{1}{2}$	-1	-1

[13] R. Kikuchi, Phys. Rev. **81**, 988 (1951), hereafter called TCP I.

[14] This formula is also discussed in Kurata, Kikuchi, and Watari, J. Chem. Phys. **21**, 434 (1953). Equation (4.1) gives Bethe's approximation when applied to the Ising model.

Eq. (4.1). The magnetic energy of the system is equal to

$$M[D,H]=\mu HN(x_2-x_1)=-2\mu HN\xi_1, \quad (4.5)$$

where μ is the nuclear magnetic moment.

Inserting Eqs. (4.3), (4.4), and (4.5) into Eqs. (3.12) and (3.13), and transforming the summation into an integral, one obtains

$$-\frac{F}{NkT}=\frac{2\xi_1\mu H}{kT}+\frac{\ln G}{N}+\int d\mathbf{r}\ln\Omega, \quad (4.6)$$

where

$$\Omega=(\tfrac{1}{2}+2\xi_1^2)\vartheta_3(x;e^{-\tau})\vartheta_3(y;e^{-\tau})\vartheta_3(z;e^{-\tau})+\tfrac{1}{2}$$
$$-2\xi^2+(\tfrac{1}{2}-2y_2-2\xi_1^2)2e^{-\tau}\sum\cos2\pi x, \quad (4.7)^{15}$$

with

$$\sum\cos2\pi x\equiv\cos2\pi x+\cos2\pi y+\cos2\pi z, \quad (4.8)$$

and

$$\int d\mathbf{r}\equiv\int_{-\frac{1}{2}}^{\frac{1}{2}}dx\int_{-\frac{1}{2}}^{\frac{1}{2}}dy\int_{-\frac{1}{2}}^{\frac{1}{2}}dz. \quad (4.9)$$

Differentiation of Eq. (4.6), with respect to ξ_1, gives

$$5\ln\frac{x_1}{x_2}-3\ln\frac{y_1}{y_3}+2\frac{\mu H}{kT}+\int d\mathbf{r}4\xi_1[\vartheta_3(x)\vartheta_3(y)\vartheta_3(z)$$
$$-1-2e^{-\tau}\sum\cos2\pi x]/\Omega=0. \quad (4.10)^{16}$$

When H is zero, $\xi_1=0$ is a solution of this equation.[15] Another possibility, $\xi_1\neq0$, corresponds to a ferromagnetic state, to which we shall refer later, but first we solve the problem of the nonferromagnetic case. As we are interested in the magnetic susceptibility which is the property defined for vanishing magnetic field, the second equation $\partial F/\partial y_2=0$ is to be solved for y_2 under the condition $\xi_1=0$. Then one obtains

$$3\ln\frac{1-2y_2}{2y_2}=\int\frac{d\mathbf{r}}{\Omega_0}2e^{-\tau}\sum\cos2\pi x, \quad (4.11)$$

where Ω_0 is Ω of Eq. (4.7) with $\xi_1=0$. The magnetic susceptibility χ is defined from Eq. (4.10) as

$$\frac{1}{\chi}=\lim_{H\to0}\frac{H}{2\mu\xi_1}=\frac{kT}{\mu^2}\left[\frac{3}{1-2y_2}-5-\int d\mathbf{r}\{\vartheta_3(x)\vartheta_3(y)\vartheta_3(z)\right.$$
$$\left.-1-2e^{-\tau}\sum\cos2\pi x\}/\Omega_0\right]. \quad (4.12)^{16}$$

As it seems hopeless to solve Eq. (4.11) analytically, it was solved numerically,[17] to obtain y_2 and hence χ of Eq. (4.12) as functions of τ. Then in order to compare the results with observations, τ was converted to actual temperature. Assuming that the density[18] of He^3 is 0.08 g/cc and is independent of temperature, Eq. (2.19) is written as

$$T=2.0(m/m')\tau, \quad (4.13)$$

where m is the true mass of a He^3 atom, and m' is its effective mass introduced by Feynman in F I. As T is proportional to τ, if one first plots χ against τ, it is easily converted to a function of T, only by changing the temperature scale, the scale factor depending on m'/m. In Fig. 2, $\chi T/C$ and $y_2-0.25$ so obtained assuming $m'/m=5.5$ are plotted against T with solid curves. This value of the ratio is taken in order to give the best fit of the observations by Fairbank, Ard, and Walters[19] which are also shown in Fig. 2 with black circles. Fairly good agreement with the experiments is also obtained for the values of m'/m in the range

$$5.0<(m'/m)_{sc}<5.5. \quad (4.14)$$

Comparing the treatment of the Ising model in TCP I and the present calculation, one knows that the ferromagnetic Curie temperature, if it exists, is to be given by the condition: $\chi\to\infty$. From the curve in Fig. 2, it is clear that this condition is not satisfied, and hence there is no ferromagnetic state, within the temperature range of the present calculation.[20]

The value of m'/m in Eq. (4.14) seems to be much too large, when one recalls that the value of the corresponding ratio for He^4 was estimated by Feynman[1] as

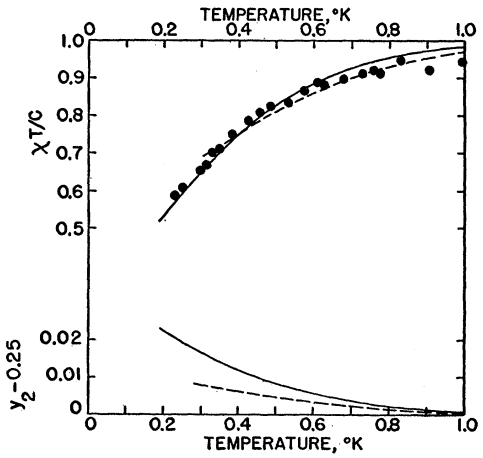


FIG. 2. Plots of nuclear magnetic susceptibility χ and the parameter y_2 defined in Table II against temperature. C in $\chi T/C$ is the normalization constant such that $\lim_{T\to\infty}\chi T/C=1$. The black circles are experiments by Fairbank, Ard, and Walters.[19] The solid curves are the calculated ones based on the simple cubic lattice treatment ("pair" and "square" approximations) and $m'/m=5.5$. The broken curves are the calculated ones based on the face-centered cubic lattice treatment ("tetrahedron" approximation) and $m'/m=3.5$.

[15] ξ_1, introduced in Table III, is a long-range order parameter. x, y, and z without subscripts are cartesian coordinates and are to be distinguished from the parameters, x_i's and y_i's, defined in Tables I and II.

[16] $\vartheta_3(x;e^{-\tau})$ is simply written as $\vartheta_3(x)$, dropping the second argument $e^{-\tau}$, as it would not cause any confusion.

[17] For the numerical integration, Weddle's rule was used. This is to approximate $\int_{-3}^{3}f(x)dx$ by $(3/10)[f(-3)+5f(-2)+f(-1)+6f(0)+f(1)+5f(2)+f(3)]$. See, for instance, D. R. Hartree, *Numerical Analysis* (Oxford University Press, Oxford, 1952), p. 101.

[18] E. C. Kerr, Phys. Rev. **96**, 551 (1954).

[19] Fairbank, Ard, and Walters, Phys. Rev. **95**, 566 (1954).

[20] L. Goldstein and M. Goldstein, J. Chem. Phys. **18**, 538 (1950).

about 1.5 and was 1.3 by the author's calculation.[2] It was suspected that this ratio might change when one improves the approximation and uses a square as the basic figure, just as was done for the Ising model in Sec. D of TCP I. But the actual calculation based on the "square" approximation gave practically the same curve[21] for $\chi T/C$ as a function of τ. This shows on one hand that the result shown in Fig. 2 is practically the rigorous solution of the model we used in this section, and on the other that the value of m'/m in (4.14) is inherent in the model itself and does not depend on the approximation used in this section.

Therefore, as the next step to examine m'/m, the face-centered cubic lattice is taken with a hope that it might give a lower value of this ratio. It will be explained in the next section.

5. FACE-CENTERED CUBIC LATTICE TREATMENT

The calculation based on the face-centered cubic lattice is essentially the same as that for the simple cubic lattice. But in order to make clear the approximation used in the process of the calculation, some of the equations will be listed below.

Equations (2.1) to (2.14) remain valid for this case, but as the reciprocal of a face-centered cubic lattice is a body-centered cubic lattice, Eq. (2.15) should be modified accordingly. Equation (2.17) is replaced by

$$\mathbf{B}(\mathbf{k},\mathbf{k})=\Theta\left(\frac{\kappa}{K},\frac{\lambda}{K},\frac{\mu}{K};e^{-2\tau}\right), \tag{5.1}$$

where

$$\begin{aligned}\Theta(x,y,z;t)\equiv&\vartheta_3(2x;t)\vartheta_3(2y;t)\vartheta_3(2z;t)\\&+\vartheta_2(2x;t)\vartheta_2(2y;t)\vartheta_3(2z;t)\\&+\vartheta_2(2x;t)\vartheta_3(2y;t)\vartheta_2(2z;t)\\&+\vartheta_3(2x;t)\vartheta_2(2y;t)\vartheta_2(2z;t),\end{aligned} \tag{5.2}$$

and $\vartheta_2(x;t)$ is another theta function defined by

$$\vartheta_2(x;t)\equiv\sum_{n=-\infty}^{\infty}t^{(n+\frac{1}{2})^2}e^{\pi i(2n+1)x}. \tag{5.3}$$

Consequently, Eq. (3.13) is changed into

$$\begin{aligned}\mathbf{B}(\mathbf{k},\mathbf{k})=&\sum_{\mathbf{x}}\exp(-a\mathbf{x}^2-2\pi i\mathbf{x}\cdot\mathbf{k})[p(\mathbf{x})-p_\infty]\\&+p_\infty\Theta\left(\frac{\kappa}{K},\frac{\lambda}{K},\frac{\mu}{K};e^{-2\tau}\right).\end{aligned} \tag{5.4}$$

As it is known that the "pair" approximation for the face-centered cubic lattice is poor[22] because the correlation among the first shell points is neglected, the tetrahedron approximation[23] will be taken up in this section. Then in order to proceed further, one uses besides Tables I and II probability parameters defined in Table IV. These parameters, z_i's, are expressed as functions of ξ_1 and y_2 of Table III and additional two variables, ξ_2 and z_3, as shown in Table V.

For the free energy F, the expression (4.6) is used with different definitions for G and Ω:

$$\frac{\ln G}{N}=6\sum_{i=1}^{3}\nu_i y_i\ln y_i-2\sum_{i=1}^{5}\gamma_i z_i\ln z_i-5\sum_{i=1}^{2}x_i\ln x_i, \tag{5.5}$$

and

$$\begin{aligned}\Omega=&(\tfrac{1}{2}+2\xi_1^2)\Theta(x,y,z;e^{-2\tau})+\tfrac{1}{2}-2\xi_1^2\\&+(\tfrac{1}{2}-2y_2-2\xi_1^2)2e^{-\tau}\sum\cos2\pi(x+y),\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}\sum\cos2\pi(x+y)\equiv&\cos2\pi(x+y)+\cos2\pi(y+z)\\&+\cos2\pi(z+x).\end{aligned} \tag{5.7}$$

TABLE IV. Definition of parameters, z_i's, for a tetrahedron. γ_i is the number of configurations having the same probability z_i. This table is the same as Table VII(a) of TCP I with different notations.

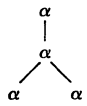
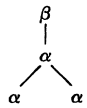
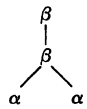
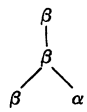
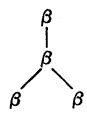
Configuration	Probability	γ_i
	z_1	1
	z_2	4
	z_3	6
	z_4	4
	z_5	1

TABLE V. Relations among parameters. The meaning of the table is the same as Table III.

	1	ξ_1	y_2	ξ_2	z_3
z_1	$\frac{1}{2}$	1	-2	-1	1
z_2			$\frac{1}{2}$	$\frac{1}{2}$	-1
z_4			$\frac{1}{2}$	$-\frac{1}{2}$	-1
z_5	$\frac{1}{2}$	-1	-2	1	1

[21] The deviation was only 0.3% even at the lowest temperature calculated, $\tau=0.5$.

[22] See, for instance, Fig. 9 of TCP I for the two-dimensional triangular net.

[23] This corresponds to Sec. H of TCP I.

Equation (5.5) follows from Eq. (H1.2) of TCP I. The next procedure is to minimize F with respect to independent variables to obtain

$$3\ln\frac{y_2}{y_1}-2\ln\frac{z_2}{z_1}=4e^{-\tau}\int\frac{d\mathbf{r}}{\Omega_0}\sum\cos 2\pi(x+y), \quad z_1z_3^3=z_2^4, \tag{5.8}$$

corresponding to Eq. (4.11). In place of Eq. (4.12), one has

$$\frac{1}{\chi}=\frac{kT}{\mu^2}\left[5+\frac{14y_2-4}{(1-2y_2)(1-3y_2)}-4\int d\mathbf{r}\{\Theta(x,y,z;e^{-2\tau})-1-2e^{-\tau}\sum\cos 2\pi(x+y)\}/\Omega_0\right]. \tag{5.9}$$

The two equations (5.8) are solved for y_2 and z_3 numerically and these values are inserted in (5.9) to evaluate χ. The results of the calculation are shown in Fig. 2 with dotted curves. In the process of arriving at these curves, the temperature scale was changed from τ to T by the relation

$$T=1.6(m/m')\tau, \tag{5.10}$$

and the value $m'/m=3.5$ was used. The range of the value of this ratio which gives fairly good fit with the experiments is

$$3.2<(m'/m)_{\text{fcc}}<3.8. \tag{5.11}$$

In order to make sure of these numbers, a calculation taking into account the second and the third neighbors was carried out. It was found that this improved approximation gave essentially the same values as in (5.11), and one can conclude, for this lattice also, that the values in (5.11) are very close to the rigorous results of the model.

6. DISCUSSION

Equations (4.14) and (5.11) show that $(m'/m)_{\text{fcc}}$ is smaller than $(m'/m)_{\text{sc}}$. This can be interpreted as follows. As one sees from the lower curves of Fig. 2, y_2 is greater than $\frac{1}{4}$. This means that $\alpha-\beta$ (and $\beta-\alpha$) pairs appear more often than $\alpha-\alpha$ and $\beta-\beta$ pairs; in other words the system shows an "antiferromagnetic" tendency. Wannier[25] proved that the Ising model of a two-dimensional triangular net does not exhibit antiferromagnetic state whereas the square net[26] does. This is interpreted as due to the fact that triangular net is divided into three equivalent sublattices whereas the square net divides into two. The same argument holds for the three-dimensional case and one expects that the antiferromagnetic state is difficult to be realized for the face-centered cubic lattice which is divided into four sublattices whereas it is easier for the simple cubic lattice which has two sublattices. This is the reason why one obtains smaller values of y_2 and hence smaller deviation of $\chi T/C$ from unity for the face-centered cubic lattice than the simple cubic, corresponding to the same value of τ. The larger the $\chi T/C$ is, the lower the m'/m becomes.

Another factor to lower $(m'/m)_{\text{fcc}}$ compared to $(m'/m)_{\text{sc}}$ is the change of the numerical factors, 2.0 and 1.6, of Eqs. (4.13) and (5.10), respectively, which are inversely proportional to square of the hypothetical lattice constant, as is seen in Eq. (2.19). When one assumes no holes, the lattice constant for the face-centered cubic lattice is larger than that for the simple cubic in order to give the same macroscopic density. The smaller this numerical factor is, the smaller the m'/m becomes.

Even though one knows the values of (4.14) and (5.11) for the two regular lattices, for the present there is no practical way of calculating the value of m'/m for a random distribution of atoms, though probably it is not too unreasonable to guess that the true value would lie in between the two.

The final problem which still remains is why the value of m'/m should be so large (3.2~5.5) in order to fit the observation. Probably the answer lies in the passage from Eq. (2.4) to Eq. (2.5) as Feynman pointed out,[4] but it will not be discussed further in this paper.

It is certainly of much importance to calculate the entropy of the system in order to discuss the nature of the approximation and also to check Pomeranchuk's prediction,[27] but the entropy which we can calculate using the method of this paper is only related to q of Eq. (3.3) and is not the total entropy of the system, as the factor Q_0 of that equation is unknown. Since the interpretation of the entropy derived from q is not settled yet, it will not be reported here.

As a conclusion we can say that the partition function which Feynman proposed can explain at least qualitatively the properties of liquid He^4 and also of liquid He^3 without contradiction.

ACKNOWLEDGMENTS

Thanks are due Professor R. P. Feynman, Professor J. E. Mayer, and Professor M. H. Cohen for enlightening discussions.

[24] These are the equations obtained by minimizing the free energy F with respect to y_2 and z_3 and then putting ξ_1 and ξ_2 equal to zero. The other two equations, which correspond to Eq. (4.10) and are derived by minimizing F with respect to ξ_1 and ξ_2 were not listed here, though Eq. (5.9) is derived from these two equations. As was mentioned in connection to Eq. (4.10) we know that $\xi_1=0$ and $\xi_2=0$ are possible solutions of these unlisted two equations for the vanishing value of H, and we are interested only in these values of ξ's. ξ_1 and ξ_2 are both long-range order parameters and their nonvanishing values correspond to ferromagnetic state which does not occur in our problem.

[25] G. H. Wannier, Phys. Rev. **79**, 357 (1950).

[26] L. Onsager, Phys. Rev. **65**, 117 (1944).

[27] I. Pomeranchuk, J. Exptl. Theoret. Phys. (U.S.S.R.) **20**, 1919 (1950); Osborne, Abraham, and Weinstock, Phys. Rev. **94**, 202 (1954); L. Goldstein, Phys. Rev. **96**, 1455 (1954).

APPENDIX. DISCUSSION OF EQ. (3.8)

For simplicity let us look at the part of Eq. (3.8) related to the x-component. Then the problem is to evaluate the value of the function

$$g(\kappa)=\sum_{k\epsilon C} e^{2\pi ik\kappa/K}, \qquad (A.1)$$

where k and κ are the same as in Eq. (2.15). In the complex number plane, $g(\kappa)$ is the vector connecting the origin to the end point of the succession of vectors of the summand. When one assumes that points belonging to the group C are scattered at random, the problem is looked upon as random walk in the complex number plane, and the probability $\bar{W}(\mathbf{r})d\mathbf{r}$ of finding the vector $g(\kappa)$ lying within the interval $(\mathbf{r}, \mathbf{r}+d\mathbf{r})$ from the origin is given by

$$\bar{W}(\mathbf{r})d\mathbf{r}=(\pi n)^{-1}\exp(-|\mathbf{r}|^2/n)d\mathbf{r}. \qquad (A.2)^{28}$$

In this equation n is the total number of arrows considered, i.e., $n=N^{\frac{1}{2}}p(C)$. As $p(C)$ is independent of N, when N becomes larger $\bar{W}(\mathbf{r})$ approaches to the delta function, $\delta(\mathbf{r})$, closer and closer. Therefore for very large N, Eq. (3.8) is justified.

[28] See, for instance, S. Chandrasekhar, Revs. Modern Phys. **15**, 1 (1943).

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Photoelectric Mixing of Incoherent Light*

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Beats have been obtained between incoherent light sources by mixing Zeeman components of a visible spectral line at a photosurface. Periodicity in emission was observed through the excitation of a 3-cm cavity. Because of incoherence between the spectral lines and incoherence between the beats from different photocathode areas, the signal-to-shot-noise ratio at the cavity is only 3×10^{-5} but the beats were modulated optically, while maintaining constant total intensity and our receiver was able to yield a signal-to-noise ratio of two at the indicator. The basic idea is that, in the photoelectric process, the emission probability for electrons is proportional to the square of the resultant electric field amplitude, implying an interference between light originating in independent sources. This is a point of view which does not appear to be tested in any other experiment involving quantum effects. The experiment also demonstrates that any time delay between photon absorption and electron release must be significantly less than 10^{-10} second.

I. INTRODUCTION

THE combination of two wave trains of slightly different frequencies is equivalent to a wave of the average frequency modulated by the difference frequency. This is evidenced in the phenomenon of acoustical beats and is responsible for the operation of superheterodyne radio receivers. The periodic variation in intensity which occurs at a fixed point on the image of a Michelson interferometer when one of the mirrors is moving may also be interpreted as beats between the light reflected from the stationary mirror and light which has had its frequency changed by reflection from the moving mirror.[1–3] However, the problem of beating incoherent light waves, i.e., light waves which originate in different sources, is quite different, and since the publication of the original suggestion,[4,5] it has been argued[6] that the observation of such beats is impossible. These arguments, when examined carefully, really provide reasons why beats between incoherent light sources are difficult, rather than impossible, to detect. Were optical lines much sharper than they are, or possible to produce, without great broadening, in much greater intensity than present techniques permit, beats between incoherent light waves would be easy to observe.

Following the publication of the original suggestion[4] for this experiment, Ruark,[7] calling attention to an

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§ Now at North American Aviation, Downey, California.

[1] A. Righi, J. Physique **2**, 437 (1883) describes an ingenious production of light beats. To demonstrate that light which has passed through a rotating Nichol prism may be resolved into two circularly polarized beams, one increased and one decreased in frequency with respect to the incident light, he performed an experiment, in its essence a double-slit interference experiment, in which one slit was illuminated by light of reduced frequency and the other slit by the increased frequency, both altered to be plane polarized in the same plane. The moving fringe pattern he observed, and interpreted as beats, is, in principle, no different from those produced by moving a mirror of a Michelson interferometer.

[2] E. Rüchardt, Optik **6**, 238 (1950), raises an objection to this point of view based on a requirement for coherence over a beat period. His objection is adequately answered in reference 3.

[3] C. V. Fragstein, Optik **8**, 289 (1951).

[4] Forrester, Parkins, and Gerjuoy, Phys. Rev. **72**, 728 (1947).

[5] Gerjuoy, Forrester, and Parkins, Phys. Rev. **73**, 922 (1948).

[6] L. R. Griffith, Phys. Rev. **73**, 922 (1948).

[7] A. Ruark, Phys. Rev. **73**, 181 (1948).