Relativistic Corrections for High-Energy p-p Scattering

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It is shown that the treatment of the collision of two charged particles by means of a first-order Born approximation and Møller's matrix element involves an inconsistency connected with the infinite cross section for small angle scattering. It is then shown that an energy formula derived for the two-body interaction by means of an early form of the Heisenberg-Pauli quantum electrodynamics makes it possible to construct a relativistic two-body extension of the nonrelativistic one-body Mott-Gordon solution. This extension is good only to order e^2 but arguments are given for believing that the angle-dependent and e^2-containing factors are partially correct for the more important terms. The Gordon sphere construction naturally leads to such factors and the consideration of small angle collisions in the laboratory system leads to a similar result. The latter suggests the possible existence of correction terms. The explicit superposition of partial waves is avoided by noting a formal similarity of the relativistic and nonrelativistic problems for principal non-spin-dependent terms. Contributions of the spin-dependent terms are worked out, also avoiding explicit summation by employing a momentum space representation and noting that once the main terms are taken care of by the Gordon sphere construction, the spin-dependent terms can be treated as a perturbation on account of their more rapid fall-off with distance.

The possibility of dealing with first-order phase shifts by means of a phase shift matrix is discussed in connection with Eqs. (17)–(17.2′) and the coupling of states with different L but the same J in relation with Eq. (9.8) and Eqs. (17.1)–(17.6). The definition of phase shifts in the relativistic problem, neglecting meson production, is discussed in connection with Eq. (8).

1. INTRODUCTION AND NOTATION

THE collision of two electrons has been treated relativistically by Møller,[1] taking into account first-order effects in e^2 only. He showed that high logical consistency results through the employment of his matrix element,

$$\mathfrak{M}=(e^2h^2/\pi)(a_1^0a_2^0,[1-\boldsymbol{\alpha}_{\mathrm{I}}\boldsymbol{\alpha}_{\mathrm{II}}]a_1a_2)/[(\mathbf{p}_1-\mathbf{p}_1^0)^2-(E_1-E_1^0)^2/c^2], \quad (1)$$

in momentum space. The quantities for the initial state are designated by zero superscripts, a_1 and a_2 stand respectively for Dirac spinor amplitudes of the two electrons with unit density normalization, the $\mathbf{p}$ denote momentum, E the energy and the remaining quantities have standard meaning. By means of this formula a treatment of Coulomb scattering has been given recently by Garren[2] who included the effect of the anomalous magnetic moment of the proton as well. Garren's considerations regarding the phase factors analogous to those present in the nonrelativistic problem do not appear to have been justified, however, except on the grounds of there being no large effect caused by such phase factors and related Coulomb phase shifts so that the distinction between relativistic and nonrelativistic values could be argued to be unimportant.

The data analysis problem is complex, and high accuracy in phase shift determinations will perhaps be unobtainable for some time. It is possible therefore that a more accurate knowledge of the relativistic corrections will not be needed in the immediate future. It has been felt, nevertheless, that it is not possible to predict which features will be important eventually and that the relativistic effects should be more firmly established and examined. An attempt to do so is being made in the present paper.

Indefiniteness in the employment of Coulomb phases frequently called σ_L is absent in the treatment below and the employment of the same value of the parameter frequently called η in different parts of the scattering matrix is justified by this work within an approximation which may be claimed to be better than to within the first power of e^2 for some of the terms. Singlet scattering is treated with relatively little uncertainty.

Use is made of the formal similarity between the phase shift treatment of the nonrelativistic and the relativistic problems provided the latter is considered in the center-of-mass system of the two colliding particles. It is shown in this connection that the phase shift is an exact concept in the relativistic treatment provided corrections for the emission of photons and of mesons are neglected. A direct extension of Gordon's nonrelativistic treatment gives then a definite answer for scattering in the singlet case. One of the terms in the scattering matrix for the triplet case is identical with that representing singlet scattering. This term involves the progressive phase shifts which depend on the interparticle distance r and contain the expression $\eta \ln 2kr$. This circumstance is connected with the infinity in the total cross section and the large small-angle scattering characteristic of the Coulomb field. This term contains the standard nonrelativistic answer.

For triplet states the additional terms of the scattering matrix can be treated by the same method. Some of these are finite everywhere but some are infinite at zero scattering angle. The latter arise from spin-orbit interactions.

It is shown in Sec. 3 that the nonrelativistic problem can be treated by the phase-shift method in the first approximation in the parameter η, i.e., to within the

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[1] C. Møller, Z. Physik **70**, 786 (1931); Ann. Physik **14**, 531 (1932).

[2] A. Garren, Phys. Rev. **96**, 1709 (1954); thesis, Carnegie Institute of Technology, 1955 (unpublished).

first power of e^2. This is the discussion connected with Eqs. (6) to (6.6). In the same section, in relation to Eqs. (9) through (13.7), there is a proof of the equivalence of the use of the relativistic value of η to the Gordon sphere method as applied to the relativistic case.

On the other hand, it has been found difficult to obtain a clear treatment of the scattering problem by means of Møller's matrix element employing the ordinary iteration method. The difficulties are discussed in Sec. 2. It is shown there that the employment of (1) in the first-order Born approximation is self-contradictory and leads to the occurrence of the incident wave in the scattered wave with a formally infinite coefficient. It is not surprising that such a paradoxical result is obtained because there exists no solution of the nonrelativistic Coulomb scattering problem with the wave at infinity asymptotic to a plane incident plus a scattered outgoing wave. This situation is discussed below in relation to the infinity in the total scattering cross section and it is shown that its origin lies in the fact that even for a screened Coulomb field a non-negligible number of particles is removed from the screening sphere by the scattering. In Møller's introduction of $\mathfrak{M}$, a limiting process with a screening sphere was used as an intermediate step but has not been followed out once $\mathfrak{M}$ was obtained.

The first-order Born-type perturbation calculation is incorrect in view of the large effect which the Coulomb field has on the incident plane wave. According to Mott and Gordon,[3] the nonrelativistic Coulomb wave is asymptotic to

$$\exp\{i[kz+\eta\ln k(r-z)]\}-[\eta/k(r-z)]\times\exp\{i[kr-\eta\ln k(r-z)+2\sigma_0]\}, \quad (1.1)$$

involving the quantity η in the exponents in combination with quantities which become infinite as the direction of incidence is approached. While a power series expansion in e^2 can be made, it becomes inapplicable at sufficiently small scattering angles. The technique of solving the problem by considerations confined to momentum space appears not to have been worked out and the problem has been attacked therefore by a direct extension of the procedure used by Gordon in his treatment of the nonrelativistic case.

In Sec. 4 the reduction to an equivalent wave equation operator is carried out leading to the operator of Eqs. (16) through (16.4). The relativistic form of the interaction operator shows the presence of the spin-orbit terms in I_2 of Eq. (16.2) and of the tensor type as well as additional scalar type terms in I_3. This expansion has its origin in the possibility of expressing the energy of two fermions interacting through the electromagnetic field to the first order in e^2 and in all other orders regarding retardation effects. This expression has been derived from the Heisenberg-Pauli early form of quantum electrodynamics. The old results are stated in Eqs. (10) and (10.1) of the text. Their application to the present problem is conveniently carried out in the center-of-mass system because of the following circumstances: (a) The unperturbed states referred to by subscripts s, t may be considered to be plane waves of the same energy or else linear combinations of such plane waves. The wavelength corresponding to exchange integrals which enters Eq. (10.1) is therefore infinite and the exchange terms enter with the same factors as the direct interaction terms. This simplification is characteristic of the center-of-mass frame of reference. (b) The Coulomb phase shifts which enter this calculation are formally similar to the phase shifts of the nonrelativistic problem so that the Mott-Gordon solution can be adopted directly for the singlet scattering.

The detailed calculation of phase shifts arising from I_2 and I_3 is avoided by the employment of the momentum space form of the interaction operator, dealt with in Eqs. (12) to (12.8). This procedure suffices for the calculation of effects not covered by the singlet scattering considerations to within the first order of the parameter η. The results are collected in Eqs. (18) to (20.5). The distinction between the treatment of the singlet scattering type effects and the remaining ones is reconsidered in Eqs. (22) to (22′). It is shown in this connection that the momentum space treatment, used as a calculational short cut, is equivalent to a phase shift treatment to within the uncertainty in the knowledge of the phase shifts caused by the terms contained in $I-I_1$ combined with the fact that the operator of Eq. (10) is not supposed to be more accurate than within the first power of e^2. It is not claimed that the part of the scattering matrix obtained through the use of I_1 which applies directly to the singlet scattering and partly to the triplet is exact, but it is shown that through its inclusion one takes care of all progressively varying phase shifts. These phase shifts are caused by the long-range character of the Coulomb field and are therefore related to the infinite small-angle scattering. On the other hand, the discussion of Eq. (21) shows that the phase-shift approach is not interfered with by the infinity in the total scattering cross section. It may be used, therefore, for the calculation of the other infinite part of the cross section which arises in the spin-orbit interaction I_2. The latter introduces a milder infinity than that of I_1 and involves no progressively varying phase shifts.

Garren's considerations regarding the effect of the anomalous part of the proton magnetic moment have been checked by other procedures and, since the results do not differ essentially from Garren's, the inclusion of considerations regarding these effects appears unjustifiable. It may be remarked that in the case of small-angle scattering which is of main importance in the interpretation of data, the problem is approximately a one-body problem because the momentum required by

[3] N. F. Mott, Proc. Roy. Soc. (London) **A118**, 542 (1928); W. Gordon, Z. Physik **48**, 180 (1928).

the particle which is at rest in the laboratory system is small for small-angle scattering. Thus, in the laboratory system, one of the particles may be considered as having a nonrelativistic velocity while the scattered particle is subjected to an essentially static external field. By this procedure the anomalous magnetic moment effects can be understood very simply, and the simplicity of the form of the relativistic η with the entrance of the velocity of the incident proton referred to as v' as in Eq. (13.7) is also immediately obvious. This fact gives one additional confidence in the treatment of the singlet part of the scattering cross section which may be surmised to be more accurate than the present derivation would indicate.

It would appear that the extensive use of Møller's matrix element in modern quantum electrodynamics may require caution, as follows from Sec. 2 below. The present paper brings to a somewhat sharper focus the question of the practicability of expansions arranged strictly in powers of e^2 rather than in terms of combinations such as are encountered in the Mott-Gordon formulas for the nonrelativistic Coulomb wave. The connection of the screened-field considerations with the internal consistency of handling the problem by iteration procedures in momentum space is brought out in the text below in connection with Eqs. (4.1) and (4.3).

The results presented here are subject to many limitations, some of which have already been discussed. Among the additional ones, one may mention the infrared-catastrophe and the wave function distortion effect. The latter may be expected principally in comparisons of n-p and p-p interactions. It can enter in the following ways.

The whole Coulomb effect may be treated in first order only. In this case the phase shifts should be calculated employing wave functions distorted by the nuclear potential as a starting point for Coulomb phase shift calculation. For high enough L the distortion effect will be slight but for the smaller L it is not negligible. For all of the singlet scattering this effect is fully taken into account by calculating the effect of the nuclear potential as a phase shift to be superposed on the Coulombian phase as is usually done in the treatment of anomalies in Coulomb scattering. The same applies to to the part of the triplet scattering matrix arising in I_1.

Alternatively, the Coulomb effect may be treated as though it were known exactly. This is the procedure generally used in nonrelativistic problems. In the relativistic case there would be little justification for calculating wave functions corresponding to all the terms in the effective potential since complications arise in calculations with tensor-force-like terms. Thus it is not likely that more than the singlet-type interaction will be treated exactly, especially since the progressive phase shifts are taken care of in these terms so that there is no practical advantage in incorporating spin-orbit and tensor-type terms in the "exact" part of the calculation. In either case, therefore, one is concerned with wave function distortion effects on the spin-orbit and tensor-type terms. A correction should be made for the difference between the phase shifts caused in these terms with and without distortion. This correction consists in adding to the scattering matrix the difference in the contributions of a singlet (L,J) state with and without wave-function distortion.

The paper is concluded with a comparison of the results of the approximate two-body treatment with the exact relativistic calculations of Mott for the one-body case. The separation of angles displayed in Eqs. (23), (23.1) leads to two sets of phase shifts as in Eqs. (24.2), (24.3). It is then shown by means of Eqs. (27) and (27′) that for small-angle collisions the one- and two-body solutions are in agreement regarding the occurrence of angle-dependent phase factors.

Notation

v = nonrelativistic velocity of incident particle in the laboratory system.
v_I = velocity of either particle in the center-of-mass system.
v' = velocity of incident particle in the laboratory system.
E_I = total energy of either particle in the center-of-mass system.
M = rest of mass of particle.
μ = reduced mass of the interacting particles.
$p=k\hbar$ = momentum of either particle.
$\xi=\mathbf{p}/(E_I+M)$
$\pi=(\boldsymbol{\sigma}\cdot\mathbf{p})$, where $\boldsymbol{\sigma}$ is the Pauli spin matrix for the relevant particle.
$\eta=e^2/\hbar v$.
$\eta_r=e^2/\hbar v'$.
$\rho=kr$.
$\{1/r\}$ = function differing from $1/r$ only in the neighborhood of the origin, where it is defined to be finite.
L = orbital angular momentum quantum number; relativistic states are indexed by the value of L for the large component.
F_L = regular solution of the differential equation for r times the radial wave function. The superscript c refers to this function in the Coulomb field.
G_L = irregular solution corresponding to F_L.
$H_L=G_L+iF_L$.
σ_L = Coulomb phase shift $=\arg\Gamma(L+1+i\eta)$.
K_L = nonrelativistic phase shift for particle without spin.
$\delta_{L,J}$ = phase shift for state of given L, J.
δ^α, δ^β = independent phase shifts arising in coupled L states of same J.
$\psi(\mathbf{r})$ = wave function in coordinate space.
$C_\mathbf{p}$ = wave function in momentum space.
$\mathfrak{F}_{L,J}$ = radial part of the wave function in presence of the interaction.
$Y_{L,\mu-m}$; $\mathcal{Y}^{L,J}{}_\mu$ = angular and spin-angular parts of the wave function, normalized as in Eq. (8.3) and (9.3) of the text.
$\begin{pmatrix} L,J \\ \mu-m,m \end{pmatrix}$ = vector addition coefficient for spin $\frac{1}{2}$.
$\mathbf{s}=\sin(\theta/2)$, where θ is the scattering angle in the center-of-mass system.
$\mathbf{c}=\cos(\theta/2)$.

2. MOMENTUM SPACE

As has been brought out in the Introduction, standard discussions give only a partial answer to the problem of

the Coulomb wave. In the present section the reasons for this deficiency will be discussed. It will be seen that the incompleteness of existing discussions is caused in part by the inconvenience of mathematical manipulations in momentum space. This inconvenience is connected with the infinite forward scattering which is characteristic of the Rutherford formula. A calculation by the usual first-order Born method becomes inapplicable therefore. The relativistic and nonrelativistic problems do not differ in essential respects regarding the difficulty of handling the divergence at small scattering angles in the first Born approximation and the problem will therefore be discussed for the nonrelativistic case in this section.

The wave equation describing the motion of one particle in a Coulomb field and described in coordinate space is

$$\left[-\frac{\hbar^2}{2\mu}\Delta+\frac{e^2}{r}\right]\psi=E\psi. \tag{2}$$

The wave function in momentum space, $C_\mathbf{p}$, introduced by

$$\psi(\mathbf{r})=h^{-\frac{3}{2}}\int C_\mathbf{p}e^{i\mathbf{kr}}d\mathbf{p}, \quad \mathbf{p}=\mathbf{k}\hbar, \tag{2.1}$$

satisfies

$$\left(\frac{p^2}{2\mu}-E\right)C_\mathbf{p}+\int V(\mathbf{p},\mathbf{p}')C_{\mathbf{p}'}d\mathbf{p}'=0, \tag{2.2}$$

where

$$V(\mathbf{p},\mathbf{p}')=\frac{e^2/(\pi h)}{(\mathbf{p}-\mathbf{p}')^2} \tag{2.3}$$

is the matrix element

$$V(r)=e^2/r \tag{2.4}$$

in momentum space. An incoming plane wave

$$\psi^{(0)}=\exp(i\mathbf{k}_0\mathbf{r}) \tag{3}$$

corresponds according to Eq. (3) to

$$C_\mathbf{p}{}^{(0)}=h^{-\frac{3}{2}}\int\exp\{i(\mathbf{k}_0-\mathbf{k})\mathbf{r}\}d\mathbf{r}=\delta(\mathbf{p}-\mathbf{p}_0), \tag{3'}$$

so that the first-order correction to $C_\mathbf{p}{}^{(0)}$ is

$$(E_\mathbf{p}-E-i\epsilon)C_\mathbf{p}{}^{(1)}=-\int V(\mathbf{p},\mathbf{p}')\delta(\mathbf{p}'-\mathbf{p}_0)d\mathbf{p}'$$
$$=-V(\mathbf{p},\mathbf{p}_0), \tag{3.1}$$

where

$$E_\mathbf{p}=p^2/2\mu, \tag{3.1'}$$

corresponding to a first-order correction to the wave function,

$$\psi^{(1)}=-\int V(\mathbf{p},\mathbf{p}_0)(E_p-E-i\epsilon)^{-1}e^{i\mathbf{kr}}d\mathbf{p}. \tag{3.2}$$

The positive small constant ϵ secures the absence of incoming waves in $\psi^{(1)}$ in the usual manner, and one obtains the asymptotic behavior

$$\psi^{(1)}\sim-4\pi^2\mu\hbar V(p\mathbf{r}/r,\mathbf{p}^0)e^{ikr}/r, \tag{3.3}$$

which is often all that is needed. The manipulations with ϵ are, on the other hand, equivalent to a direct solution of Eq. (2) subject to the boundary condition of $\psi\sim\exp(i\mathbf{k}_0\mathbf{r})+$outgoing waves. In order that this method be applicable it is necessary that $C_\mathbf{p}{}^{(1)}$ obtained by means of it be square-integrable, which is the case in many problems. The direct solution of Eq. (2) for the case of a central V gives

$$\psi^{(1)}=\sum_L i^L(2L+1)P_L(\mathbf{k}^0\mathbf{r}/k^0r)F_L{}^{(1)}(kr)/kr, \tag{3.4}$$

with

$$F_L{}^{(1)}(kr)=-(2\mu/k\hbar^2)\Big\{H_L(kr)\times\int_0^r F_L(kr')V(r')F_L(kr')dr'+F_L(kr)\times\int_r^\infty F_L(kr')V(r')H_L(kr')dr'\Big\}, \tag{3.5}$$

where F_L, G_L are regular and irregular functions neglecting the Coulomb interaction, normalized so that $F'G-G'F=1$ and with a difference in asymptotic phase of $\pi/2$ leading to $G_L+iF_L\equiv H_L$ being an outgoing wave. By means of the definition (2.1), one obtains $C_\mathbf{p}{}^{(1)}$ in terms of $\psi^{(1)}$ as

$$C_\mathbf{p}{}^{(1)}=4\pi h^{-\frac{3}{2}}k^{-2}\sum_L(2L+1)P_L(\mathbf{k}^0\mathbf{k}/k^2)\times\int_0^\infty F_L(kr)F_L{}^{(1)}(kr)dr. \tag{3.6}$$

From Eq. (3.5), it follows that

$$F_L{}^{(1)}\sim-(2\mu/\hbar^2k)H_L(kr)\int_0^\infty F_L{}^2(kr')V(r')dr', \tag{3.7}$$

and

$$\psi^{(1)}\sim-[2\mu/\hbar^2k^2](e^{ikr}/r)\sum_L(2L+1)P_L(\mathbf{k}^0\mathbf{r}/kr)\times\int_0^\infty F_L{}^2(kr')V(r')dr', \tag{3.8}$$

while for $|\mathbf{p}|=|\mathbf{p}^0|$, the Legendre function expansion of the exponential in the integral which gives (2.3) in terms of (2.4) yields

$$V(\mathbf{p},\mathbf{p}^0)=(4\pi/h^3k^2)\sum_L(2L+1)P_L(\mathbf{kk}^0/k^2)\times\int_0^\infty F_L{}^2(kr')V(r')dr'. \tag{3.9}$$

Comparison of Eq. (3.8) with Eq. (3.9) gives

$$\psi^{(1)}\sim-(2\pi\mu h/r)e^{ikr}V(\mathbf{p},\mathbf{p}^0) \tag{4}$$

provided the direction of the momentum $\mathbf{p}$ is taken to be the same as that of $\mathbf{r}$. Comparison of Eq. (4) with Eq. (3.3) shows the equivalence of the two considerations. While this equivalence is well known and has been dealt with by Mott and Massey, it nevertheless appeared desirable to present the main steps as a basis for the following discussion. It is also necessary to call attention to another well known fact, *viz.*, the occurrence of phase shifts in their Taylor form (first Born approximation) through the integrals in Eqs. (3.8) and (3.9).

The treatment of the problem in coordinate space by means of Eqs. (3.4) through (3.9) does not apply for a Coulomb field. This fact has been discovered independently by Gordon[3] and Mott.[3] The assumption of the existence of a definite phase shift is implicit in Eq. (3.4) and the result as represented in Eq. (3.8) illustrates, through the presence of first-order phase shifts, the dependence of the Rayleigh-Faxèn-Holtsmark method on this assumption. The coordinate space treatments of the Coulomb field by Mott and Gordon show that solutions of this form do not exist. Accordingly one may expect that the momentum space treatment will not apply either. In fact, according to (3.2) and (2.3), $C_{\mathbf{p}}^{(1)}$ contains the factor $1/(\mathbf{p}-\mathbf{p}_0)^2$ and the probability of scattering having taken place depends on the divergent $\int(\mathbf{p}-\mathbf{p}_0)^{-4}d\mathbf{p}$. This divergence is the result of infinite small-angle scattering. It makes the method inapplicable to the Coulomb potential. For other $V(\mathbf{p}-\mathbf{p}_0)$, such a divergence need not occur and there are many cases in which the method works. The question arises as to whether the divergence interferes with the momentum calculation only formally and whether correct results may not nevertheless be obtainable by means of the $C_{\mathbf{p}}^{(1)}$ of Eq. (3.2). This question is a natural one in view of the success of Møller's treatment[1] of the scattering problem in momentum space. In this treatment the transition probabilities are calculated by means of $|C_{\mathbf{p}}^{(1)}|^2$. One could argue that the consideration can be carried out for a screened Coulomb field as may be done by changing $1/r$ to $e^{-\alpha r}/r$ with a small α. Instead of $(\mathbf{k}-\mathbf{k}_0)^{-2}$, there occurs then $[(\mathbf{k}-\mathbf{k}_0)^2+\alpha^2]^{-1}$ and the result converges. For very small $\alpha=1/a$, however, such a procedure is nevertheless incorrect because if the chance of finding the system in a scattered state becomes comparable with that of finding it in $\mathbf{p}_0$ the calculation becomes inapplicable. For a sufficiently large screening radius a, the chance of small-angle scattering becomes large and the value of e^2 becomes severely restricted. According to Rutherford's scattering formula the number of small-angle scattering processes taking place for a scattering angle $\theta>\theta_0$ is approximately

$$4\pi(e^2/\mu v^2)^2/\theta_0^2. \tag{4.1}$$

If the center of force is enclosed by a screening sphere of radius a, then the minimum deflection θ_0 corresponds to

$$\theta_0\cong e^2/Ea, \tag{4.2}$$

and according to (4.1) and (4.2) the number of particles scattered per second from the wave represented by Eq. (3) is approximately

$$4\pi(e^2/\mu v^2)^2(Ea/e^2)^2=\pi a^2. \tag{4.3}$$

Some details of geometry of the action of the screening sphere are neglected in the simultaneous application of Eqs. (4.1) and (4.2), but it is seen from (4.3) that the incident wave consisting of one particle/(cm^2 sec) has removed from it by action of the Coulomb field inside the screening sphere approximately the number of particles incident on the sphere. It would be wrong therefore to neglect the distortions of the incident wave by the Coulomb wave, and the employment of $[(\mathbf{k}-\mathbf{k}_0)^2+\alpha^2]^{-1}$ is unjustifiable. As a increases, θ_0 decreases and the principal contributions to the scattered intensity arise from very-small-angle scattering. It is clear that in the limit $\theta_0\to 0$ the representation of the incident wave by means of Eq. (3) becomes impossible. The inconsistency of the first order iteration procedure in momentum space becomes even more apparent from the following considerations.

If the momentum-space first order method is used in the calculation of $\psi^{(1)}$ by means of (3.2), the first order correction to the wave function is found to be proportional to an integral containing integration over angles, as in

$$\int(\mathbf{k}-\mathbf{k}_0)^{-2}e^{i\mathbf{k}\mathbf{r}}d\Omega_{\mathbf{k}} = (2\pi/kk_0)\sum i^L(2L+1)P_L(\mathbf{k}_0\mathbf{r}/k_0r) \times Q_L((k^2+k_0^2)/(2kk_0))F_L(kr)/(kr), \tag{5}$$

where the Q_L are Legendre functions of the second kind. Since these contain terms in $\log[(k+k_0)/(k-k_0)]$, the integration in Eq. (3.2) brings in $\log\epsilon$ so that $\psi^{(1)}$ diverges. Since

$$Q_L\left(\frac{k^2+k_0^2}{2kk_0}\right)=\tfrac{1}{2}P_L\left(\frac{k^2+k_0^2}{2kk_0}\right) \times\log\left[\left(\frac{k+k_0}{k-k_0}\right)^2\right]+\cdots, \tag{5.1}$$

where the terms left out contain the P_L only, and since for $k=k_0$ the P_L in this formula becomes unity, the terms in $\log(k-k_0)$ are seen to reproduce the incident wave but with an infinite coefficient. The application of the iteration procedures is seen to be inconsistent. It will be shown in the next section that the difficulties of the direct iteration procedure can be avoided by a change of method.

The relation of Eq. (3.6) to the exact Mott-Gordon solution is seen from the expression for the wave func-

tion in momentum space corresponding to the latter, *viz.*,

$$C_{\mathbf{p}}{}^{c}=4\pi h^{-\frac{3}{2}}\sum_{L}(2L+1)P_{L}(\mathbf{k}\mathbf{k}_{0}/kk_{0})(kk_{0})^{-1}$$
$$\times\exp(i\sigma_{L})\int_{0}^{\infty}F_{L}(kr)F_{L}{}^{c}(k_{0}r)dr, \quad (3.6')$$

where the σ_L are the usual constant Coulomb phase shifts and $F_L{}^c$ is the regular Coulomb function for angular momentum L. If in this formula one sets $\sigma_L=0$ and $F_L{}^c=F_L$ and employs

$$\int_{0}^{\infty}F_{L}(k'r)F_{L}(kr)dr=(\pi/2)\delta(k-k'),$$

there results

$$(C_{\mathbf{p}}{}^{c})^{0}=h^{\frac{3}{2}}\delta(\mathbf{p}-\mathbf{p}_{0}), \quad (3.6'')$$

but for any finite η the δ function is smeared out by the phases in $F_L{}^c$ as well as the σ_L outside $F_L{}^c$. For small η the smearing out is small, however, and in the calculation of perturbing effects one may then replace (3.6') by (3.6'').

It is seen that the exact $C_{\mathbf{p}}{}^c$ can be constructed as in (3.6') and the approximation of Eq. (3.6) and that it is essential to employ in the exact solution the factors $\exp(i\sigma_L)$. The procedure which leads to (3.3) by the momentum-space method fails in the Coulomb case but the phase shifts nevertheless give the coordinate-space and momentum-space wave functions.

3. COULOMB PHASE SHIFT METHOD

It will first be shown that the nonrelativistic Coulomb phase shifts can be calculated by means of a first-order procedure forming an immediate extension of the Taylor formula.[4] By this means, one can then form an immediate extension of Gordon's construction of the Coulomb wave. It will then be shown that a similar procedure may be used relativistically if results to order e^2 only are required. The relativistic treatment is carried out by making use of a simplification in energy expressions arising in the center-of-mass system. A general connection between energy and phase shift will be made use of.

For the nonrelativistic problem for a particle without spin, the phase shift caused by the field inside a sphere of radius r can be calculated by means of[5]

$$K_{L}=-\int(V/E)F_{L}{}^{2}d\rho, \quad (6)$$

where E is the energy of the particle. In the case of two particles, E should be replaced by the energy of relative motion. Making use of

$$e^{2}/rE=2\eta/\rho \quad (6.1)$$

and employing the value of F_0 unperturbed by the Coulomb field, there results

$$\delta K_{0}=-\eta\int_{0}^{\rho}\frac{1-\cos(2\rho)}{\rho}d\rho=-\eta[C+\ln(2\rho)-\mathrm{Ci}(2\rho)],$$

where C is the Euler constant 0.5772..., and $\mathrm{Ci}(2\rho)$ is the cosine integral. For large ρ,

$$\delta K_{0}\sim-\eta[C+\ln 2\rho-(1/2\rho)\sin 2\rho]. \quad (6.2)$$

In the usual notation, the phase of the Coulomb wave at large r is

$$\rho-L\pi/2-\eta\ln 2\rho+\sigma_{L}, \quad \sigma_{L}=\arg\Gamma(L+1+i\eta). \quad (6.3)$$

For $\eta\ll 1$, one has to first order in η the value of the Coulomb phase shift

$$\sigma_{0}\cong-C\eta. \quad (6.4)$$

For sufficiently large ρ, comparison of (6.2), (6.3), and (6.4) shows agreement of the phase for $L=0$. For $\eta=0$, employment of recurrence relations yields

$$(L+1)\int_{0}^{\rho}(F_{L+1}{}^{2}-F_{L}{}^{2})d\rho/\rho=-(F_{L+1}{}^{2}+F_{L}{}^{2})/2. \quad (6.5)$$

Applying this formula to Eq. (6) and making use of the validity of the expression for the phase as in (6.3), it is seen from (6.5) that the addition to the value for $\eta=0$ caused by η in first order is

$$2\eta\int_{0}^{\infty}(F_{L+1}{}^{2}-F_{L}{}^{2})d\rho/\rho=\eta/(L+1)$$
$$\cong\arg(L+1+i\eta)\cong\eta/(L+1). \quad (6.6)$$

This completes the verification of the agreement of phase shifts calculated as a first-order perturbation with those obtained from exact hypergeometric function solutions. Logically the verification could perhaps be dispensed with if it were not for the fact that the phase will be used below, including the $\eta\ln(2\rho)$ part. For large ρ this part of the phase becomes large, while the assumption has been made in the derivation of Eq. (6) that $|\delta K_L|\ll 1$. A verification of the consistency of the method is desirable therefore.

Gordon's paper[3] and a few additional considerations show that the Coulomb wave as generally used may be considered to be formed by means of the following limiting process. The radius R of the screening sphere is made large in comparison with the thickness of the transition region; the latter thickness is kept large in comparison with the wavelength so as to minimize reflections at the transition region. The incident $\psi^{(0)}$ of Eq. (3) is modified by the sphere and its interior. For any R the wave function *inside* the sphere is a constant

[4] H. M. Taylor, Proc. Roy. Soc. (London) **A136**, 605 (1932).
[5] Breit, Condon, and Present, Phys. Rev. **50**, 825 (1936).

of absolute value 1 times a series the terms of which are as in

$$\psi^c=\sum_0^\infty i^L(2L+1)P_L(\cos\theta)e^{i\sigma_L}F_L{}^c/\rho \tag{7}$$

except for the fact that beyond a certain L the terms are modified by the action of the screening sphere and are much smaller in absolute value. The values of L at which this takes place correspond approximately to classical angular momenta at which the screening sphere would be just missed by the incoming particle. The asymptotic form of $F_L{}^c$ is

$$F_L{}^c\sim\sin(\rho-L\pi/2-\eta\ln 2\rho+\sigma_L), \tag{7.1}$$

and the sum on the right side of Eq. (7) is thus seen to be obtainable by regarding $\sigma_L-\eta\ln(2kR)$ in accordance with (7.1) as the phase shift caused by the screening sphere with its interior and then discarding the factor $e^{-i\eta\ln 2kR}$ in the formula for ψ^c. Equation (7) is equivalent to the standard hypergeometric formula for ψ^c and the asymptotic form which gives Rutherford's formula. It may be noted that ψ^c is the wave function inside the Gordon sphere, *i.e.*, in the region of a pure Coulomb field. The applicability of ψ^c to immediate calculation of scattering is contingent on the possibility of escape from the screening sphere without refraction and reflection. It may thus be shown that for $R\to\infty$ the standard use of ψ^c for the calculation of scattering becomes justifiable under most circumstances even though ψ^c does not represent the wave outside the screening sphere.

Employing the first-order phase shifts caused by e^2/r in integrals such as that on the right of Eq. (6), one obtains the phase shifts to first order in η. The possibility of carrying out a calculation to this order does not depend therefore on the availability of exact solutions such as the usual hypergeometric functions. The fact just mentioned might be considered to be trivial were it not for the care which must be exercised in the employment of first-order solutions, which has been discussed in Sec. 2 of the present paper. The result of employing Eq. (7) with first order phase shifts is free of the inconsistencies discussed in connection with the iteration procedure for the wave as a whole. The difficulty of large scattering at small angles is absent in the partial wave analysis method because the wave is constructed for any angle.

In order to extend the method to the relativistic case, it is necessary to be able to calculate the relativistic phase shifts. Since the definition of phase shifts in the relativistic problem does not seem to have been clearly stated in the literature, these will now be introduced. The two-body wave function ψ will be considered in the system of the center of mass. For sufficiently large values of the distance r between the two particles, the wave function satisfies the equation for free particles. In order to secure this condition the particles must lie outside the Gordon sphere, which in the present case is taken in the space of $\mathbf{r}_{II}-\mathbf{r}_I$ of the interparticle displacement vector in the system of the center of mass. These equations are

$$\begin{aligned}(E-2M)\Psi+\pi_I\chi_I+\pi_{II}\chi_{II}=0,\quad (c=1)\\ \pi_I\Psi+E\chi_I+\pi_{II}\varphi=0,\\ \pi_{II}\Psi+E\chi_{II}+\pi_I\varphi=0,\\ \pi_{II}\chi_I+\pi_I\chi_{II}+(E+2M)\varphi=0,\end{aligned} \tag{8}$$

where the velocity of light c has been set $=1$ and where the representation of the Dirac matrices has been taken to be that of Dirac's first papers, since it has the advantage of easy passage to the nonrelativistic limit. The function Ψ is "large" in both particles, χ_I small in I and large in II, χ_{II} large in I and small in II, φ small in both. The π are defined in the list of notation. The function Ψ is just like the nonrelativistic function for two particles with spin and is suitable for the classification of states in the usual spectroscopic terminology of ${}^1S_0$, ${}^3S_1$, ${}^1P_0$, ${}^3P_0$, ${}^3P_1$, ${}^3P_2$, ... states. Disregarding for the moment the phenomenon of coupling between states of different L and the same J, every state ${}^sL_J(s=1,3)$ is characterized by an asymptotic form of Ψ which consists of a linear combination of spin functions ${}^s\chi_m$, angular space functions $Y_{L,\mu-m}$, and a radial space function so that

$$\Psi=\mathcal{Y}^{L,J}{}_\mu\,\mathfrak{F}_{L,J}, \tag{8.1}$$

where

$$\mathfrak{F}_{L,J}\sim\text{const}\sin(kr-L\pi/2+\delta_{L,J})/r, \tag{8.2}$$

and

$$\mathcal{Y}^{L,J}{}_\mu=\sum_m\begin{pmatrix}L,J\\ \mu-m,\ m\end{pmatrix}Y_{L,\mu-m}\chi_m \tag{8.3}$$

is the linear combination of products of space angular functions and spin functions corresponding to total angular momentum J. The asymptotic form of $\mathfrak{F}_{L,J}$ determines the phase shift $\delta_{L,J}$ outside the Gordon sphere. The asymptotic form of Ψ determines through (8) the asymptotic forms of χ_I, χ_{II}, φ as well. The $\mathcal{Y}^{L,J}{}_\mu$ entering these functions are not the same as those of the Ψ and the ${}^sL_J$ classification is seen to give an incomplete view of ψ in much the same way as the specification ${}^2p_{\frac{1}{2}}$ of a Dirac electron in a central field does not mean that the orbital angular momentum quantum number is 1, the specification of the state in terms of good quantum numbers being possible only in terms of Dirac's k and the total angular momentum. In order to identify the state it is not necessary however to use good quantum numbers exclusively, the specification $n\,{}^2p_{\frac{1}{2}}$ having a unique meaning for example. On account of interactions involving spin and space coordinates simultaneously, it is not possible to have in general only states of the form (8.1), states of different L and the same J becoming in general coupled to each other. If one neglects the formation of mesons or any other essentially inelastic processes, it is possible to

describe the system by means of real phase shifts,[6] the same phase shift affecting the phase of the functions for both L by the same amount.

The change in the energy of a system can be connected with the value of a phase shift.[7] In the reference just quoted, this connection has been studied nonrelativistically. Without change of principle some modifications result in a relativistic consideration. A large sphere of radius R_Q is used to quantize the wave function. The four components of Ψ are made to vanish on the surface of the quantizing sphere. The Gordon sphere of radius $R \ll R_Q$ encloses a region in the space of $\mathbf{r}=\mathbf{r}_{\rm I}-\mathbf{r}_{\rm II}$ inside of which electromagnetic and specifically nuclear interactions are located, the region within which the latter take place being supposed to be much smaller than the Gordon sphere.

At a distance R_1 between R and R_Q,

$$R<R_1\ll R_Q, \tag{9}$$

the phase of the function determined by interactions inside R and boundary conditions at $r=0$ has a value $\varphi(R_1)$. The existence of the phase, such as that occurring on the right side of Eq. (8.2) does not depend of course on the possibility of describing the interaction by means of a potential.[6]

Since $R_1>R$ it is always possible to find a region inside the sphere of radius R_1 in which the application of a small static perturbing potential is physically possible. Such a perturbing potential, adiabatically applied, produces a change in energy of the state, the latter being defined by the boundary conditions at $r=0$ and $r=R_Q$, the number of nodes of Ψ between these values of r remaining constant. For values of $r>R$, the phase is a definite concept for all but the higher L for which asymptotic forms such as (8.2) cease to hold. For these the phase can be defined by increasing r. A limitation is reached at $\mu v R_Q \sim 10\hbar L$, but since $\hbar/\mu v$ is of nuclear dimensions the values of L which must be exceeded in order that the concept of phase should break down are seen to be so large as to make this logical difficulty have no practical consequence, the omission of very large L introducing very narrow diffraction patterns. For any J of practical interest one thus has at R_1 a well-defined phase, $\varphi(R_1)$, the sine of which gives the asymptotic form of $\mathfrak{F}$. The perturbing potential δV produces a change $\delta\varphi(R_1)$ in $\varphi(R_1)$ which may be identified with the change in phase shift $\delta(\delta_{L,J})$. Since for an adiabatic change in the wave function the phase at R_Q must be left unchanged and since from R_1 to R the change in phase occurs only through a change in k, one has, considering that $R_Q \gg R_1$,

$$\delta\varphi(R_Q)=R_Q(dk/dE)\delta E+\delta(\delta_{L,J}); \tag{9.1}$$

while, by first-order perturbation theory for energy,

$$\delta E=\int_0^{R_Q}(\psi,(\delta V)\psi)_{sa}r^2dr\Big/\int_0^{R_Q}(\psi,\psi)_{sa}r^2dr, \tag{9.2}$$

where the subscript sa on the scalar product sign means that the scalar product is taken over spin and angles only. Since the normalization integral occurring in the denominator of Eq. (9.2) is contributed to mainly by the region $R_1<r<R_Q$ and since the spin-angular function of Eq. (8.1) may be subjected to the normalization

$$(\mathcal{Y}^{L,J}{}_\mu,\ \mathcal{Y}^{L,J}{}_\mu)_{sa}=1, \tag{9.3}$$

there follows from (9.1)

$$\delta(\delta_{L,J})=-2\frac{dk}{dE}\int_0^\infty(\psi^N,(\delta V)\psi^N)_{sa}d\mathbf{r}, \tag{9.4}$$

provided the normalized function ψ^N has a "large-large" part Ψ such that

$$\langle(\Psi,\Psi)_{sa}\rangle_r\sim(1/2r^2)\langle(\Psi,\Psi)_{sa}\rangle_r/\langle\langle(\psi^N,\psi^N)_{sa}\rangle_r, \tag{9.5}$$

where $\langle\ \rangle_r$ means an average over r taken through a length of many wavelengths. The right side of this equation is arranged to be homogeneous in Ψ and ψ^N taken together and the equation is therefore suitable for normalizing Ψ. This normalization is such that $\langle(\psi^N,\psi^N)_{sa}\rangle_r\sim 1/(2r^2)$, so that the f in the denominator of (9.2) is $R_Q/2$ and (9.4) follows from the requirement $\delta\varphi(R_Q)=0$ and (9.1). From (8), it follows on the other hand that

$$(\psi^N,\psi^N)_s=[4E_{\rm I}^2/(E_{\rm I}+M)^2](\Psi,\Psi)_s, \tag{9.6}$$

where subscript s indicates that the scalar product applies to the spin indices only. From (9.5) it follows that apart from an irrelevant constant factor of absolute value 1,

$$\Psi\sim[(E_{\rm I}+M_{\rm I})/2E_{\rm I}](\mathcal{Y}^{L,J}{}_\mu/r)\times\sin(kr-L\pi/2+\delta_{L,J}). \tag{9.7}$$

In the case of coupled states this requirement has the obvious generalization

$$\begin{pmatrix}\Psi_{L,J^\alpha}\\ \Psi_{L+2,J^\alpha}\end{pmatrix}\sim\frac{E_{\rm I}+M}{2E_{\rm I}r}\mathcal{Y}^{L,J}{}_\mu\begin{pmatrix}a_L\\ a_{L+2}\end{pmatrix}\times\sin(kr-L\pi/2+\delta_{L,J^\alpha}), \tag{9.8}$$

with

$$|a_L|^2+|a_{L+2}|^2=1. \tag{9.9}$$

For each pair of coupled L there are two independent solutions corresponding to two real phase shifts δ^α, δ^β. The existence of two real phase shifts in the sense of (9.8) is immediately obvious from the fact that there exist in the quantizing sphere two linearly independent energy states arising from $J=L+1$ with coupling of L and $L+2$ to each other. In $R_1<r<R_Q$ the change in energy affects k as it occurs in both components $\Psi_{L,J}$

[6] G. Breit, University of Pennsylvania Bicentennial Conference (University of Pennsylvania Press, Philadelphia, 1941).
[7] G. Breit, Revs. Modern Phys. 23, 238 (1951).

and $\Psi_{L+2,J}$; the phase at R_1 is consequently affected equally for both, and since at R_Q the difference in φ caused by a change from L to $L+2$ is equivalent to a change in sign of the wave function, the phase at R_1 for the two components must be the same. In this case, $(\psi^N,(\delta V)\psi^N)_{sa}$ consists of

$$\sum a_m^* a_n(\psi_{m,J}{}^N,(\delta V)\psi_{n,J}{}^N) \tag{9.4'}$$

where the four large-large components of $\psi_{L,J}{}^N$ form $\Psi_{L,J}$ and where $\psi_{L+2,J}{}^N$ and $\Psi_{L+2,J}$ are similarly related. Expression (9.4') is to be substituted in place of the integrand in (9.4), the indices m, n taking on the values L, $L+2$ independently.

The effects of the Coulomb field will now be calculated. The calculation will be made first as though the Coulomb field is present alone. Nuclear interactions are supposed to be superposed on the electromagnetic field and to cause additional phase shifts. The starting point is thus the field-free case, and the phase shift to calculate is that caused by electromagnetic interactions inside the Gordon sphere. According to Eq. (9.1) the quantity δE is of primary interest, the occurrence of δV in Eq. (9.4) being only a consequence of the relationship of δV to δE. The calculation of δE can be carried out rather simply by making use of an old result[8] according to which the operator

$$(e^2/r)(1-\boldsymbol{\alpha}_{\mathrm{I}}\boldsymbol{\alpha}_{\mathrm{II}}) \tag{10}$$

gives energy changes correctly to first order in e^2 provided the exchange integrals $A_{st;ts}$ arising in the treatment of identical particles are modified through the inclusion of factors

$$\cos(2\pi r_{PP'}/\lambda_{st}),\quad \lambda_{st}=c/|E_s-E_t|, \tag{10.1}$$

where s, t designate single-particle states with energies E_s, E_t, and points P, P' are variable points in the coordinate space of the particle over which the double integral representing $A_{st;ts}$ is carried out. An appreciable simplification occurs in the center-of-mass system, which makes it possible to use the operator of Eq. (10) directly.

The center-of-mass system is defined by

$$(\mathbf{p}_{\mathrm{I}}+\mathbf{p}_{\mathrm{II}})\psi=0. \tag{11}$$

The unperturbed wave function is therefore of the form

$$\psi=h^{-\frac{3}{2}}\int C_{\mathbf{p}}\exp\{i\mathbf{k}(\mathbf{r}_{\mathrm{I}}-\mathbf{r}_{\mathrm{II}})t\}d\mathbf{p}. \tag{11.1}$$

It consists of a superposition of plane waves with correlated momenta for particles I and II. The momentum for particle I is $\hbar\mathbf{k}$ and the momentum wave function $C_{\mathbf{p}}$ depends on the 16 combinations of spin indices for the two particles. Since the wave function ψ is supposed to be that for two noninteracting particles in field-free space, the expansion (11.1) obviously exists, as may be verified by constructing the eigenfunctions explicitly in the above form. The analysis is furthermore restricted to one value of $|\mathbf{k}|$ because each elementary solution corresponds to the same total energy,

$$E=2E_{\mathrm{I}}=2[M^2+p_{\mathrm{I}}{}^2]^{\frac{1}{2}}, \tag{11.2}$$

there being no interaction between the particles. This simplification would be absent if the zero-order eigenfunctions were taken for an interacting field which would require the presence of different E_{I} and consequently of different $|\mathbf{k}|$ in the momentum space representation. The first order change in energy caused by the electromagnetic interaction may thus be calculated in the center-of-mass system as though there were no retardation effects, which enter only through the factors (10.1) which are unity since the $E_s-E_t=0$.

In principle the problem is seen to be solved. The phase shifts may be computed by means of (9.4) and (9.4'). They will contain progressively varying parts such as appear in the nonrelativistic problem of Eqs. (6) through (6.6). The phase shifts with such parts give modifications of the wave like that in Eq. (7). There will appear some additional phase shifts which can be treated by copying the usual treatment of anomalies caused in nonrelativistic Coulomb scattering by the presence of non-Coulombian potentials.

In view of the fact that according to (3.6'') the replacement of η by zero leads to a δ function in momentum space, the effect of these additional terms can be calculated in first order in η by replacing the Coulomb wave by a δ function in momentum space. This procedure will be used in the present paper. On the other hand, the effect of phase shift containing terms varying with $\ln r$ will be taken into account by the Gordon sphere construction. This construction yields a Coulomb wave, the δ function approximation to which is used for the calculation of effects of remaining phase shifts.

4. REDUCTION TO EQUIVALENT WAVE EQUATIONS

By means of Eq. (8), one obtains the spin scalar product for the norm as in Eq. (9.6). There is needed also the expectation value of the effective interaction energy. The calculation can be shortened through the partial use of momentum space. The employment of this space does not imply, however, a change of plan. It is only a short cut in the evaluation of the effects of the phase shifts.

For two plane waves ψ, ψ', one obtains for the spin scalar product:

$$(\psi,[1-(\boldsymbol{\alpha}_{\mathrm{I}}\boldsymbol{\alpha}_{\mathrm{II}})]\psi')_s=N(\Psi,J\Psi')_s, \tag{12}$$

where

$$N=[2E_{\mathrm{I}}/(E_{\mathrm{I}}+M)]^2 \tag{12.1}$$

and

$$J=J_1+J_2+J_3+J_4+J_5, \tag{12.2}$$

with

$$NJ_1=1+\xi^2\xi'^2+3(\xi^2+\xi'^2), \tag{12.3}$$

[8] G. Breit, Phys. Rev. 34, 375, 553 (1929).

$$NJ_2=2[((\xi'-\xi)\mathbf{S})^2-\tfrac{1}{3}(\xi'-\xi)^2\mathbf{S}^2], \quad (12.4)$$

$$NJ_3=2\{\tfrac{1}{3}[\xi\times\xi']^2\mathbf{S}^2-([\xi\times\xi']\cdot\mathbf{S})^2\}, \quad (12.5)$$

$$NJ_4=-\tfrac{2}{3}\mathbf{S}^2[(1+\xi^2)(1+\xi'^2)-1+\xi^2+\xi'^2-(\xi\cdot\xi')^2-4(\xi\cdot\xi')], \quad (12.6)$$

$$NJ_5=i[6+2(\xi\cdot\xi')]([\xi\times\xi']\cdot\mathbf{S}), \quad (12.7)$$

$$\xi=\mathbf{p}/(E_I+M), \quad \xi'=\mathbf{p}'/(E_I+M). \quad (12.8)$$

Here $\mathbf{p}$, $\mathbf{p}'$ are respectively the momenta of ψ, ψ'. Since the results are needed only for states of equal energy, $p=p'$. The spin quantum number S, the eigenvalue of $\mathbf{S}^2$, is taken to be the same for the two plane waves and the expressions apply both for singlet and triplet waves. By conservation of parity the consideration of these possibilities suffices.

For the singlets, J_1 is the only nonvanishing part of J. A simple calculation yields

$$J_1=1+(p/E_I)^2. \quad (13)$$

It may be noted incidentally that J_1-1 is the square of the classical velocity of either particle. Since J_1 is independent of the direction of $\mathbf{p}$ and since N multiplies $(\Psi,\Psi)_s$ in Eq. (9.6) just as it multiplies $(\Psi,J\Psi')_s$ in Eq. (12.1), the quantity J_1 enters as a factor in the calculation of the phase shifts, since it enters the integral in Eq. (9.4). The effect of J_1 is seen to arise in the expression for the energy. The factor dk/dE in front of the integral also contains a relativistic effect, as is seen from

$$E=2E_I=2[M^2+\hbar^2k^2]^{\frac{1}{2}}, \quad (13.1)$$

which gives

$$2dk/dE=E_I/(\hbar^2k). \quad (13.2)$$

Nonrelativistically, the latter formula becomes

$$2dk/dE=M/(\hbar Mv/2)=2/(\hbar v), \quad \text{(NR)}. \quad (13.2')$$

The factor $(E_I+M)/(2E_I)$ in (9.7) enters to the second power in the integral in Eq. (9.4) through Ψ but is canceled by N in Eq. (12). The remaining part of (9.7) is just like that of the nonrelativistic problem, but with the relativistic k entering F_L. The two factors containing relativistic effects combine to give

$$(E_I^2+p^2)/\hbar^2E_Ik, \quad (13.3)$$

the nonrelativistic value of which is $M/\hbar p=2/\hbar v$. The singlet phase shift accumulating in the Gordon sphere is thus

$$K_L=-\frac{E_I^2+p^2}{\hbar pE_I}\int F_L^2\frac{e^2}{r}dr. \quad (13.4)$$

In other respects the construction of the modified plane wave takes place as sketched in connection with Eqs. (6.1) to (7.1) and (8) to (9.8). The only differences are in the relativistic k which enters F_L and in the replacement

$$2\eta\rightarrow\left(\frac{E_I^2+p^2}{\hbar pE_I}\right)e^2=2\eta_r, \quad (13.5)$$

as may be seen either by noting that in the nonrelativistic approximation the factor multiplying $\int(F_L^2/r)dr$ is 2η as in Eqs. (6), (6.1) or by comparing the factor (13.3) with its nonrelativistic value. The quantity η_r can be conveniently expressed in terms of v', the velocity of the incident proton as measured in the laboratory system, in which the second proton is initially at rest. The relation of v' to the velocity v_I of either proton measured in the center-of-mass system before the collision is

$$v'=2v_I/(1+v_I^2), \quad v_I=p/E_I. \quad (13.6)$$

Comparison with (13.5) shows that

$$\eta_r=e^2/\hbar v'. \quad (13.7)$$

The singlet scattering is thus described in the center-of-mass system by the Mott-Gordon ψ^c with the replacement of η by η_r and with the relativistic $k=p/\hbar$, replacing the classical k. While it was expedient to relate the relativistic and nonrelativistic cases by noting their formal similarity, an explicit construction in terms of partial waves could be gone through. If it is desired to do so, Eq. (21), which is needed below for another purpose, will be found useful.

Triplet scattering differs from the singlet through the presence of terms involving $\mathbf{S}$, i.e., of J_2, J_3, J_4, J_5 in Eq. (12.2). These quantities are associated with additional phase shifts which play the part of phase shifts caused by deviations from the Coulomb potential in the nonrelativistic theory of anomalous Coulomb scattering. Analyzing the 16-component function in plane waves as in Eq. (2.1), one can make use of Eq. (12) in a calculation of the expectation value of the quantity (10). The quantities $J_2, \cdots, J_5$ appear now in a double integral over $\mathbf{k}$ and $\mathbf{k}'$ which has to be integrated over $\mathbf{r}$ to obtain the phase shifts. Since there are also present the factors $\exp(i\mathbf{k}'\mathbf{r})$ and $\exp(-i\mathbf{kr})$, the components of $\mathbf{p}$ and $\mathbf{p}'$ contained in ξ, ξ' can be replaced by means of $\hbar\partial/i\partial x, \cdots, \hbar\partial/i\partial z'$ provided the order of the factors is arranged as follows:

$$[\exp(-i\mathbf{kr})]\xi_l\cdots \quad \cdots\xi_m'\exp(i\mathbf{k}'\mathbf{r}), \quad (14)$$

the unprimed and primed variables appearing on the left and right respectively. The replacements

$$\xi_l\rightarrow(\hbar/i\partial x_l)/(E_I+M); \quad \xi_m'\rightarrow(\hbar/i\partial x_m')/(E_I+M) \quad (14')$$

can now be made, provided the operators are kept in the order mentioned, next to the respective exponentials. The factor e^2/r can be inserted in the middle of the expression and the expectation value of expression (10) can be calculated so as to obtain the phase shifts. The step from (14) to (14') involves partial integrations with respect to the x_l. The parts outside the integral disappear in the usual manner provided ψ vanishes at ∞ and provided the singularities of the integrand in the finite region of space do not make the part outside the integral arising in the partial integration go through

infinity. In the latter case wrong answers may result. Since $1/r$ is infinite at $r=0$, it will be modified into a finite function

$$\{1/r\}\sim 1/r. \tag{15}$$

This function differs from $1/r$ at very small distances only. Here it is taken to be finite and mildly rounded. Since it is probable that the employment of $1/r$ at very small distances is unjustifiable on account of the proton's interaction with various fields[9] the employment of $\{1/r\}$ is preferable to that of $1/r$. The partial integrations give vanishing parts outside the integrals, there being no need to exclude $r=0$ from the region covered by the integral. The replacement of the ξ is thus justifiable. Integration over $\mathbf{p}$ and $\mathbf{p}'$ gives the wave function. Differential operators which have originated in ξ' and ξ appear respectively to the right and left of $\{1/r\}$. It then becomes convenient to move some of the operators through $\{1/r\}$ to the right, so as to exhibit interactions with familiar physical connotations. It is thus found that

$$(\psi,[1-\boldsymbol{\alpha}_{\mathrm{I}}\boldsymbol{\alpha}_{\mathrm{II}}]\{(1/r)\}\psi)=N\int(\Psi,I\Psi)_s d\mathbf{r}, \tag{16}$$

where the scalar product on the left side of the equation is taken over all variables while that under the integral sign on the right is taken over spin variables only. The quantity I is

$$I=I_1+I_2+I_3, \tag{16.1}$$

where, denoting d/dr by a prime,

$$NI_1=\{1/r\}+p^2\{1/r\}p^2/(E_{\mathrm{I}}+M)^4 +3[p^2\{1/r\}+\{1/r\}p^2]/(E_{\mathrm{I}}+M)^2, \tag{16.2}$$

$$NI_2=2\hbar(E_{\mathrm{I}}+M)^{-2}\left\{3\frac{\{1/r\}'}{r}+(E_{\mathrm{I}}+M)^{-2} \times\left[\frac{\{1/r\}'}{r}p^2-\frac{\hbar^2\partial(\{1/r\}'/r)}{\partial r} \times\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\right]\right\}(\mathbf{L}\cdot\mathbf{s}), \tag{16.2'}$$

$$NI_3=-2\hbar^2(E_{\mathrm{I}}+M)^{-2}\left[\left(\frac{\{1/r\}'}{r}-\Delta\{1/r\}\right)\mathbf{S}^2 +\frac{(\mathbf{r}\cdot\mathbf{S})^2}{r}\frac{d(\{1/r\}'/r)}{dr}\right]+2\hbar^2(E_{\mathrm{I}}+M)^{-4} \times\left\{\frac{d\{1/r\}'/r}{rdr}A+\frac{\{1/r\}'}{r}[p^2\mathbf{S}^2-(\mathbf{p}\cdot\mathbf{S})^2]\right\}, \tag{16.3}$$

with

$$A=\mathbf{L}\cdot(\mathbf{L}\cdot\mathbf{S})\mathbf{S}+i\hbar[(\mathbf{r}\cdot\mathbf{S})(\mathbf{p}\cdot\mathbf{S})-(\mathbf{r}\cdot\mathbf{p})\mathbf{S}^2]. \tag{16.4}$$

[9] W. Heisenberg, Festschrift zur Feier des 200 jährigen Bestehens der Akad. Wiss. in Göttingen (1951); Comm. Pure Appl. Math. 4, No. 1 (1951); Z. Naturforsch. **5a**, 251, 367 (1950); **6a**, 281 (1951).

Here I_1, I_2 correspond, respectively, to J_1, J_5. The effect of I_1 is included in the construction of ψ^c which is identical with that discussed for the singlet state. If I consisted entirely of I_1 the only difference between the singlets and triplets would be in the spin function which can be present as a factor in Ψ. Since $\mathbf{S}$ is contained in I_2 and I_3 the spin dependence becomes more complicated on account of these terms. It may be taken into account by calculating additions to phase shifts caused by I_2 and I_3. These additions will not require another change in the effective value of η such as took place for the singlet state. The reason for this qualitative difference between I_1 and $I-I_1$ is that the latter does not give rise to the progressively varying phase shift terms such as $-\eta\ln 2\rho$, which require the presence of terms in $1/r$ in integrals of the type occurring in (9.4). It is not necessary therefore to modify the Gordon sphere construction in taking these phase shifts into account except for their inclusion in the values of the final phase shifts as in the theory of anomalous Coulomb scattering, modified however by the presence of coupling between states with different L and the same J.

The spin-orbit effects are trustworthy to within the first order in e^2 only. It is therefore not possible to make an exact calculation by means of J of Eq. (12) or I of Eq. (16). The calculation of the effects of (16.2'), (16.3), (16.4) on the phase shifts and hence on the scattering matrix could in principle be carried out by combining the effects of specifically nuclear potentials with those of additions to the Hamiltonian that would reproduce the effects of (16.3), (16.4) exactly and then keeping only first order terms in e^2. Such a procedure would be cumbersome and perturbation methods suggest themselves. This part of the calculation could conceivably be performed in two arrangements: (a) obtaining the effects of the specifically nuclear potential first, including the calculation of the wave function. Having the wave function it is possible to calculate the phase shift changes caused by terms in e^2 and to obtain the anomalies in the scattering arising from terms in (16.2'), (16.3), (16.4). (b) Obtaining by means of (16.2'), (16.3), (16.4) changes in the Coulomb wave to first order in e^2, outside the region of specifically nuclear interactions, calculating the combined effect within the region of nuclear forces, and joining the two branches of the wave function in the usual manner. Procedure (b) is obviously the more elaborate. Procedure (a) can be carried out as soon as there is available a wave function with neglect of the generalized spin-orbit terms. The consideration of this procedure shows that the first-order Coulomb effect as a whole as well as the spin-orbit terms cannot be calculated without the knowledge of the wave function corresponding to specifically nuclear interactions alone. The application of Eq. (6), with V representing the Coulombian and related effects, obviously gives different results depending on whether or not F_L^2 includes the changes owing to the presence

of nuclear potentials. If the calculations were made in the order (b), these effects would be present also because the first-order change in the wave function caused by the spin-orbit forces affects the additional phase shift caused by the nuclear potential.

In applications to high-energy scattering, the Coulomb effects are relatively small and an approximation in the evaluation of the first-order effects is permissible. It is obviously much simpler to calculate the generalized spin-orbit e^2 terms in the absence of the nuclear potential and to add their effect directly to the scattering matrix. This simplification will be used from now on in the present paper as a temporary expedient in securing results of a not too elaborate form. It is possible, however, to take into account the effects of nuclear wave function distortion by subtracting from the scattered wave the first order effect of a particular set of partial waves and adding the change caused by calculating the phase shifts by means of wave functions distorted by the nuclear potential. It may be remarked that the presence of related effects of interaction of meson and Coulomb fields would have to be considered also.

In the calculation of the approximation just introduced, there are present Coulombian effects only. The derivation of I treats all Coulombian effects on the same footing, equal trust being put in all terms. On the other hand, at low energies the $\{1/r\}$ part of I_1 is known to be nearly exact. There is some sense therefore in attaching more significance to I_1 than to I_2, I_3 and similarly to J_1 than to the other J. These nonrelativistically exactly valid terms have been seen to follow from the Gordon sphere construction also in the relativistic case. The approximation under discussion will be used therefore for the remaining terms only. These include tensor-like interactions coupling states of different L and the same J to each other. The application of the phase-shift method to these cases will now be discussed.

The effective energy operator for phase shift calculation multiplied by suitable constants such as are found in Eqs. (6) and (9.4) will be called h'. The fact that for each of the two coupled states α and β there is a definite phase shift δ^α is expressed by

$$(\psi_\mu{}^\alpha, h'\psi_\mu{}^\beta)=\delta_{\alpha\beta}\delta^\alpha, \tag{17}$$

where the $\psi_\mu{}^\alpha$ are the eigenstates corresponding to definite phase shifts. They are expressible as

$$\psi_\mu{}^\alpha=\sum_L a_L{}^\alpha F_L \mathcal{Y}^{LJ}{}_\mu, \tag{17.1}$$

so that

$$\sum_{L\Lambda} a_L{}^{\alpha *} h_{L\Lambda}{}' a_\Lambda{}^\beta=\delta_{\alpha\beta}\delta^\alpha, \tag{17.2}$$

with

$$h_{L\Lambda}{}'=(F_L\mathcal{Y}^{LJ}{}_\mu, h'F_\Lambda\mathcal{Y}^{\Lambda J}{}_\mu). \tag{17.2'}$$

Since according to (17.2') the $h_{L\Lambda}{}'$ is Hermitean, Eq. (17.2) defines an eigenvalue problem for a Hermitean operator, and the $a_L{}^\alpha$ can be made to satisfy the unitary conditions

$$\sum_L a_L{}^{\alpha *} a_L{}^\beta=\delta_{\alpha\beta}, \quad \sum_\alpha a_L{}^{\alpha *} a_\Lambda{}^\alpha=\delta_{L\Lambda}, \tag{17.3}$$

the first of which is necessary for normalization and orthogonality of the $\varphi_\mu{}^\alpha$ in (17.1) such as are required by the connection with the energy value treatment in a large sphere discussed in connection with Eq. (9). The phase shifts δ^α obtainable from (17.2) give a change in the eigenstate

$$\psi_\mu{}^\alpha \rightarrow \sum_L a_L{}^\alpha (F_L+\delta^\alpha H_L)\mathcal{Y}^{LJ}{}_\mu, \tag{17.4}$$

only first order effects in δ^α being taken into account. Multiplying this relation by $a_\Lambda{}^{\alpha *}$ and summing over α, one has

$$F_\Lambda \mathcal{Y}^{\Lambda J}{}_\mu \rightarrow F_\Lambda \mathcal{Y}^{\Lambda J}{}_\mu+\sum_L \left(\sum_\alpha a_\Lambda{}^{\alpha *}\delta^\alpha a_L{}^\alpha\right) H_L \mathcal{Y}^{LJ}{}_\mu. \tag{17.5}$$

Changing L to L' and Λ to Λ' in Eq. (17.2) and then multiplying by $a_L{}^\alpha a_\Lambda{}^{\beta *}$ and summing over α, β by means of (17.3), one finds

$$h_{L\Lambda}{}'=\sum_\alpha a_L{}^\alpha \delta^\alpha a_\Lambda{}^{\alpha *}, \tag{17.5'}$$

and hence

$$F_\Lambda \mathcal{Y}^{\Lambda J}{}_\mu \rightarrow F_\Lambda \mathcal{Y}^{\Lambda J}{}_\mu+\sum_L h_{L\Lambda}{}' H_L \mathcal{Y}^{LJ}{}_\mu. \tag{17.6}$$

The $h_{L\Lambda}{}'$ are thus seen to enter in place of the $\mathcal{T}_{\Lambda L}$ of Eq. (3.2) of Breit, Ehrman, and Hull.[10] In order that $(\mathcal{T})$ be symmetric (h') must be real, which is the case for real F_L and the usual choice of the $\mathcal{Y}^{LJ}{}_\mu$.

Equation (17.6) could have been derived by making a first order Born approximation calculation for the two coupled states, and its form is directly clear from this fact. However, the connection with the Gordon sphere construction would not have been apparent without tracing the transformations as has been done above. The close similarity of (h') to an energy matrix is thus brought into evidence.

5. APPROXIMATE EVALUATION

The evaluation made below is approximate in the sense of neglecting the effect of wave function distortion caused by specific internucleon interactions on the value of the first order effects.

The term J_1 of J of Eqs. (12), (12.3) gave I_1 of I and has been seen to give the Gordon sphere treatment of the Mott-Gordon nonrelativistic formula, provided the quantity η is suitably modified as in Eq. (13.7). The Mott-Gordon wave constructed in this manner will be taken here as the unperturbed wave. The remaining effects will be evaluated by noting that the contributions of $J_2, \cdots, J_5$ are formally like those of a perturbing part of a Hamiltonian in momentum space. One can evaluate the contribution to the scattering matrix S therefore by employing the known form of the part of S caused by J_1 and ascertaining the other contributions by comparison.

Since the effects of $J_2, \cdots, J_5$ are not certain except in order e^2, the part of S caused by J_1 will be used

[10] Breit, Ehrman, and Hull, Phys. Rev. **97**, 1051 (1955).

neglecting all but first order effects in η in this part of S. One has

$$J_5/J_1=i[(E_I+M)(2E_I+M)/(2E_I^2-M^2)]\times\{1-(E_I-M)(1-x)/[2(2E_I+M)]\}\times([(\mathbf{p}/p)\times(\mathbf{p}'/p')]\cdot\mathbf{S}), \quad (18)$$

with

$$x=\cos\theta. \quad (18')$$

Making use of

$$S^c=(-\eta_r/2k\mathbf{s}^2)\exp\{i[\Phi-\eta\ln\mathbf{s}^2]\},\qquad \Phi=kr-\eta_r\ln(2kr)+2\sigma_0, \quad (18.1)$$

as in the paper of Berit and Hull,[11] one finds by means of Eq. (18) that there is a contribution $\Delta\alpha_1$ to the quantity α_1 of Breit and Ehrman[12]:

$$\Delta'\alpha_1=\Delta'\alpha_4=-(e^2k/4E_I)\times\{2(2E_I+M)/[(E_I+M)(1-x)]-(E_I-M)/(E_I+M)\}\exp(-i\eta_r\ln\mathbf{s}^2), \quad (18.2)$$

where the inclusion of $\exp(-i\eta_r\ln\mathbf{s}^2)$ has not been justified by the calculation as presented so far but can be inferred by a consideration of small angle scattering in the laboratory system. The whole term is principally of interest for $\theta\ll1$. For purposes of reference, the scattering matrix will be written out here in the notation of these references:

$$S-S^c=(e^{i\Phi}/k)\begin{pmatrix}\alpha_2, & 2^{-\frac{1}{2}}\alpha_4e^{-i\varphi}\sin\theta, & \alpha_3e^{-2i\varphi}\sin^2\theta\\ 2^{-\frac{1}{2}}\alpha_1e^{i\varphi}\sin\theta, & \alpha_5, & -2^{-\frac{1}{2}}\alpha_1e^{-i\varphi}\sin\theta\\ \alpha_3e^{2i\varphi}\sin^2\theta, & -2^{-\frac{1}{2}}\alpha_4e^{i\varphi}\sin\theta, & \alpha_2\end{pmatrix} \quad (19)$$

rows and columns being labeled with spin magnetic quantum numbers in the order 1, 0, -1 from left to right as well as from top to bottom. In the form of Eq. (12.7), identification with Garren's form[2] of the spin-orbit interaction caused by the Coulomb field is readily made.

For the evaluation of effects of J_2 and J_3 it is convenient to have available

$$((\mathbf{p}'-\mathbf{p})\mathbf{S})^2-\tfrac{1}{3}(\mathbf{p}'-\mathbf{p})^2\mathbf{S}^2=\tfrac{2}{3}p^2\mathbf{s}^2\begin{pmatrix}3\mathbf{s}^2-1, & -3\sqrt{2}\mathbf{sc}e^{-i\varphi}, & 3\mathbf{c}^2e^{-2i\varphi}\\ -3\sqrt{2}\mathbf{sc}e^{i\varphi}, & 2(1-3\mathbf{s}^2), & 3\sqrt{2}\mathbf{sc}e^{-i\varphi}\\ 3\mathbf{c}^2e^{2i\varphi}, & 3\sqrt{2}\mathbf{sc}e^{i\varphi}, & 3\mathbf{s}^2-1\end{pmatrix}, \quad (19.1)$$

$$=2p^2\mathbf{s}^2\mathfrak{M}_2$$

and

$$\tfrac{1}{3}\mathbf{S}^2[(\mathbf{p}/p)\times(\mathbf{p}'/p')]^2-([(\mathbf{p}/p)\times(\mathbf{p}'/p')]\cdot\mathbf{S})^2=-\tfrac{1}{6}\sin^2\theta\begin{pmatrix}-1, & 0, & -3e^{-2i\varphi}\\ 0, & 2, & 0\\ -3e^{2i\varphi}, & 0, & -1\end{pmatrix}. \quad (19.2)$$

$$=\tfrac{1}{2}(\sin^2\theta)\mathfrak{M}_3$$

Employing these matrices and the expressions for J_2, J_3 again in comparison with J_1, one obtains as contributions to S the following combination:

$$(\Delta_2+\Delta_3)S=(e^2/4E_I)e^{i\Phi}\times\{-\mathfrak{M}_2-[(E_I-M)/(E_I+M)]\mathbf{c}^2\mathfrak{M}_3\}. \quad (20)$$

The effect of J_4 is to add a constant to all diagonal matrix elements of S, *viz.*,

$$\Delta S_{mm}=(e^2/4E_I)\times\{(8/3)+(4/3)[(E_I-M)/(E_I+M)]\mathbf{c}^2\}e^{i\Phi}. \quad (20.1)$$

Collecting all contributions except for that in Eq. (18.2) and comparing with the form of S in terms of $\alpha_1, \alpha_2, \cdots, \alpha_5$, one finds for the effect of $J-J_5$:

$$\Delta\alpha_1=\Delta\alpha_4=(e^2k/4E_I), \quad (20.2)$$

$$\Delta\alpha_2/\Delta\alpha_1=3-\mathbf{s}^2+[(E_I-M)/(E_I+M)]\mathbf{c}^2, \quad (20.3)$$

$$\Delta\alpha_3/\Delta\alpha_1=-E_I/[2(E_I+M)\mathbf{s}^2], \quad (20.4)$$

$$\Delta\alpha_5/\Delta\alpha_1=2+2\mathbf{s}^2+2[(E_I-M)/(E_I+M)]\mathbf{c}^2. \quad (20.5)$$

It will be noted that $\Delta\alpha_1, \cdots, \Delta\alpha_5$ are finite at $\theta=0$ and that according to (19) the contributions to S caused by them are finite also. According to Eqs. (18.2) and (19) the contributions to S caused by $\Delta'\alpha_1$ and $\Delta'\alpha_4$ become infinite as $1/\theta$ at $\theta=0$. The question arises as to whether this circumstance vitiates the treatment. Were the treatment based on an equation in momentum space such as Eq. (2.2), there would indeed be doubt regarding the legitimacy of using the terms. The considerations used here are equivalent, however, to a construction of the wave function by means of the Gordon sphere making use of phase shifts, and the question does not arise as is seen by recalling that: (a) The standard nonrelativistic Coulomb wave has a scattered wave part which can be calculated as the result of phase shift effects in the Gordon sphere. (b) The calculation by means of Eq. (12) as in Eqs. (18) to (20.5) gives the effects on S which would exist if the remainder of the wave were a plane wave and is for this reason equivalent to a calculation by means of phase

[11] G. Breit and M. H. Hull, Phys. Rev. **97**, 1047 (1955).

[12] G. Breit and J. B. Ehrman, Phys. Rev. **96**, 805 (1954); see also reference 10.

shifts employing the additions of the phase caused by $J_2, \cdots, J_5$ to the first order in these additional phase shifts, i.e., to the same accuracy to which the starting point of the calculation justifies their use. These two points will now be presented more fully.

Regarding (a), one verifies by standard procedures that excepting $\theta=0$ [see (3.6′), (3.6″)],

$$\frac{\eta \exp\{-i\eta \ln(1-x)\}}{1-x}=i\sum_L (L+\tfrac{1}{2})P_L(x) \times\exp\{i[2(\sigma_L-\sigma_0)-\eta \ln 2\}, \quad (21)$$

and a consideration with the Gordon sphere brings in the quantity on the right side in such a manner as to give the quantity on the left as the well-known factor in the expression for the Coulomb scattered wave. The left side is infinite at small angles but the phase shift construction holds nevertheless. The objection to an infinite amplitude arises only if the scattered wave is calculated by means of the assumption that the incident wave is the only one that need be considered in solving for the scattered wave by first order iteration in momentum space.

The calculation of S took place employing phase shifts. Since the triplet states of odd L for two particles with equal masses are antisymmetric, the phase shifts are applicable to p-p scattering. The matrix S is applicable to nonidentical particles.[11,12] For identical particles in triplet states the convenient matrix is[11,12]

$$S^a(\theta)=[S(\theta)-S(\pi-\theta)]/\sqrt{2}.$$

Regarding (b), the essential features can be described by considering the asymptotic form of the scattered part of Coulomb wave for a spinless particle:

$$\psi_s{}^c\sim-[\eta/k(r-z)]\exp\{i[kr-\eta \ln k(r-z)+2\sigma_0]\} +\sum[(2L+1)/\rho]P_L \sin K_L \times\exp\{i(\rho-\eta \ln 2\rho+K_L+2\sigma_L)\}. \quad (22)$$

For non-Coulombian scattering, an incident e^{ikz} gives a scattered wave

$$\psi_s\sim\sum[(2L+1)/\rho]P_L \sin K_L \exp\{i(\rho+K_L)\}, \quad (22.1)$$

which will be considered for the same values of the small phase shifts K_L as for the Coulomb wave in Eq. (22). These small phase shifts represent the effects of $J_2, \cdots, J_5$, *i.e.*, of I_2+I_3 in (16.3). The calculation by means of $J_2+\cdots+J_5$, employing an incident plane wave and including first order effects only, gives (22.1) which is equivalent to the employment of

$$\sum_L [(2L+1)/\rho]P_L K_L e^{i\rho}. \quad (22.1')$$

Should these phase shifts be used to the same order in (22), their contribution would be

$$\sum_L [(2L+1)/\rho]P_L K_L e^{i(\rho-\eta \ln 2\rho)}e^{2i\sigma_L}. \quad (22')$$

The method used gives the phase shifts to the first order in terms of free wave functions as a starting point —as in Eqs. (6) to (6.6). Being calculated by means of free waves they correspond to the $J_2+\cdots+J_5$ calculation in momentum space. Since these phase shifts are not available more accurately, the strict employment of (22) is not justifiable, for it would imply the knowledge of phase shifts caused by $J_2+\cdots+J_5$ in higher orders. In (22′) the inclusion of the effect of the σ_L produces formally an effect of second order in e^2 or η. Its inclusion would not be justified by the present calculation because corrections to the K_L in the next order of η could offset the refinement caused by the inclusion of the σ_L.

The argument given against the inclusion of higher order effects in the spin-orbit terms has the appearance of disqualifying the solution for singlet scattering, i.e., the part of the solution corresponding to J_1 alone. It is indeed true that more accurate calculations can be expected to give corrections of order η^2 in S and the usual Coulomb wave formula already contains terms of order η^2. On the other hand, the summation of the right side of (21) with the relativistic η is the only solution of the problem including J_1 free of contradiction. If one were to disregard the terms in ξ^2 and ξ^4 entering J_1, there would be left over progressive phase shifts involving $\ln r$ which cannot be managed by a procedure like that in Eqs. (22) to (22′). In this respect J_1 requires a different treatment from the remainder of J.

The inclusion of $\exp(-i\eta \ln s^2)$ in S^c and the validity of S^c to higher than the first order in e^2 for small-angle scattering may be justified by noting that in the laboratory system these collisions appear to a good approximation as though they took place from a fixed center of force. The recoil energy is approximately Θ^2 times the incident energy so that for $\Theta=\frac{1}{6}$ and $E=300$ Mev the recoil proton has an energy under 10 Mev. Scattering from a fixed center has been treated relativistically by Mott.[13] His results as they appear in the papers just quoted are hard to follow for the application made below and have been worked out for an attractive rather than repulsive field of force. They may be presented as follows. The wave function of a Dirac particle in a Coulomb field may be represented in terms of spherical harmonics and radial functions as shown by Darwin.[14] For the special case of total angular momentum having a z component $\frac{1}{2}$, the solutions have the form

$$\psi_L{}^a=\begin{bmatrix} i[L/(2L-1)]^{\frac{1}{2}}Y_{L-1}{}^0 f(r) \\ i[(L-1)/(2L-1)]^{\frac{1}{2}}Y_{L-1}{}^1 f(r) \\ [L/(2L+1)]^{\frac{1}{2}}Y_L{}^0 g(r) \\ -[(L+1)/(2L+1)]^{\frac{1}{2}}Y_L{}^1 g(r) \end{bmatrix}, \quad (J=L-\tfrac{1}{2},\ s=L), \quad (23)$$

$$\psi_L{}^b=\begin{bmatrix} i[(L+1)/(2L+3)]^{\frac{1}{2}}Y_{L+1}{}^0 f(r) \\ -i[(L+2)/(2L+3)]^{\frac{1}{2}}Y_{L+1}{}^1 f(r) \\ [(L+1)/(2L+1)]^{\frac{1}{2}}Y_L{}^0 g(r) \\ [L/(2L+1)]^{\frac{1}{2}}Y_L{}^1 g(r) \end{bmatrix}, \quad (J=L+\tfrac{1}{2},\ s=-L-1), \quad (23.1)$$

[13] N. F. Mott, Proc. Roy. Soc. (London) **A124**, 425 (1929); **A135**, 429 (1932).

[14] C. G. Darwin, Proc. Roy. Soc. (London) **A118**, 654 (1928).

with

$$Y_L{}^0=[(2L+1)/4\pi]^{\frac{1}{2}}P_L(\cos\Theta),$$
$$Y_L{}^1=-[(2L+1)/4\pi]^{\frac{1}{2}}[L(L+1)]^{-\frac{1}{2}}\times e^{i\Phi}\sin\Theta\, P_L{}'(\cos\Theta). \quad (23.1')$$

The Dirac quantum number k is here denoted by s. Here Θ, Φ are polar angles in the laboratory system. The azimuthal angle Φ is here denoted by the same symbol as Φ of Eq. (18.1), no confusion being likely to arise due to this double use of Φ. The original Dirac choice of α matrices is used here but the choice of radial functions is slightly different. If one employs $\hbar/Mc$, Mc^2 as units of length and energy, f and g satisfy

$$\left(\epsilon-1-\frac{\gamma}{r}\right)g+\left(\frac{d}{dr}+\frac{1-s}{r}\right)f=0,$$
$$\left(\epsilon+1+\frac{\gamma}{r}\right)f-\left(\frac{d}{dr}+\frac{1+s}{r}\right)g=0. \quad (23.2)$$

The energy of the single particle is denoted by ϵ. The radial equations have the solutions:

$$f-i[(\epsilon-1)/(\epsilon+1)]^{\frac{1}{2}}g=Cr^{\beta-1}e^{i\kappa r}M\left(\beta+1+\frac{i\epsilon\gamma}{\kappa},\ 2\beta+1,\ -2i\kappa r\right), \quad (23.3)$$

$$f+i[(\epsilon-1)/(\epsilon+1)]^{\frac{1}{2}}g=Cr^{\beta-1}e^{i(\kappa r+\chi)}M\left(\beta+\frac{i\epsilon\gamma}{\kappa},\ 2\beta+1,\ -2i\kappa r\right), \quad (23.4)$$

where

$$\beta=(s^2-\gamma^2)^{\frac{1}{2}},\quad \kappa=(\epsilon^2-1)^{\frac{1}{2}}, \quad (23.5)$$

and

$$e^{i\chi}=\left(\beta+\frac{i\epsilon}{\kappa}\gamma\right)\Big/\left(s+\frac{i\gamma}{\kappa}\right). \quad (23.6)$$

These expressions give different asymptotic forms for cases (a) and (b) of Eqs. (23), (23.1), *viz.*,

$$\rho g^a\sim\sin[\rho-(L\pi/2)-\eta\ln2\rho+\sigma_L{}^a],$$
$$\rho g^b\sim\sin[\rho-(L\pi/2)-\eta\ln2\rho+\sigma_L{}^b], \quad (24)$$

where

$$\eta=\epsilon\gamma/\kappa, \quad (24.1)$$

$$\sigma_L{}^a=(\pi/2)(L-\beta_L)+\arg\Gamma(\beta_L+1+i\epsilon\gamma/\kappa)+\tfrac{1}{2}\tan^{-1}(\gamma/\kappa L)-\tfrac{1}{2}\tan^{-1}(\epsilon\gamma/\kappa\beta_L), \quad (24.2)$$

$$\sigma_L{}^b=(\pi/2)[L+1-\beta_{L+1}]+\arg\Gamma(\beta_{L+1}+i\epsilon\gamma/\kappa)+\tfrac{1}{2}\tan^{-1}(\epsilon\gamma/\kappa\beta_{L+1})-\tfrac{1}{2}\tan^{-1}[\gamma/\kappa(L+1)], \quad (24.3)$$

where

$$\beta_L=(L^2-\gamma^2)^{\frac{1}{2}}.$$

The Mott solution corresponds to taking

$$\psi=(4\pi)^{\frac{1}{2}}\sum_L i^L[L^{\frac{1}{2}}\psi_L{}^a\exp(i\sigma_L{}^a)+(L+1)^{\frac{1}{2}}\psi_L{}^b\exp(i\sigma_L{}^b)]. \quad (25)$$

For small-angle collisions the main contributions arise from large values of L, so that a first approximation is obtained by setting

$$\beta_L\cong L, \quad (26)$$

resulting in

$$\sigma_L{}^a\cong\sigma_L{}^b\cong\arg\Gamma(L+1+i\eta). \quad (26.1)$$

These values inserted in Eqs. (24), (25) give for the third component of ψ,

$$\rho\psi_3\sim\sum_L i^L(2L+1)P_Le^{i\sigma_L}\times\sin(\rho-L\pi/2-\eta\ln2\rho+\sigma_L), \quad (26.2)$$

use having been made of (23), (23.1), and (23.2). The terms in the summation are understood to be modified at large L because for any ρ one can find a sufficiently large L to invalidate the asymptotic forms. For such L the considerations made by means of Eqs. (24.2), (24.3) do not apply. On the other hand, for large L and given ρ the radial equation for g becomes

$$\left[\left(\epsilon+1-\frac{\gamma}{r}\right)\left(\epsilon-1-\frac{\gamma}{r}\right)-\frac{\gamma/r^2}{\epsilon+1-\gamma/r}\times\left(\frac{d}{dr}+\frac{1+s}{r}\right)+\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{s(s+1)}{r^2}\right]g=0. \quad (26.3)$$

The last three terms in this equation are just like corresponding terms in the nonrelativistic equation including these terms as well, and the combinations with terms in ϵ^2-1 and $-2\epsilon\gamma/r$ resulting from the first term in the equation may be written as

$$\left[1-\frac{2\eta}{\kappa r}+\frac{d^2}{d(\kappa r)^2}+\frac{2d}{(\kappa r)d(\kappa r)}-\frac{s(s+1)}{(\kappa r)^2}\right]g=0, \quad (26.4)$$

with η as in (24.1). This equation is just like the nonrelativistic one except for the occurrence of the relativistic η and κ. The asymptotic forms obtained from Eqs. (24), (24.2), (24.3) in the approximation of Eq. (26) are seen to be consistent with it. This circumstance may be expected from the fact that for large $L\cong s$ the region with $r<$ than $r\cong|s|$ is not important. For large $|s|\cong r$, however, the terms which have been omitted in (26.3) in order to obtain (26.4) are of relative order γ/s^2, some of them involving ϵ besides. The approach of the relativistic phases to the classical ones is thus readily understandable.

As L increases in Eq. (26.2), it reaches a value $\sim r$ after which the terms as written have to be modified because of the inapplicability of asymptotic forms. The discussion for large L which has just been gone through in connection with Eqs. (26.3), (26.4) shows that the difference in the left sides of these equations is of the order γ/L^2 for values of r close to the classical turning point of the centrifugal barrier. The decrease in the absolute value of the omitted terms takes place therefore in nearly the same manner in the nonrela-

tivistic and relativistic cases, and the agreement is improved as $\Theta\to 0$ because contributions from the small L become increasingly less important. The right side of Eq. (26.2) may be replaced therefore by the nonrelativistic formula for the Coulomb wave. In the small-angle approximation the Coulomb wave contains the factor $\exp\{-i\eta\ln(1-\cos\Theta)\}$. This factor is related to the phases obtained in the c.m. system by

$$2\mathbf{s}^2=(\gamma+1)\sin^2\Theta/[1+\tfrac{1}{2}(\gamma-1)\sin^2\Theta]\cong(\gamma+1)[1-\tfrac{1}{2}(\gamma-1)\Theta^2]\sin^2\Theta, \quad (27)$$
$$\delta=(1-v'^2/c^2)^{-\frac{1}{2}}.$$

For small scattering angles, one has therefore

$$\exp\{-i\eta\ln\sin^2\Theta\}\cong[1-\tfrac{1}{2}(\gamma-1)i\eta\Theta^2]\times\exp\{-i\eta\ln[2\mathbf{s}^2/(\gamma+1)]\}, \quad (27')$$

and to a good approximation the factor $\exp(-i\eta\ln\mathbf{s}^2)$ is reproduced. On the other hand there is seen to be a term in $\eta\Theta^2$ coming in as a correction so that exact agreement is not proved. The calculation just presented is very similar to that which gave Eq. (5.27) of Mott's second paper. The role played by large L for small Θ would not have been clear however with a direct use of Mott's paper.

The consideration of the small-angle scattering in the laboratory system is seen to support conclusions drawn from the two-body phase shift approach.

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Phase Shifts for Relativistic Corrections in High-Energy p-p Scattering*

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Scattering phase shifts for the relativistic corrections to the ordinary Coulomb interaction between two protons are calculated in the first Born approximation. The scattering matrix resulting from these phase shifts is obtained and shown to agree with the results of the preceding paper.

In the preceding paper,[1] the relativistic corrections to the Coulomb scattering of two identical particles were discussed in terms of the equation derived by Breit for first order changes in the energy.[2] It was shown there that the matrix element of the interaction energy of the two particles, H', may be written as

$$H'=e^2\left(\psi^*\frac{1-\boldsymbol{\alpha}_{\rm I}\boldsymbol{\alpha}_{\rm II}}{r}\psi'\right)=e^2\left(\Psi^*\frac{1}{r}\{1+\xi^2\xi'^2+3(\xi^2+\xi'^2)+2i\{3+(\boldsymbol{\xi}\cdot\boldsymbol{\xi}')([\boldsymbol{\xi}\times\boldsymbol{\xi}']\cdot\mathbf{S})\}-2(\boldsymbol{\xi}-\boldsymbol{\xi}')^2\mathbf{S}^2+2[(\boldsymbol{\xi}'-\boldsymbol{\xi})\cdot\mathbf{S}]^2-2([\boldsymbol{\xi}\times\boldsymbol{\xi}']\cdot\mathbf{S})^2\}\Psi'.\right) \quad (1)$$

The notation is the same as that in the preceding paper; in particular $\boldsymbol{\xi}$ is related to the momentum of one of the particles in the center-of-mass system by

$$\boldsymbol{\xi}=\mathbf{p}/(E_{\rm I}+M),$$

the primed and unprimed symbols referring to values appropriate to the incident and final wave function, respectively. Furthermore, the spinor Ψ is that component of the wave function of the relative motion ψ which is large if both particles are of positive energy. In the treatment of the preceding paper,[1] the singlet and main nonspin dependent triplet terms were treated by means of phase shifts and in coordinate space, while the remaining terms were evaluated in first Born approximation in momentum space after this procedure was shown to be equivalent, to first order in e^2, to a phase shift treatment. It is of some interest to consider these spin dependent terms also in terms of phase shifts. By doing so a verification of the argument regarding the equivalence of the direct phase shift treatment and the momentum space calculations is provided and the values of the phase shifts which are modified by specifically nuclear forces are made available. If the initial and final scattering states are represented by configuration-space wave functions, a corresponding expression may be written for H', with $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ replaced by their operator equivalents. Regrouping terms in Eqs. (16)–(16.4) of B one obtains for the matrix element

$$H'=H_a'+H_b'+H_c'+H_d', \quad (2)$$

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[1] G. Breit, preceding paper. Henceforth, this paper is referred to as B.
[2] G. Breit, Phys. Rev. **34**, 553 (1929).