tivistic and relativistic cases, and the agreement is improved as $\Theta\to 0$ because contributions from the small L become increasingly less important. The right side of Eq. (26.2) may be replaced therefore by the nonrelativistic formula for the Coulomb wave. In the small-angle approximation the Coulomb wave contains the factor $\exp\{-i\eta\ln(1-\cos\Theta)\}$. This factor is related to the phases obtained in the c.m. system by

$$2\mathbf{s}^2=(\gamma+1)\sin^2\Theta/[1+\tfrac{1}{2}(\gamma-1)\sin^2\Theta]$$
$$\cong(\gamma+1)[1-\tfrac{1}{2}(\gamma-1)\Theta^2]\sin^2\Theta, \quad (27)$$
$$\delta=(1-v'^2/c^2)^{-\frac{1}{2}}.$$

For small scattering angles, one has therefore

$$\exp\{-i\eta\ln\sin^2\Theta\}\cong[1-\tfrac{1}{2}(\gamma-1)i\eta\Theta^2]$$
$$\times\exp\{-i\eta\ln[2\mathbf{s}^2/(\gamma+1)]\}, \quad (27')$$

and to a good approximation the factor $\exp(-i\eta\ln\mathbf{s}^2)$ is reproduced. On the other hand there is seen to be a term in $\eta\Theta^2$ coming in as a correction so that exact agreement is not proved. The calculation just presented is very similar to that which gave Eq. (5.27) of Mott's second paper. The role played by large L for small Θ would not have been clear however with a direct use of Mott's paper.

The consideration of the small-angle scattering in the laboratory system is seen to support conclusions drawn from the two-body phase shift approach.

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Phase Shifts for Relativistic Corrections in High-Energy p-p Scattering*

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Scattering phase shifts for the relativistic corrections to the ordinary Coulomb interaction between two protons are calculated in the first Born approximation. The scattering matrix resulting from these phase shifts is obtained and shown to agree with the results of the preceding paper.

IN the preceding paper,[1] the relativistic corrections to the Coulomb scattering of two identical particles were discussed in terms of the equation derived by Breit for first order changes in the energy.[2] It was shown there that the matrix element of the interaction energy of the two particles, H', may be written as

$$H'=e^2\left(\psi^*\frac{1-\boldsymbol{\alpha}_{\mathrm{I}}\boldsymbol{\alpha}_{\mathrm{II}}}{r}\psi'\right)=e^2\left(\Psi^*\frac{1}{r}\{1+\xi^2\xi'^2+3(\xi^2+\xi'^2)\right.$$
$$+2i\{3+(\boldsymbol{\xi}\cdot\boldsymbol{\xi}')([\boldsymbol{\xi}\times\boldsymbol{\xi}']\cdot\mathbf{S})\}-2(\boldsymbol{\xi}-\boldsymbol{\xi}')^2\mathbf{S}^2$$
$$\left.+2[(\boldsymbol{\xi}'-\boldsymbol{\xi})\cdot\mathbf{S}]^2-2([\boldsymbol{\xi}\times\boldsymbol{\xi}']\cdot\mathbf{S})^2\}\Psi'.\right) \quad (1)$$

The notation is the same as that in the preceding paper; in particular ξ is related to the momentum of one of the particles in the center-of-mass system by

$$\boldsymbol{\xi}=\mathbf{p}/(E_{\mathrm{I}}+M),$$

the primed and unprimed symbols referring to values appropriate to the incident and final wave function, respectively. Furthermore, the spinor Ψ is that component of the wave function of the relative motion ψ which is large if both particles are of positive energy. In the treatment of the preceding paper,[1] the singlet and main nonspin dependent triplet terms were treated by means of phase shifts and in coordinate space, while the remaining terms were evaluated in first Born approximation in momentum space after this procedure was shown to be equivalent, to first order in e^2, to a phase shift treatment. It is of some interest to consider these spin dependent terms also in terms of phase shifts. By doing so a verification of the argument regarding the equivalence of the direct phase shift treatment and the momentum space calculations is provided and the values of the phase shifts which are modified by specifically nuclear forces are made available. If the initial and final scattering states are represented by configuration-space wave functions, a corresponding expression may be written for H', with ξ and ξ' replaced by their operator equivalents. Regrouping terms in Eqs. (16)–(16.4) of B one obtains for the matrix element

$$H'=H_a'+H_b'+H_c'+H_d', \quad (2)$$

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[1] G. Breit, preceding paper. Henceforth, this paper is referred to as B.
[2] G. Breit, Phys. Rev. 34, 553 (1929).

where

$$H_a' = e^2 \int \Psi^* \left[\frac{1}{r} + \frac{p^2\{1/r\}p^2}{(E_\mathrm{I}+M)^4} + \frac{3}{(E_\mathrm{I}+M)^2} \times \left(p^2\left\{\frac{1}{r}\right\} + \left\{\frac{1}{r}\right\} p^2 \right) \right] \Psi' d\mathbf{r},$$

$$H_b' = e^2 \int \Psi^* \frac{2\hbar}{(E_\mathrm{I}+M)^2} \left[\frac{3}{r}\left\{\frac{1}{r}\right\}' + \frac{1}{(E_\mathrm{I}+M)^2} \left(\frac{1}{r}\left\{\frac{1}{r}\right\}' p^2 - \hbar^2 \frac{\partial}{\partial r}\left(\frac{1}{r}\left\{\frac{1}{r}\right\}'\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\right)\right] (\mathbf{L}\cdot\mathbf{S}) \Psi' d\mathbf{r},$$

$$H_c' = -\frac{2\hbar^2 e^2}{(E_\mathrm{I}+M)^2} \int \Psi^* \left[\left(\frac{1}{r}\left\{\frac{1}{r}\right\}' - \Delta\left\{\frac{1}{r}\right\} \right) \mathbf{S}^2 + \frac{1}{r} (\mathbf{r}\cdot\mathbf{S})^2 \frac{\partial}{\partial r}\left(\frac{1}{r}\left\{\frac{1}{r}\right\}' \right) \right] \Psi' d\mathbf{r}, \tag{3}$$

$$H_d' = \frac{2\hbar^2 e^2}{(E_\mathrm{I}+M)^4} \int \Psi^* \left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\left\{\frac{1}{r}\right\}'\right) \{ (\mathbf{L}\cdot\mathbf{S})^2 + \hbar \mathbf{L}\cdot\mathbf{S} + i\hbar[(\mathbf{r}\cdot\mathbf{S})(\mathbf{p}\cdot\mathbf{S}) - (\mathbf{r}\cdot\mathbf{p})\mathbf{S}^2]\} + \frac{1}{r}\left\{\frac{1}{r}\right\}' [p^2\mathbf{S}^2 - (\mathbf{p}\cdot\mathbf{S})^2] \right] \Psi' d\mathbf{r}.$$

The matrix elements H_a', H_b' and $H_c'+H_d'$ correspond, respectively, to the operators NI_1, NI_2 and NI_3 of the preceding paper.[1] The operator $\mathbf{p}$ in these expressions stands for $(\hbar/i)\nabla$, the factors involving derivatives with respect to r coming from the commutators of $\mathbf{p}$ with r. The quantity $\{1/r\}$ is a function of r analytic at the origin and arbitrarily close to $1/r$ elsewhere. The notation $\{1/r\}'$ means $(\partial/\partial r)\{1/r\}$. The limit $\{1/r\}\rightarrow 1/r$ is taken after the integrations have been performed. A justification of this procedure is given in Sec. IV of B. The first term in (2) is spin-independent and the only one surviving in the singlet states. The second term is the matrix element of a vector operator in the space of $\mathbf{S}$, and the third involves a tensor operator in $\mathbf{S}$ space. These terms are of order $(E_\mathrm{I}-M)/(E_\mathrm{I}+M)$ relative to the first, and the fourth term is of this same order relative to the second and third.

In the first Born approximation, the wave functions ψ and ψ' represent free-particle waves, which, for the purpose of evaluating phase shifts, are chosen to be spherical. The functions Ψ then have the form

$$\Psi = (1/r) F_L(r) \mathcal{Y}^{L,J}{}_{\mu}, \tag{4}$$

in which F_L is the regular solution to the differential equation for r times the radial wave function. The normalization is such that

$$\int \mathcal{Y}^* \mathcal{Y} d\Omega = 1, \tag{5}$$

and

$$F_L \sim \sin(\kappa r - L\pi/2).$$

These functions have the properties

$$\mathbf{S}^2\Psi = S(S+1)\Psi, \quad L^2\Psi = L(L+1)\Psi, \quad p^2\Psi = \hbar^2\kappa^2\Psi, \tag{6}$$

with $\hbar^2\kappa^2 = E_\mathrm{I}^2 - M^2$. Henceforth, states of the two-particle system will be labelled by values of the orbital angular momentum L and total angular momentum J appropriate for the component Ψ.

The scattering phase shifts $\delta_{L,J}$ are related to the diagonal matrix elements for the states J, L by [see Eqs. (9.2), (9.4), (13.2), (17.2′), and (17.5′) of B]

$$\delta_{L,J} = -[(E_\mathrm{I}+M)/2E_\mathrm{I}]^2 (E_\mathrm{I}/\hbar^2\kappa) H_{L,J}'. \tag{7}$$

An additional contribution to the scattering matrix will be supplied by the off-diagonal matrix elements, which are most conveniently handled in terms of the $\mathcal{T}$ matrix of Breit, Ehrman, and Hull.[3] As is shown in Eqs. (17) to (17.6) of B, the same relation exists between the elements of H' and of $\mathcal{T}$ as that between H' and δ, given in (7).

Suitable treatment of the integral in (3), as has been discussed in B, shows that this term gives rise to the usual Coulomb phase shift, $\sigma_L = \arg\Gamma(L+1+i\eta_r)$, where

$$\eta_r = \frac{e^2}{\hbar v'} = \frac{e^2}{\hbar} \frac{(1+v_\mathrm{I}^2/c^2)}{2v_\mathrm{I}}, \tag{8}$$

and v_I, v' are the velocities of the incident particle in the center-of-momentum and laboratory systems, respectively. This in turn contributes a diagonal term to the scattering matrix of the form

$$-(\eta_r/2\kappa\mathrm{s}^2) \exp\{-i[\eta_r \ln \mathrm{s}^2 - \rho + \eta_r \ln 2\rho - 2\sigma_0]\}, \tag{9}$$

where $\mathrm{s} = \sin(\theta/2)$. The identity of the particles is meant to be treated as in Breit and Hull.[4]

A straightforward evaluation of the integrals involved for $L \geq 2$ leads to the phase shifts listed in Table I. For convenience, the following notation is introduced:

$$\mathcal{K} \equiv \kappa e^2/4E_\mathrm{I}; \quad \mathcal{E} \equiv (E_\mathrm{I}-M)/(E_\mathrm{I}+M). \tag{10}$$

TABLE I. Phase shifts $\delta_{L,J}$, in units of $(\kappa e^2/4E_\mathrm{I})$, from H_b', H_c', H_d' for $L \geq 2$.

L, J	From H_b'	From H_c'	From H_d'
$L, L-1$	$-\frac{1}{L}(3+\mathcal{E})$	$-\frac{1}{L(2L-1)}$	$-\frac{\mathcal{E}}{L(2L-1)}$
L, L	$-\frac{1}{L(L+1)}(3+\mathcal{E})$	$\frac{1}{L(L+1)}$	$\frac{\mathcal{E}}{L(L+1)}$
$L, L+1$	$\frac{1}{L+1}(3+\mathcal{E})$	$-\frac{1}{(L+1)(2L+3)}$	$-\frac{\mathcal{E}}{(L+1)(2L+3)}$

[3] Breit, Ehrman, and Hull, Phys. Rev. **97**, 1051 (1955).
[4] G. Breit and M. H. Hull, Jr., Phys. Rev. **97**, 1047 (1955).

TABLE II. Phase shifts $\delta_{L,J}$, in units of $(\kappa e^2/4E_1)$, from H_b', H_c', H_d', for $L=0, 1$.

L, J	From H_b'	From H_c'	From H_d'
0, 1	0	8/3	$\frac{2}{3}\mathcal{E}$
1, 0	$-(3+\frac{1}{3}\mathcal{E})$	-1	$\frac{1}{3}\mathcal{E}$
1, 1	$-\frac{1}{2}(3+\frac{1}{3}\mathcal{E})$	$-\frac{1}{2}$	$\frac{1}{6}\mathcal{E}$
1, 2	$\frac{1}{2}(3+\frac{1}{3}\mathcal{E})$	$-1/10$	$(7/30)\mathcal{E}$

It is not proper to put $L=0$ or 1 in the general expressions for the phase shifts given in Table I. These general formulas have been obtained by letting $\{1/r\}\rightarrow 1/r$ in the integrand of the integrals in question, a procedure justified by the fact that the integrals are uniformly convergent for $L\geq 2$. This convergence does not obtain for $L=0$, 1 so that it is not unexpected that following the prescribed procedure of letting $\{1/r\}\rightarrow 1/r$ after the integrations have been performed leads to results different from those obtained by letting $L\rightarrow 0$, 1 in the general expressions of Table I. The phase shifts for $L=0$ and $L=1$ are given in Table II.

Only H_c' and H_d' give contributions to the $\mathcal{T}$ matrix, which are handled in a similar manner. The results are shown below, the general expression this time being valid for all $L\geqslant 0$.

$$\mathcal{T}_{L,L+2}=\mathcal{T}_{L+2,L}=\mathcal{K}(1+\mathcal{E})/\{(2L+3)[(L+1)(L+2)]^{\frac{1}{2}}\}. \quad (11)$$

It is to be noted that these phase shifts represent only the lowest term in an expansion of the exact phase shifts in powers of η.

The scattering of a partial wave of specified L will be composed of contributions, described by $\delta_{L,J}$, from each of the three values of J allowed for a given L, in addition to further scattering, described by $\mathcal{T}_{L,L+2}$, from the states $L\pm 2$. In the presence of an additional interaction of the tensor type, it may prove desirable to treat the diagonal contributions to the scattering matrix differently from the nondiagonal contributions. Therefore this separation is made here in dealing with the Coulomb corrections. The various partial waves may be combined to give the total scattered wave and hence the S-matrix; convenient formulas for this are to be found in Breit and Hull[4] and in Breit, Ehrman, and Hull.[3] In this connection it may be remarked that to the lowest order in η, the scattering phase shifts δ are of course identical with the functions $Q_{L,J}=\exp(i\delta_{L,J})\sin\delta_{L,J}$ used there. These formulas describe the scattering of a nonrelativistic Coulomb wave, the purely Coulomb part of the scattering being included in the factors $e_{L,0}=\exp(2i\sigma_{L,0})$, in which $\sigma_{L,0}$ is the Coulomb phase shift of the Lth partial wave relative to that of the $L=0$ wave. According to Eq. (13) of B, however, the matrix element H_a', which is the part analogous to the nonrelativistic Coulomb interaction, gave just the usual formula for the Coulomb phase shift, save for a modification in the definition of η. It might be assumed, therefore, that using such a modified $e_{L,0}$ in the expression for the scattering of the partial waves would correspond to taking into account exactly the scattering given by H_a'. Extreme care must be taken in this regard, however. The phase shifts are known only to first order in η, as mentioned before. Thus a consistent treatment based upon an expansion in powers of η would demand that $e_{L,0}$ be replaced by its value for $\eta=0$, that is, unity. This in fact introduces considerable simplification into the summing of the infinite series involved. The series may then be reduced to sums over functions of L times P_L or its derivatives, together with a few extra terms to take account of special behavior for $L=0, 1$. The functions of L take the form of ratios of polynomials which, however, are factorable into products of terms linear in L. The theory of partial fractions may then be used to convert the summations into linear combinations of the function $F(a,x)$ and of its derivative and integral with respect to x. The function $F(a,x)$ is defined by

$$F(a,x)\equiv\sum_{L=0}^{\infty}\frac{P_L(x)}{L+a},$$

where a is an integer or rational fraction. The generating function for Legendre polynomials may be used to show that

$$F(a,x)=\int_0^1\frac{h^{a-1}dh}{(1-2hx+h^2)^{\frac{1}{2}}}, \quad (12)$$

and on performing the indicated integration, one is led to elliptic functions for the values of a in question.

Alternatively it may be noted that the combination

$$\sum_{L=0}^{\infty}\left(\frac{1}{2L+3}-\frac{1}{2L-1}\right)P_L(x)=-\tfrac{1}{2}\pi P_{\frac{1}{2}}(-x), \quad (13)$$

occurs in most of the sums. The Legendre function of the first kind of order one-half, $P_{\frac{1}{2}}$, may, in turn, be expressed in terms of elliptic functions.

If the scattering matrix is parametrized in terms of the α's introduced previously,[4] the contributions from the various parts of the interaction matrix are as follows.

From H_b':

$$\alpha_1^{(b)}=-\alpha_4^{(b)}=-\mathcal{K}(1-x)^{-1}(3+\mathcal{E}x), \quad (14)$$

where $x\equiv\cos\theta$.

From the diagonal parts of H_c':

$$\begin{aligned}\alpha_1^{(c)}&=\alpha_4^{(c)}=\mathcal{K}(1-x^2)^{-1}\\&\qquad\times[-(1+x)+(1-x)K+2xE],\\ \alpha_2^{(c)}&=\mathcal{K}(3+\tfrac{1}{2}K-E),\\ \alpha_3^{(c)}&=-\mathcal{K}(1-x^2)^{-2}[-(x+1)^2\\&\qquad-\tfrac{1}{2}(x-3)(x+1)K+(x^2+3)E],\\ \alpha_5^{(c)}&=-\mathcal{K}(-2+K-2E),\end{aligned} \quad (15)$$

where K, E are complete elliptic integrals of the first and second kinds, respectively, with arguments $k^2=\frac{1}{2}(1+x)$.

From the diagonal parts of H_d':

$$\begin{aligned}
\alpha_1^{(d)}&=\alpha_4^{(d)}=\mathcal{K}\mathcal{E}(1-x^2)^{-1}\\
&\quad\times[(x+1)(x-2)+(1-x)K+2xE],\\
\alpha_2^{(d)}&=\mathcal{K}\mathcal{E}(1+\tfrac{1}{2}K-E),\\
\alpha_3^{(d)}&=-\mathcal{K}\mathcal{E}(1-x^2)^{-2}[-(x+1)^2\\
&\quad-\tfrac{1}{2}(x-3)(x-1)K+(x^2+3)E],\\
\alpha_5^{(d)}&=-\mathcal{K}\mathcal{E}(-2x+K-2E).
\end{aligned}\tag{16}$$

From the nondiagonal parts of H_c':

$$\begin{aligned}
\Delta\alpha_1^{(c)}&=\Delta\alpha_4^{(c)}=-\mathcal{K}(1-x^2)^{-1}\\
&\quad\times[(x+1)(x-2)+(1-x)K+2xE],\\
\Delta\alpha_2^{(c)}&=-\tfrac{1}{2}\Delta\alpha_5^{(c)}=-\mathcal{K}[\tfrac{1}{2}(1-x)+\tfrac{1}{2}K-E],\\
\Delta\alpha_3^{(c)}&=-\mathcal{K}(1-x^2)^{-2}[-\tfrac{1}{2}(x-3)(x+1)^2\\
&\quad+\tfrac{1}{2}(x-3)(x+1)K-(x^2+3)E].
\end{aligned}\tag{17}$$

From the nondiagonal parts of H_d':

$$\begin{aligned}
\Delta\alpha_1^{(d)}&=\Delta\alpha_4^{(d)}=-\mathcal{K}\mathcal{E}(1-x^2)^{-2}\\
&\quad\times[(x+1)(x-2)+(1-x)K+2xE],\\
\Delta\alpha_2^{(d)}&=-\tfrac{1}{2}\Delta\alpha_5^{(d)}=-\mathcal{K}\mathcal{E}[\tfrac{1}{2}(1-x)+\tfrac{1}{2}K-E],\\
\Delta\alpha_3^{(d)}&=\mathcal{K}\mathcal{E}(1-x^2)^{-2}[\tfrac{1}{2}(x+1)^2(x-3)\\
&\quad-\tfrac{1}{2}(x-3)(x-1)K+(x^2+3)E].
\end{aligned}\tag{18}$$

Combining the diagonal and nondiagonal parts from H_c' and H_d' one obtains the following simpler expressions.

From H_c':

$$\begin{aligned}
\alpha_1^{(c)}&=\alpha_4^{(c)}=\mathcal{K},\\
\alpha_2^{(c)}&=\mathcal{K}[3-\tfrac{1}{2}(1-x)],\\
\alpha_3^{(c)}&=-\mathcal{K}[\tfrac{1}{2}(1-x)^{-1}],\\
\alpha_5^{(c)}&=\mathcal{K}(3-x).
\end{aligned}\tag{19}$$

From H_d':

$$\begin{aligned}
\alpha_1^{(d)}&=\alpha_4^{(d)}=0,\\
\alpha_2^{(d)}&=\mathcal{K}\mathcal{E}\tfrac{1}{2}(1+x),\\
\alpha_3^{(d)}&=-\mathcal{K}\mathcal{E}[\tfrac{1}{2}(1-x)^{-1}],\\
\alpha_5^{(d)}&=\mathcal{K}\mathcal{E}(1+x).
\end{aligned}\tag{20}$$

For comparison, the results of summing the partial waves for the scattering by H_b', using the exact expression for $e_{L,0}$ together with the first Born approximation phase shifts, are given below. The summation in this case has been performed by using the following integral representation for $e_{L,0}$:

$$e_{L,0}=-\frac{1}{\Gamma(i\eta)}(2)^{i\eta+\frac{1}{2}}\pi^{\frac{1}{2}}\int_0^\infty dy e^{-y}y^{i\eta-\frac{1}{2}}I_{L+\frac{1}{2}}(y),\tag{21}$$

where $I_\nu(y)$ is the modified Bessel function of the first kind of imaginary argument. This representation is obtainable from the Mellin transform of the quotient of two gamma functions. The terms not containing $\exp(-i\eta\ln\mathbf{s}^2)$ come from the $L=0$ and $L=1$ phase shifts. There results:

$$\begin{aligned}
\alpha_1=-\mathcal{K}\bigg\{&\frac{6}{1-x^2}\exp(-i\eta\ln\mathbf{s}^2)-\frac{3}{1+x}\\
&+\mathcal{E}\bigg[\frac{2}{1-x^2}\exp(-i\eta\ln\mathbf{s}^2)-\frac{1+i\eta}{1-i\eta}-\frac{1}{1+x}\bigg]\bigg\}.
\end{aligned}\tag{22}$$

Since terms of order η^2 and higher have been neglected in the calculation of the phase shifts, the appearance of terms not having $\exp(-i\eta\ln\mathbf{s}^2)$ may not be significant. Such terms may be modified in a treatment including higher order effects in η consistently.

The scattering matrix obtained here agrees in the lowest order in η with that obtained by Breit directly from Eq. (1);[1] the contributions from H_a' and H_b' agree with the result of Garren[5] to the same order in η.

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[5] A. Garren, Phys. Rev. **96**, 1709 (1954); U. S. Atomic Energy Commission Report NYO-7102, Carnegie Institute of Technology, March 1, 1955 (unpublished).