assumed by functions

$$\phi_1(w)-4i\pi^{\frac{1}{2}}w \tag{6a}$$

and

$$\phi_2(w)-8i\pi^{\frac{1}{2}}w, \tag{7a}$$

where ϕ_1 and ϕ_2 are regular and bounded in S. Because the correspondence $w\leftrightarrow 2w$ maps S onto itself, we may without loss of generality replace the 8 in (7a) by a 4, and (dividing by $4i\sqrt{\pi}$ and writing $\psi_k=\phi_k/4i\sqrt{\pi}$) we have only to prove that the value-sets of the functions

$$w-\psi_1(w), \tag{6b}$$

and

$$w-\psi_2(w) \tag{7b}$$

are not disjoint, where ψ_1 and ψ_2 are regular and bounded in S. But this is immediate: for, if $|\psi_k|\leq M$, $k=1, 2$ then by Rouche's theorem both functions (6b) and (7b) assume the value $M\sqrt{2}i$ inside the circle of center $M\sqrt{2}i$ and radius M.

Remark.—If K_i denotes the maximum of $|U(z)|$ on γ_i $(i=1, 2)$, and $\tau=\max(K_1,K_2)$, the above reasoning gives the more precise result that $F(U)$ has a branch point in the circle $|U|\leq\tau$, whence τ is an upper bound for the radius of convergence R of the virial series. It has not been deemed worth while to make a numerical estimate (which would be a straightforward task) because in any case the method is too crude to answer the interesting question of whether (1) converges at $x_1=\zeta(3/2)=2.612\cdots$, the value of the dimensionless density for which condensation is known to occur.

PHYSICAL REVIEW VOLUME 99, NUMBER 6 SEPTEMBER 15, 1955

Turbulence Spectrum in Chandrasekhar's Theory*

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(Received April 4, 1955)

The results of Chandrasekhar's recent theory of turbulence are transformed in this paper from ordinary space (used exclusively in his paper) to wave-number space. Consideration is limited to the case of stationary, homogeneous, and isotropic turbulence. A "time-dependent spectrum" is defined in terms of the scalar product of eddy velocities at two different times; this spectrum is related to Chandrasekhar's time-dependent correlation function by a Fourier transform, as in the conventional, time-independent case. For infinite Reynolds number the spectrum is obtained directly by transforming the correlation function into wave-number space; the spectrum is given by a much simpler expression than is the corresponding correlation function.

Consequently, an approximate (linearized) form of the *differential* equation for the correlation has been transformed to wave-number space, and this equation is readily solvable, even for a finite Reynolds number. For $k>1/\nu$ the spectrum vanishes, and this cutoff at large wave-numbers is interpreted as the disintegration of turbulence into laminar flow at dimensions sufficiently small for viscosity to dominate over the inertial transfer of energy.

Finally the general, nonlinear correlation equation has been transformed into an integral equation for the spectrum, but a general solution has not yet been obtained for either the spectrum or correlation.

I. INTRODUCTION

IN a recent paper Chandrasekhar[1] has presented a new theory of isotropic, homogeneous turbulence in a steady state. The basic innovations in the new theory are (1) the consideration of velocity correlations not only at two different points but also at *two different times* and (2) the hypothesis of a statistical relationship between the second-order and fourth-order correlation tensors.

Starting with the hydrodynamic equation of motion and the equation of continuity, Chandrasekhar derived his fundamental equation for $f(r,t)$, the longitudinal correlation function[2]:

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial t^2}-\nu^2 D_5{}^2\right)f=f\frac{\partial}{\partial r}D_5 f. \tag{1}$$

Here D_5, the Laplacian operator in five-dimensional space, may be written

$$D_5=\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}. \tag{2}$$

In these equations r $(=|\mathbf{r}|)$ is the separation of the two points measured in units of some arbitrary length l; t is the time interval between the measurements of velocity at the two points in the unit of $l/\langle(u_1{}^2)\rangle_{\text{Av}}{}^{\frac{1}{2}}$, where u_1 is the velocity component in the direction of $\mathbf{r}$ (say, along the x-axis); ν is the viscosity in units of $l\langle(u_1{}^2)\rangle_{\text{Av}}{}^{\frac{1}{2}}$. The longitudinal correlation function is defined in terms of velocity components by

$$f(r,t)=\frac{\langle u_1(r_0,t_0)u_1(r_0+r,\,t_0+t)\rangle_{\text{Av}}}{\langle u_1{}^2\rangle_{\text{Av}}}. \tag{3}$$

We define ϕ by

$$f=1-\phi; \tag{4}$$

then for small values of t and r (such that f does not

* The research reported in this paper was supported in part by the Geophysics Research Directorate of the Air Force Cambridge Research Center, Air Research and Development Command, under contracts with the University of Chicago.
[1] S. Chandrasekhar, Proc. Roy. Soc. (London) **A229**, 1 (1955). Hereafter this article will be referred to as "Paper I."
[2] Paper I, Eq. (46).

depart greatly from unity) Eq. (1) may be written

$$\partial^2\phi/\partial t^2=(\nu^2 D_5{}^2+D_5)\phi. \quad (5)$$

For the case of infinite Reynolds number (zero viscosity), Chandrasekhar has solved Eq. (5) and obtained[3]

$$f(r,t)=1-\left(\frac{r}{r_0}\right)^\alpha \psi\left(\frac{t}{r}\right), \quad (r\ll r_0;\, t\ll r_0) \quad (6)$$

in which[4]

$$\psi(x)=\frac{1}{2(\alpha+2)}\{|x-1|^{\alpha+2}(\alpha+2+x)+|x+1|^{\alpha+2}(\alpha+2-x)\}. \quad (7)$$

Here $x=t/r$, and in Eq. (6) r_0 is a constant of the order of the dimensions of the system in units of l.[5] In these equations α is a positive constant but is not otherwise determined. If, however, one chooses the Kolmogoroff spectrum as an initial condition, then $\alpha=\frac{2}{3}$.

In view of the interest often attached to turbulence spectra, an attempt is made in this paper to transform the correlation function of Eqs. (6) and (7) to wave-number space. In the next section the time-dependent spectrum is defined and then derived from Chandrasekhar's correlation function by means of a Fourier transform. In Sec. III, the differential equation (5) is transformed to Fourier space and is solved for the dependence of the spectrum on time when finite viscosity is considered. In Sec. IV, we transform the general, nonlinear equation (1) to wave-number space.

II. SPECTRUM FOR ZERO VISCOSITY

By analogy with the conventional definition of the turbulence spectrum, $F(k)$, we define for the time-dependent case,

$$F(k,t)=c_1k^2\{\mathbf{V}_k(t_0)\cdot\mathbf{V}_k(t_0+t)\}_{\text{Av}}, \quad (8)$$

where $\mathbf{V}_k(t)$ represents the velocity at time t for an eddy of wave-number k. Throughout this paper c_i represents proportionality constants and averages are taken over all space. Note that with the definition (8), $F(k,t)$ is not necessarily always positive, since in general it is a time correlation of eddy velocities and becomes an energy spectrum only for $t=0$. The spectrum is related to the correlation function, again analogously to the conventional case,[6] by

$$f(r,t)=3\int_0^\infty \frac{F(k,t)}{k^3r^3}(\sin kr-kr\cos kr)dk \Big/ \int_0^\infty F(k,0)dk. \quad (9)$$

[3] Paper I, Eq. (63).

[4] Paper I, Eq. (81).

[5] In Paper I, r_0 is taken as unity and it is specified instead that the unit of length is equal to the diameter of the largest eddy present; r_0 is retained here to emphasize the uncertainty in the exact value of this constant.

[6] W. Heisenberg, Z. Phvsik **124**, 628 (1948), Eq. (50).

There should be no confusion between the time-dependent spectrum for steady-state turbulence considered in this paper and the ordinary spectrum in decaying turbulence. The latter changes with *absolute* time, whereas only *relative* time is implied in the present theory. In general, for small values of t these spectra should be of comparable accuracy to the well-known Kolmogoroff spectrum (adopted herein for $t=0$) and should be applicable in such time-dependent problems as ionospheric scattering of electromagnetic radiation.[7]

In order to relate the spectrum and correlation function, we wish to express Eq. (9) in terms of a simple sine or cosine transform. Hence by successive differentiations we find

$$h(r,t)\equiv-\frac{1}{r}\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r^4\frac{df}{dr}\right)\right]=c_2\int_0^\infty kF(k,t)\sin krdk. \quad (10)$$

It is convenient to work in the variables

$$\eta=kt, \quad y=1/x=r/t. \quad (11)$$

Then, if we define

$$j(y)\equiv t^{1-\alpha}h(r,t), \quad (12)$$

where it is assumed that the correlation may be written in the form (6), the sine transform of Eq. (10) is, for $\alpha=\frac{2}{3}$,

$$F(k,t)=\frac{c_3\eta^{\frac{2}{3}}}{k^{5/3}}\int_0^\infty j(y)\sin\eta y dy. \quad (13)$$

By means of Eqs. (12), (10), and (6), j is related to $\psi(x=1/y)$, which is given by (7). A straightforward evaluation gives two limiting series:

$$j_0(y)=-\frac{110}{81}y\left[1+\frac{(2-\frac{2}{3})(3-\frac{2}{3})}{3!}y^2+\frac{(2-\frac{2}{3})(3-\frac{2}{3})(4-\frac{2}{3})(5-\frac{2}{3})}{5!}y^4+\cdots\right], \quad 0<y<1, \quad (14)$$

and

$$j_\infty(y)=+\frac{110}{27}y^{-\frac{1}{3}}\left[1+\frac{(1-\frac{2}{3})(2-\frac{2}{3})}{2!}y^{-2}+\frac{(1-\frac{2}{3})(2-\frac{2}{3})(3-\frac{2}{3})(4-\frac{2}{3})}{4!}y^{-4}+\cdots\right], \quad 1<y<\infty. \quad (15)$$

It will be noticed that neither of the above series converges as $y\rightarrow 1$, so that $j_0\rightarrow-\infty$ and $j_\infty\rightarrow+\infty$. This discontinuity arises in the highest derivative considered in Eq. (10), and its presence may be traced

[7] E.g., see R. A. Silverman and M. Balser, Phys. Rev. **96**, 560 (1954).

back to the fact that for infinite Reynolds number Eq. (5) is a wave equation (in a five-dimensional space) for unit velocity. Thus physically the discontinuity in $j(y)$ at $y=1$ (or $r=t$) reflects a singularity in the correlation function at $r=t=0$. This singularity apparently arises because we have incorrectly assumed the Kolmogoroff correlation $[1-(r/r_0)^{\frac{2}{3}}]$ to hold strictly for $t=0$. And, indeed, for $\alpha>1$ in Eq. (6) there is no discontinuity. However, the integral (13) is finite in any event and we might expect that the Kolmogoroff approximation near the origin should not have any profound effect on the spectrum, except perhaps at large wave numbers.

Substituting Eqs. (15) and (14) into (13), we obtain an expression for $F(k,t)$ involving several infinite series in powers of $\eta=kt$. If sufficient terms are carried, however, it may be shown that most of the coefficients cancel identically, with the remaining coefficients reducing to the simple expression,

$$F(k,t)=c_4k^{-5/3}\cos kt. \tag{16}$$

This equation may be checked relatively easily by taking the inverse transform as given by Eq. (9). Thus we find

$$\lim_{k_0\to 0}\frac{\dfrac{3}{r^3}\displaystyle\int_{k_0}^{\infty}k^{-14/3}\cos kt(\sin kr-kr\cos kr)dk}{\displaystyle\int_{k_0}^{\infty}k^{-5/3}dk}$$
$$=\lim_{k_0\to 0}\left\{1-\left[\frac{27}{55}\Gamma(\tfrac{1}{3})\left(\sin\frac{\pi}{6}\right)k_0^{\frac{2}{3}}\right]r^{\frac{2}{3}}\psi\left(\frac{t}{r}\right)\right\}, \tag{17}$$

where ψ is given by Eq. (7). Comparing (17) with (6), we see that the factor in square brackets in (17) may be identified with $r_0^{-\frac{2}{3}}$, where r_0 (and k_0) refer to the largest dimensions (and eddies) in the system.

III. APPROXIMATE SPECTRUM FOR FINITE VISCOSITY

The remarkable simplification achieved in the above equations for the time-dependent spectrum suggests that the basic differential equation might also be greatly simplified if transformed to wave-number space. Therefore, writing Eq. (5) [which is the linearized form of the general equation (1)] in terms of $F(k,t)$ by means of Eq. (9) (where $f=1-\phi$), we readily obtain

$$\partial^2F/\partial t^2=(\nu^2k^2-1)k^2F. \tag{18}$$

The solution to (18) may be written:

$$F(k,t)=A\cos[(1-\nu^2k^2)^{\frac{1}{2}}kt]+B\sin[(1-\nu^2k^2)^{\frac{1}{2}}kt],\quad k<1/\nu \tag{19}$$

and

$$F(k,t)=C\cosh[(\nu^2k^2-1)^{\frac{1}{2}}kt]+D\sinh[(\nu^2k^2-1)^{\frac{1}{2}}kt],\quad k>1/\nu. \tag{20}$$

In stationary turbulence there is, by definition, no "decay" in the usual sense; the time t in these equations represents the interval of time between two events. Hence the spectrum as we have defined it in Eq. (18) cannot depend on whether t is positive or negative. Therefore, in (19) and (20) we must set $B=D=0$. Moreover, we must have $C=0$, since the spectrum must stay bounded as $t\to\infty$. Hence we are left with[8]

$$\left.\begin{aligned}F(k,t)&\equiv F_\nu(k,0)\cos[(1-\nu^2k^2)^{\frac{1}{2}}kt], && k<1/\nu,\\ &=0, && k>1/\nu.\end{aligned}\right\} \tag{21}$$

This cutoff in the spectrum for $k>1/\nu$ apparently indicates that the turbulence has been completely overcome by viscosity and that for sufficiently small elements of the fluid only laminar flow exists. Heisenberg[6] has pointed out that there may be a turbulence cutoff at high wave numbers and that consequently his k^{-7} law might not hold for indefinitely small eddies. This idea has been elaborated upon by Batchelor and Townsend.[9]

It is not possible at present to specify the nature of $F_\nu(k,0)$, which appears as an integration constant in Eq. (21). Probably Heisenberg's[6] spectrum represents the closest approach to the energy distribution presently available; however, it can only be an approximation, for this initial energy spectrum must also have a cutoff at $k=1/\nu$. That is, from (21) we have

$$F(1/\nu,t)=F_\nu(1/\nu,0)=0. \tag{22}$$

The cosine factor in (21) seems to indicate an oscillation of the eddy velocities between the limits $\pm|\mathbf{V}_k(t=0)|$. However, since we are restricted to stationary turbulence, the *absolute values* of the velocities cannot be a function of time. That is, the fluctuation of $F(k,t)$ with time merely reflects a constant rotation of the eddy $\mathbf{k}$ in space, with a period of

$$T=2\pi/[(1-\nu^2k^2)^{\frac{1}{2}}k]. \tag{23}$$

For $k=0$ the rotation period is infinite and decreases to a minimum of $T_{\min}=4\pi\nu$ for eddies with $k=1/\sqrt{2}\nu$. For smaller eddies the viscosity hinders the rotation more and more until finally at $k=1/\nu$ there is no turbulent motion whatever. Thus the cosine term is interpreted as nothing more than a consequence of isotropy in three-dimensional turbulence.

With this picture the eddies are perfectly stable and retain their identity indefinitely. Apparently, quantitative information on the finite lifetimes of eddies [as well as further information on the initial spectrum, $F_\nu(k,0)$] must await a solution to the more exact

[8] One's first inclination is to write $F(k,t)=C\exp[-(\nu^2k^2-1)^{\frac{1}{2}}kt]$ for $k>1/\nu$; however, this solution is unacceptable for the reasons stated above. We are indebted to Professor Chandrasekhar for drawing our attention to this point and for demonstrating that the spectrum must vanish instead for $k>1/\nu$.

[9] G. K. Batchelor and A. A. Townsend, Proc. Roy. Soc. (London) **A199**, 238 (1949).

differential equation (1) or its equivalent in wave number space, which is derived in the next section.

IV. GENERAL EQUATION FOR THE SPECTRUM

In this section, we wish to express the general differential equation (1) in terms of the turbulence spectrum, $F(k,t)$. Equation (9) may be written in terms of Bessel functions, J_s, as follows[6]:

$$f(r,t)=c_5\int_0^\infty \frac{F(k',t)}{(k'r)^{\frac{3}{2}}}J_{\frac{3}{2}}(k'r)dk', \tag{24}$$

where

$$c_5=3(\pi/2)^{\frac{1}{2}}\Big/\int_0^\infty F(k,0)dk. \tag{25}$$

Combining (24) with a recurrence relation for Bessel functions,

$$\frac{d}{dz}[z^{-s}J_s(z)]=-z^{-s}J_{s+1}(z), \tag{26}$$

we obtain for the left-hand side of Eq. (1),

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial t^2}-\nu^2 D_5{}^2\right)f=-\frac{c_5}{r^2}\int_0^\infty\left[\frac{F}{k'}\left(\frac{\partial^2}{\partial t^2}-\nu^2k'^4\right)\right] \times J_{\frac{5}{2}}(k'r)(k'r)^{\frac{1}{2}}dk'. \tag{27}$$

The right-hand side of Eq. (1) similarly transforms to

$$f\left(\frac{\partial}{\partial r}D_s f\right)=-c_s{}^2\int_0^\infty F(k',t)\frac{J_{\frac{3}{2}}(k'r)}{(k'r)^{\frac{3}{2}}}dk' \times\int_0^\infty \frac{k''^3F(k'',t)}{r}\frac{d}{dk''}\left[\frac{J_{\frac{3}{2}}(k''r)}{(k''r)^{\frac{3}{2}}}\right]dk''. \tag{28}$$

Integrating the second integral in (28) by parts and setting Eq. (27) equal to (28), we have, for a spectrum that vanishes sufficiently rapidly as $k\rightarrow\infty$,

$$\int_0^\infty\left[\frac{1}{k'}\left(\frac{\partial^2 F}{\partial t^2}-\nu^2k'^4F\right)\right]J_{\frac{5}{2}}(k'r)(k'r)^{\frac{1}{2}}dk' =-\frac{c_5}{r^2}\int_0^\infty\int_0^\infty F(k't)\frac{\partial}{\partial k''}[k''^3F(k'',t)] \times\frac{J_{\frac{3}{2}}(k'r)J_{\frac{3}{2}}(k''r)}{(k'k'')^{\frac{3}{2}}}dk'dk''. \tag{29}$$

Next we multiply both sides of Eq. (29) by $(kr)^{\frac{1}{2}}J_{\frac{5}{2}}(kr)$ and integrate over r from zero to infinity. Then applying Hankel's inversion theorem[10] to the left-hand side of (29), we find, with the aid of Eq. (25),

$$\left(\frac{\partial^2}{\partial t^2}-\nu^2k^4\right)F(k,t) =-\frac{\frac{1}{k}\int_0^\infty\int_0^\infty K(k,k',k'')F(k't)\frac{\partial}{\partial k''}[k''^3F(k'',t)]dk'dk''}{\int_0^\infty F(k,0)dk}, \tag{30}$$

where

$$K(k,k',k'')=\frac{3(\pi/2)^{\frac{1}{2}}k^{\frac{5}{2}}}{(k'k'')^{\frac{3}{2}}}\int_0^\infty\frac{J_{\frac{5}{2}}(kr)J_{\frac{3}{2}}(k'r)}{r^{\frac{3}{2}}} \times J_{\frac{3}{2}}(k''r)dr. \tag{31}$$

The kernel K has been evaluated[11] as follows:

$$K(k,k',k'')=\begin{cases}0, & (k'-k'')^2>k^2,\\ 1, & (k'+k'')^2<k^2,\\ \frac{1}{4}(\cos\theta-1)^2(\cos\theta+2), \end{cases} \tag{32}$$

otherwise, where

$$\cos\theta=(k'^2+k''^2-k^2)/2k'k''. \tag{33}$$

One additional integration by parts of Eq. (30) gives

$$\left(\frac{\partial^2}{\partial t^2}-\nu^2k^4\right)F(k,t) =\frac{\frac{1}{k}\int_0^\infty\int_0^\infty k''^3F(k't)F(k''t)\frac{\partial K(k,k',k'')}{\partial k''}dk'dk''}{\int_0^\infty F(k,0)dk}. \tag{34}$$

Although $\partial K/\partial k''$ is a discontinuous function, Eq. (34) may be more amenable to numerical computations than Eq. (30).

Equation (30) or (34), with K as given by (32), is the equivalent in wave-number space of Chandrasekhar's general equation (1). The relatively simple form of Eq. (32) for the kernel, K, suggests that (34) may prove useful for investigating the spectrum of turbulence in Chandrasekhar's new theory.

V. ACKNOWLEDGMENTS

Professor Chandrasekhar kindly suggested that we undertake this extension to his recent work. We are also deeply grateful to him for access to his paper prior to publication and for numerous illuminating discussions on turbulence theory.

[10] I. N. Sneddon, *Fourier Transforms* (McGraw-Hill Book Company, Inc., New York, 1951), p. 48.

[11] G. N. Watson, *A Treatise on Bessel Functions* (University Press, Cambridge, 1944), second edition, p. 411, Eq. (2).