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Virial Series of the Ideal Bose-Einstein Gas

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B. Widom has conjectured that the radius of convergence of the virial series of the ideal Bose-Einstein gas is infinite. The present note shows this to be false.

Widom[1] considers the virial series of the ideal Bose-Einstein gas, that is, the power series

$$x_2=\sum_{n=1}^{\infty} a_n x_1{}^n, \qquad (1)$$

where

$$x_s=\sum_1^{\infty} n^{-(s+\frac{1}{2})}y^n,$$

and shows that R, the radius of convergence of (1), is ≥ 0.257. He conjectures further that R is infinite, i.e., that (1) converges for all x_1. We show here that this conjecture is false.[2]

Our first step follows Newman. Let

$$x_2=f(x_1), \quad f \text{ entire.} \qquad (2)$$

Then $dx_2/dy=f'(x_1)(dx_1/dy)$ whence, since $y(dx_2/dy)=x_1$, we have

$$\frac{dy}{y}=\frac{f'(x_1)}{x_1}dx_1=\left[\frac{1}{x_1}+g(x_1)\right]dx_1,$$

where g is entire. Integrating, we get $y=x_1e^{G(x_1)}$, where G is entire. Thus (2) implies that y is an entire function of x_1, and it suffices to show (writing z for y): *The function-element defined by the power series*

$$u(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^{\frac{3}{2}}}, \quad |z|<1 \qquad (3)$$

is not inverse to an entire function.

To show this we must obtain the analytic continuation of $u(z)$ outside the unit circle. This is best done using the integral representation

$$u(z)=\frac{4}{3\sqrt{\pi}}z\int_0^{\infty}\frac{dt}{\exp(t^{\frac{2}{3}})-z},$$

from which one easily deduces: All singularities of the complete analytic function $U(z)$ (gotten by continuing $u(z)$ onto its entire Riemann surface) lie above $z=0$ and $z=1$. $z=1$ is an algebraic branch point of order one on every sheet, and $z=0$ is a logarithmic branch point on every sheet but the original. When $u(z)$ is continued once around $z=1$ it passes into the new function element

$$u_1(z)=u(z)-4i(\pi \log z)^{\frac{1}{2}}. \qquad (4)$$

From this information we can complete the proof. Let γ_1 be a path going from $z=0$ once around $z=1$ and back to a point z of $0<|z|<1$, and let γ_2 be a path going from $z=0$ once around $z=1$, then once around $z=0$, then again around $z=1$ and back to a point z of $0<|z|<1$. The result of continuing $u(z)$ along γ_1 is given by (4), from which we deduce that the result of continuing $u(z)$ along γ_2 is the element

$$u_2(z)=u(z)-4i(\pi\log z)^{\frac{1}{2}}-4i[\pi(\log z+2\pi i)]^{\frac{1}{2}}. \qquad (5)$$

Suppose now that γ_1 terminates in the point z_1, and γ_2 in z_2, and $u_1(z_1)=u_2(z_2)=\alpha$. From (4) and (5) we see that $z_1\neq z_2$. Then, if Γ_1 and Γ_2 are the images of γ_1 and γ_2 in the U-plane, the result of continuing the function $F(U)$ [inverse to $U(z)$] from $U=0$ to $U=\alpha$ along the paths Γ_1 and Γ_2 is to obtain two distinct determinations at $U=\alpha$, namely z_1 and z_2. Thus the inverse function $F(U)$ could not even be single-valued, and *a fortiori* not entire.

To complete the proof, we prove the existence of γ_1 and γ_2 with the desired properties. By letting γ_1 run around the origin in all possible ways after its return to $0<|z|<1$, $w=(\log z)^{\frac{1}{2}}$ takes on all values in the sector S: $\frac{1}{4}\pi<\arg w<\frac{3}{4}\pi$ and therefore $u_1(z)$ takes on all values

$$u(\exp(w^2))-4i\pi^{\frac{1}{2}}w. \qquad (6)$$

Similarly, by letting γ_2 run around the origin in all possible ways after its final return to $0<|z|<1$, $u_2(z)$ takes on all values

$$u(\exp(w^2))-4i\pi^{\frac{1}{2}}w-4i\pi^{\frac{1}{2}}(w^2+2\pi i)^{\frac{1}{2}}. \qquad (7)$$

It remains only to show that the sets of values assumed by the functions (6) and (7), respectively, as w ranges over S, are not disjoint. Since $|u(\exp(w^2))|<\zeta(3/2)$, and since $(w^2+2\pi i)^{\frac{1}{2}}$ differs from w by a bounded function in S, it suffices to prove the same for the values

[1] B. Widom, Phys. Rev. **96**, 16 (1954).

[2] This was proved recently by Dr. D. J. Newman of the Republic Aviation Corporation, who showed $R\leq 64$. Newman's proof (not published) uses a generalization of Picard's theorem on entire functions, which we are able to avoid.

Since the submission of this note, the author has been informed of prior work of Dr. W. H. J. Fuchs of Cornell University, who has obtained the more precise result $12.56\cdots\leq R\leq 27.73\cdots$. The methods of Fuchs and the present author are essentially the same, our presentation (of the more limited result) being, however, more compact and transparent.

assumed by functions

$$\phi_1(w)-4i\pi^{\frac{1}{2}}w \tag{6a}$$

and

$$\phi_2(w)-8i\pi^{\frac{1}{2}}w, \tag{7a}$$

where ϕ_1 and ϕ_2 are regular and bounded in S. Because the correspondence $w\leftrightarrow 2w$ maps S onto itself, we may without loss of generality replace the 8 in (7a) by a 4, and (dividing by $4i\sqrt{\pi}$ and writing $\psi_k=\phi_k/4i\sqrt{\pi}$) we have only to prove that the value-sets of the functions

$$w-\psi_1(w), \tag{6b}$$

and

$$w-\psi_2(w) \tag{7b}$$

are not disjoint, where ψ_1 and ψ_2 are regular and bounded in S. But this is immediate: for, if $|\psi_k|\leq M$, $k=1, 2$ then by Rouche's theorem both functions (6b) and (7b) assume the value $M\sqrt{2}i$ inside the circle of center $M\sqrt{2}i$ and radius M.

Remark.—If K_i denotes the maximum of $|U(z)|$ on γ_i $(i=1, 2)$, and $\tau=\max(K_1,K_2)$, the above reasoning gives the more precise result that $F(U)$ has a branch point in the circle $|U|\leq\tau$, whence τ is an upper bound for the radius of convergence R of the virial series. It has not been deemed worth while to make a numerical estimate (which would be a straightforward task) because in any case the method is too crude to answer the interesting question of whether (1) converges at $x_1=\zeta(3/2)=2.612\cdots$, the value of the dimensionless density for which condensation is known to occur.

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Turbulence Spectrum in Chandrasekhar's Theory*

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The results of Chandrasekhar's recent theory of turbulence are transformed in this paper from ordinary space (used exclusively in his paper) to wave-number space. Consideration is limited to the case of stationary, homogeneous, and isotropic turbulence. A "time-dependent spectrum" is defined in terms of the scalar product of eddy velocities at two different times; this spectrum is related to Chandrasekhar's time-dependent correlation function by a Fourier transform, as in the conventional, time-independent case. For infinite Reynolds number the spectrum is obtained directly by transforming the correlation function into wave-number space; the spectrum is given by a much simpler expression than is the corresponding correlation function.

Consequently, an approximate (linearized) form of the *differential* equation for the correlation has been transformed to wave-number space, and this equation is readily solvable, even for a finite Reynolds number. For $k>1/\nu$ the spectrum vanishes, and this cutoff at large wave-numbers is interpreted as the disintegration of turbulence into laminar flow at dimensions sufficiently small for viscosity to dominate over the inertial transfer of energy.

Finally the general, nonlinear correlation equation has been transformed into an integral equation for the spectrum, but a general solution has not yet been obtained for either the spectrum or correlation.

I. INTRODUCTION

In a recent paper Chandrasekhar[1] has presented a new theory of isotropic, homogeneous turbulence in a steady state. The basic innovations in the new theory are (1) the consideration of velocity correlations not only at two different points but also at *two different times* and (2) the hypothesis of a statistical relationship between the second-order and fourth-order correlation tensors.

Starting with the hydrodynamic equation of motion and the equation of continuity, Chandrasekhar derived his fundamental equation for $f(r,t)$, the longitudinal correlation function[2]:

$$\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial t^2}-\nu^2 D_5{}^2\right)f=f\frac{\partial}{\partial r}D_5 f. \tag{1}$$

Here D_5, the Laplacian operator in five-dimensional space, may be written

$$D_5=\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}. \tag{2}$$

In these equations r $(=|\mathbf{r}|)$ is the separation of the two points measured in units of some arbitrary length l; t is the time interval between the measurements of velocity at the two points in the unit of $l/\langle(u_1{}^2)\rangle_{\text{Av}}{}^{\frac{1}{2}}$, where u_1 is the velocity component in the direction of $\mathbf{r}$ (say, along the x-axis); ν is the viscosity in units of $l\langle(u_1{}^2)\rangle_{\text{Av}}{}^{\frac{1}{2}}$. The longitudinal correlation function is defined in terms of velocity components by

$$f(r,t)=\frac{\langle u_1(r_0,t_0)u_1(r_0+r,\, t_0+t)\rangle_{\text{Av}}}{\langle u_1{}^2\rangle_{\text{Av}}}. \tag{3}$$

We define ϕ by

$$f=1-\phi; \tag{4}$$

then for small values of t and r (such that f does not

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[1] S. Chandrasekhar, Proc. Roy. Soc. (London) **A229**, 1 (1955). Hereafter this article will be referred to as "Paper I."
[2] Paper I, Eq. (46).