# Simple Nonrelativistic Model for Single Meson Production*

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Single pseudoscalar meson production is assumed to be described by a transition matrix element taken between initial and final two-nucleon wave functions defined as solutions to Schrödinger's equation with a phenomenological potential included. The effective interaction is a linear combination of the two possible nonrelativistic invariants. The angular distribution and energy dependence is considered in detail for the reaction $p+p\rightarrow\pi^{+}+d$. Using the Jastrow potential to define the proton wave functions, the large anisotropy can be explained by including the $D$ state of the deuteron. Results become less satisfactory at increasing energies. The predicted energy dependence of the total cross section for proton beam energies from 311 Mev to 515 Mev agrees well with experiment. The strong dependence of the cross section on the incident proton momentum contributes to the agreement.

An order of magnitude estimate of the alternate reaction, $p+p\rightarrow\pi^{+}+n+p$, predicts, at 340 Mev (laboratory system), a branching ratio slightly greater than one.

Finally, the absorption rate of $\pi^{-}$ mesons by deuterons from a $K$-shell orbit, i.e., $\pi^{-}+d\rightarrow 2n$, is calculated. This gives an estimate of the strength of the $S$ meson wave part of the interaction and is consistent with the restrictions imposed by the results obtained for the reaction $p+p\rightarrow\pi^{+}+d$.

## 1. INTRODUCTION

THIS paper is an attempt to understand single meson production or absorption due to nucleon-nucleon collisions near threshold in terms of a simple, nonrelativistic model. Since, by far, most of the experimental information available is on the process

$$p+p\leftrightarrows\pi^{+}+d, \tag{1.1}$$

we consider this reaction in most detail. We also estimate the order of magnitude of the alternate reaction,

$$p+p\rightarrow\pi^{+}+n+p, \tag{1.2}$$

that is predicted by the model. As a final check on the consistency of our assumptions, we calculate the absorption rate of negative pions by a deuteron from a $K$-shell Bohr orbit,

$$\pi^{-}+d\rightarrow n+n. \tag{1.3}$$

The latter reaction is of particular significance since it involves the interaction with the meson in a pure $S$ state.

A summary of all the available data on pion production by nucleons is given in a recent paper by Rosenfeld.[1] In particular, the important features of reaction (1.1) are a rapid increase of the total cross section with energy (roughly, $\sigma(\pi^{+},d)$ increases a little less rapidly than $\eta^{3}$, the cube of the meson momentum) and a strong anisotropy in the angular distribution of the mesons in the center-of-mass system. More precisely, we can write

$$d\sigma/d\Omega_{q}=A(1+\xi\cos^{2}\theta), \tag{1.4}$$

with $3\lesssim\xi\lesssim 6$ for energies ranging from 324 Mev to 515 Mev (in the laboratory system). $\theta$ is the angle between the outgoing meson and the proton beam, in the center-of-mass frame. Unfortunately, the experimental errors in measuring $\xi$ are rather large, usually varying from 25 to 100%. The total cross section is known to better accuracy (see Fig. 1). There is relatively little data available on reaction (1.2). The angular distribution is probably roughly the same as for reaction (1.1) while the total cross section, $\sigma(\pi^{+},n,p)$, increases more rapidly than $\sigma(\pi^{+},d)$. Two measurements of the branching ratio at 340 Mev give[1]

$$\sigma(\pi^{+},n,p)/\sigma(\pi^{+},d)=0.82\pm0.25;\ 0.54\pm0.18. \tag{1.5}$$

Since the latter measurement gave absolute cross sections that are inconsistent with other measurements in that energy region, we would expect the higher value to be more probable.

The model, first suggested by Chew, Goldberger *et al.*,[2] consists of writing the transition matrix element for the process in the form

$$\mathfrak{M}=\langle\psi_{f}|T|\psi_{i}\rangle, \tag{1.6}$$

where $\psi_{f}$, and $\psi_{i}$ are nonrelativistic wave functions describing the two-nucleon states before and after the production (or absorption) of a meson. They are defined as solutions to Schrödinger's equation with phenomenological potentials included, chosen to give the correct properties of the deuteron and two nucleon scattering at the energies involved. $T$ is chosen to be the simplest possible interaction operator that is a nonrelativistic invariant function of the nucleon operators, $\boldsymbol{\sigma}^{(i)}$, $\boldsymbol{\nabla}_{i}$, and $\boldsymbol{\tau}^{(i)}$, where $\boldsymbol{\sigma}^{(i)}$ represents the three nonrelativistic Pauli spin matrices operating on the $i$th nucleon and $\boldsymbol{\tau}^{(i)}$ the corresponding isotopic spin matrices. $T$ is assumed to be a linear function in the

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[1] A. H. Rosenfeld, Phys. Rev. **96**, 130, 139 (1954).

[2] Chew, Goldberger, Steinberger, and Yang, Phys. Rev. **84**, 581 (1951).

meson wave function, $\boldsymbol{\phi}$, which is chosen to be a pseudoscalar. Assuming charge independence,

$$T=\alpha\left[\sum_{i=1}^{2}\boldsymbol{\sigma}^{(i)}\cdot\boldsymbol{\nabla}_i[\boldsymbol{\tau}^{(i)}\cdot\boldsymbol{\phi}(i)]\right]+\beta\left[\sum_{i=1}^{2}[\boldsymbol{\tau}^{(i)}\cdot\boldsymbol{\phi}(i)]\boldsymbol{\sigma}^{(i)}\cdot\boldsymbol{\nabla}_i\right]. \quad (1.7)$$

$\boldsymbol{\phi}(i)$ shall usually be chosen to be a plane wave evaluated at the coordinate of the $i$th nucleon. $\alpha$ and $\beta$ are parameters to be chosen by experiment and may be energy-dependent. Whatever factors are necessary to give a normalization of one meson per unit volume are assumed to be included in the $\alpha$ and $\beta$. We do not restrict ourselves to the usual field-theoretic normalization factor of $1/\sqrt{E_q}$, where $E_q$ is the meson total energy. It is hoped that the nonlocal features of the process might be absorbed in the energy dependence of $\alpha$ and $\beta$. It is found that the best fit of the energy dependence of reaction (1.1) is obtained with $\alpha$ and $\beta$ energy-independent.

The requirement that the $T$ be Hermitian leads to the conditions,

$$\alpha=a+\tfrac{1}{2}ib,\quad \beta=ib;\quad a,b \text{ real.} \quad (1.8)$$

Since the transition probability per unit time is given by

$$\omega=(2\pi/\hbar)\int\sum|\mathfrak{M}|^2\rho(E_f)\delta(E_i-E_f)dE_f, \quad (1.9)$$

where $\sum|\mathfrak{M}|^2$ represents the appropriate sum and average over nucleon spin states, we can define the dimensionless quantities $g_a^2$ and $g_b^2$;

$$g_a^2=(a/\hbar c)^2(\mu c/\hbar)^3;\quad g_b^2=(b/\hbar c)^2(\mu c/\hbar)^3. \quad (1.10)$$

Finally, it should be noted that the term in $T$ proportional to $\alpha$ leads to $P$-wave meson emission, while the $\beta$ term leads to predominantly $S$-wave emission. Since the experimental results indicate that $P$-wave emission is strongly favored, we shall require the condition on $\alpha$ and $\beta$,

$$|\alpha|\gg|\beta|. \quad (1.11)$$

## 2. CALCULATION OF $p+p\rightarrow\pi^++d$

We evaluate $\sum|\mathfrak{M}|^2$ in the center-of-mass frame where the total momentum of the system is zero. Using the coordinates, $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, $\mathbf{R}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$, our final nucleon state, a deuteron, is represented by the wave function,

$$|\psi_f{}^m\rangle=e^{i\mathbf{k}\cdot\mathbf{R}}(4\pi r_D)^{-\frac{1}{2}}r^{-1}[u(r)+8^{-\frac{1}{2}}S_{12}w(r)]\chi_1{}^m\Lambda_0. \quad (2.1)$$

$\Lambda_0$ is the antisymmetric, isotopic spin zero eigenstate for the two nucleons. We must choose the antisymmetric state so that $\psi_f$ will be totally antisymmetric in space, spin, and isotopic spin variables. $S_{12}$ is the usual tensor operator, so that $S_{12}\chi_1{}^m$ represents the ${}^3D_1$ angular momentum state. $r_D$ is given by

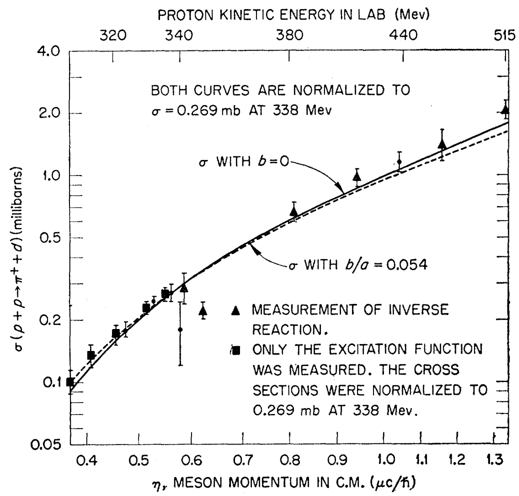


FIG. 1. Comparison of theoretical curves with experimental values of the total cross section for the reaction $p+p\rightarrow\pi^++d$. The solid curve assumes only $P$ meson wave emission in the model used. The dotted curve assumes enough $S$ meson wave emission to raise curve at lower energies to improve agreement.

$$r_D=\hbar(MB)^{-\frac{1}{2}}=4.31\times10^{-13}\text{ cm},$$

where $M$ is the nucleon mass and $B$ is the binding energy of the deuteron. The normalization of one deuteron per unit volume leads to the condition

$$\int[u^2(r)+w^2(r)]dr=r_D;$$

and, since $r_D$ is a rough measure of the size of the deuteron, $u$ and $w$ are of the order of magnitude of one.

By writing $\phi(i)$ as $e^{-i\mathbf{q}\cdot\mathbf{r}_i}$, using (2.1) for $\psi_f$, and denoting the initial proton state, without isotopic spin included, by $\psi_p$, $\mathfrak{M}$ becomes, after isotopic spin and center-of-mass coordinate integrations,

$$\begin{aligned}\mathfrak{M}=&(4\pi r_D)^{-\frac{1}{2}}\langle r^{-1}[u(r)+8^{-\frac{1}{2}}S_{12}w(r)]\chi_1{}^m|i\alpha\\&\times\cos(\mathbf{q}\cdot\mathbf{r}/2)(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{q}+\alpha\sin(\mathbf{q}\cdot\mathbf{r}/2)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\\&\cdot\mathbf{q}-\beta\cos(\mathbf{q}\cdot\mathbf{r}/2)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}+i\beta\sin(\mathbf{q}\cdot\mathbf{r}/2)\\&\times(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla}|\psi_p\rangle(2\pi)^3\delta(\mathbf{k}+\mathbf{q}). \quad (2.2)\end{aligned}$$

Since the final nucleon state is a deuteron with fixed parity and angular momentum, the selection rules are particularly simple if we restrict the meson to only $S$ and $P$ wave emission. In (2.2) this is equivalent to expanding $\sin(\mathbf{q}\cdot\mathbf{r}/2)$, $\cos(\mathbf{q}\cdot\mathbf{r}/2)$ in spherical harmonics and neglecting terms in $q^2$ and higher. It turns out that, in this model, the effective parameter in the expansion in powers of $q$ is $q/k_0\approx\frac{1}{5}$, $k_0$ the incident proton wave number. Hence, neglecting terms in $(q/k_0)^2$ and higher amount to errors of not more than a few percent. There errors will be of the same order as those arising from relativistic effects. With this restriction, the only transition that can contribute to the triplet matrix element, $\mathfrak{M}_t$, (when the initial protons are in the triplet state) is from the $^3P_1$ proton state with $S$-wave meson emission. In (2.2), this arises from the term in $T$,

$$\beta \cos(\mathbf{q}\cdot\mathbf{r}/2)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{\nabla},$$

which, since it leads to a term independent of $q/k_0$, will result in a large isotropic contribution to $d\sigma/d\Omega$ unless $\beta$ is chosen small compared to $\alpha$. If (1.11) is kept in mind, the major contribution to $\mathfrak{M}_s$, the singlet transition, comes from the term proportional to $\alpha$. Only the $^1S_0$, $^1D_2$ proton states can contribute to $\mathfrak{M}_s$.

If we expand the proton wave functions in partial waves, we need only retain the $S$ and $D$ singlet states and the triplet $P$ state with total angular momentum one. These can be written, denoting the singlet and triplet states of $|\psi_p\rangle$ by $|\psi_s\rangle$, $|\psi_t{}^{m'}\rangle$ respectively,

$$\psi_s\rangle=[2e^{i\delta_0}(k_0r)^{-1}u_0(r)-10e^{i\delta_2}(k_0r)^{-1} \times u_2(r)P_2(\theta)+\cdots]\chi_0\rangle, \quad (2.3\text{a})$$

$$\psi_t{}^{m'}\rangle=[2ie^{i\delta_1}(k_0r)^{-1}u_1(r)\tfrac{3}{2}(k_0r)^{-1} \times(\mathbf{S}\cdot\mathbf{r})(\mathbf{S}\cdot\mathbf{k}_0)+\cdots]\chi_1{}^{m'}\rangle, \quad (2.3\text{b})$$

$\mathbf{S}=\frac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$ and $u_l$ has been defined such that

$$u_l \underset{r\to\infty}{\longrightarrow} \sin(k_0r-\pi/2l+\delta_l).$$

We have used the method of Rarita-Schwinger[3] to represent the $^3P_1$ angular momentum state in (2.3b). Substituting (2.3a), (2.3b) in (2.2) and squaring, we obtain, after performing the angular integrations and spin sums,

$$\begin{aligned}\tfrac{1}{4}\textstyle\sum_m|\mathfrak{M}_s|^2+\tfrac{1}{4}\textstyle\sum_{m,m'}|\mathfrak{M}_t|^2 &=(16\pi/r_D)k_0^{-2}\{(q/k_0)^2|\alpha|^2[(S_0)^2+\tfrac{1}{2}(D_2)^2\\ &\quad-\sqrt{2}(S_0D_2)\cos(\delta_2-\delta_0)+(\tfrac{1}{2}(D_2)^2+\sqrt{2}(S_0D_2)\\ &\quad\times\cos(\delta_2-\delta_0))3\cos^2\theta]+2|\beta|^2(P_1)^2\}. \quad (2.4)\end{aligned}$$

The radial integrals in (2.4) are defined as

$$S_0=\int_0^\infty u_0(x)u(x)j_0(qx/2k_0)dx, \quad x=k_0r$$

$$S_2=\int_0^\infty u_2(x)u(x)j_0(qx/2k_0)dx,$$

$$D_2=\int_0^\infty u_2(x)w(x)j_0(qx/2k_0)dx,$$

$$D_0=\int_0^\infty u_0(x)w(x)j_0(qx/2k_0)dx,$$

$$P_1=\int_0^\infty u_1(x)\frac{d}{dx}\left[\frac{u(x)}{x}\right]xj_0(qx/2k_0)dx. \quad (2.5)$$

[3] W. Rarita and J. Schwinger, Phys. Rev. **59**, 550 (1941).

For the sake of conciseness, we have left out of (2.4) terms arising from $|\mathfrak{M}_s|^2$ containing $\beta$ (and the integrals $S_2$, $D_0$), and terms arising from $|\mathfrak{M}_t|^2$ containing $\alpha$. With $|\beta|\ll|\alpha|$, these terms are less important but would greatly complicate (2.4). The effect of these terms has been investigated and will be discussed later. As discussed earlier, we have consistently dropped terms involving higher powers of $q$. We have made fairly crude approximations in terms involving $\beta$. Integrals of the form

$$\int_0^\infty u_0(x)xj_0(qx/2k_0)\frac{du(x)}{dx}dx,$$

that appear in the singlet terms have been dropped. These are about twenty-five percent of integrals like $S_0$. Terms in the triplet transition involving the deuteron $D$ state have also been dropped. These approximations are justified since the condition (1.11) on $\beta$ makes the over-all error in (2.4) much smaller. A more careful calculation would not be consistent with the crudeness of the model and the large uncertainties in the experimental data.

With these approximations, it is clear that (2.4) results in a $\cos^2\theta$ dependence only if we include the $D$ state of the deuteron. Since the percentage of $D$ state is small, we might expect $D_2$ to be no more than twenty percent of $S_0$ so that the value of $\xi$ [see Eq. (1.4)], predicted by our model, would be too low. This is not the case. The fact that the proton wave functions oscillate many times in the region where the deuteron wave function is large enables $D_2$ to compete with $S_0$. This is because the $D$ state proton wave function, $u_2$, which is coupled to the deuteron $D$ state, has a longer wavelength than $u_0$ in the region where $w(x)$ is maximum. In particular, this is true if we replace $u_0$, $u_2$ by Born approximation wave functions and approximate $u$ and $w$ by Hulthén wave functions. For example, at 340 Mev we find

$$S_0=0.45;\quad D_2=0.42 \text{ (Born approximation).}$$

Here we approximated $w(x)$ by $\frac{1}{5}u(x)$.

If we examine (2.4) more carefully, however, we see that $\xi$ is also sensitive to the difference between the $D$ and $S$ proton phase shifts, $\delta_2-\delta_0$. Therefore, the only consistent approach would be to use the "exact" wave functions for $u_0$ and $u_2$ in evaluating the integrals (2.5). It is found that the effects of an interaction on $u_0$ and $u_2$ can seriously modify the Born approximation results.

It should be noted that our model does not lead to a $D$ to $S$, or $S$ to $D$ singlet to triplet nucleon transition if we neglect terms in $\beta$. (These transitions are represented by the integrals $S_2$, $D_0$, for example.) These transitions would be proportional to $j_2(qx/2k_0)$, which is negligible in the energy range we consider, i.e., threshold to 515 Mev in the laboratory system. It is interesting that it is not necessary to include the $D$ to $S$ transition to obtain a large anisotropy in the angular distribution.

We come now to the evaluation of the radial integrals. The proton wave functions are defined by using the Jastrow singlet and triplet potentials[4] in Schrödinger's equation.

Singlet well:

$$V=\infty,\quad r<r_0,\quad r_0=0.60\times10^{-13}\text{ cm},$$
$$V=-V_0\exp(r-r_0/r_s),\quad r>r_0,\quad r_s=0.40\times10^{-13}\text{ cm},$$
$$V_0=375\text{ Mev.}\tag{2.6a}$$

Triplet well:

$$V=-V_tS_{12}\exp(-r/r_t),\quad r_t=0.75\times10^{-13}\text{ cm},$$
$$V_t=50.8\text{ Mev.}\tag{2.6b}$$

While this well is not completely satisfactory, it gives the best available fit of high energy proton-proton scattering data.[5] While the poor agreement with polarization experiments[5] casts considerable doubt on the validity of the triplet well, the general features of the singlet well, i.e., the hard core and deep, long-tailed well, are of particular interest. The use of (2.6) is further justified by the fact that the transition from the proton singlet state gives the major contribution to the process.

Solutions for the wave functions $u_0$, $u_2$ and their phase shifts were found numerically at laboratory energies of 340, 372, and 437 Mev. This corresponds to energies, in the center-of-mass frame, of 163, 182, and 207 Mev, with corresponding meson energies of 22, 40, and 62 Mev. Numerical tables of the deuteron wave functions were obtained from the work of Feshbach, Schwinger, Harr,[6] and Pease.[7] They use a combination of tensor and central Yukawa wells,

$$\begin{aligned}V(r)&=-V_0\{[1-\tfrac{1}{2}g+\tfrac{1}{2}g(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]e^{-r/r_c}+\gamma S_{12}e^{-r/r_t}\},\\ V_0&=46.96\text{ Mev},\quad g=0.005,\quad \gamma=0.5085,\\ r_c&=1.18\times10^{-13}\text{ cm},\quad r_t=1.70\times10^{-13}\text{ cm}.\end{aligned}\tag{2.7}$$

The five parameters were obtained from comparison with experimental values for the binding energy of the deuteron, its quadrupole moment, and the triplet and singlet neutron-proton scattering lengths and effective ranges. Charge independence of nuclear forces is assumed to obtain the central force range $r_c$ from proton-proton low-energy scattering data. The tensor range, $r_t$, is fixed by the work of Pease and Feshbach[7] on the binding energy of triton. The percentage of $D$ state obtained is 3.1 percent. Equation (2.7) fails to predict high-energy scattering data. The apparent inconsistency in the well shapes (2.7) and (2.6) is not very serious, especially since the shapes and relative amplitudes of $u(r)$ and $w(r)$ are largely independent of the potential wells assumed, provided they yield the correct properties of the deuteron.

[4] R. Jastrow, Phys. Rev. **81**, 165 (1951).
[5] L. J. B. Goldfarb and D. Feldman, Phys. Rev. **88**, 1099 (1952).
[6] Feshbach, Schwinger, and Harr, Harvard University Report HUX-5. Computation Laboratory, Harvard, 1949 (unpublished).
[7] R. L. Pease and H. Feshbach, Phys. Rev. **78**, 135 (1950).

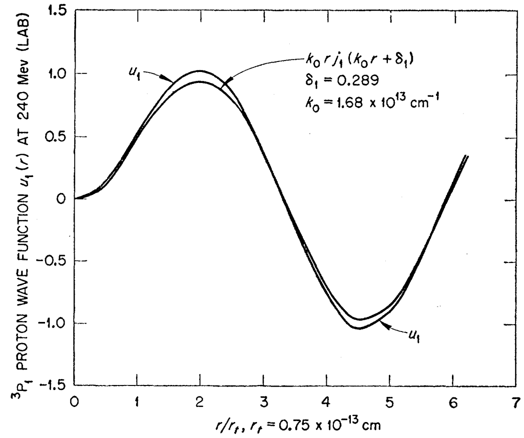


FIG. 2. Comparison of exact $^3P_1$ proton wave function at 240 Mev, obtained using the Jastrow potential, with a "modified" Born approximation wave function.

Since less accuracy is required in evaluating the triplet terms, we represented the $^3P_1$ wave function, $u_1$, analytically. The triplet well given by (2.6b) is relatively weak so that a Born approximation would seem valid. We calculated the $^3P_1$ phase shift exactly at 240 Mev to check the work of Goldfarb and Feldman[5] who calculate the phase shift, using (2.6b) and Born approximation wave functions. Agreement was to a few percent. In order to get a better approximation for $u_1$ in the region of interaction, we modified the Born approximation slightly, using

$$u_1=k_0rj_1(k_0r+\delta_1).\tag{2.8}$$

Figure 2 compares (2.8) with the exact solution at 240 Mev. For a given energy we obtain $\delta_1$ from the Born approximation. Finally, in order to evaluate $P_1$ analytically, we approximate the deuteron wave function by

$$\begin{aligned}u(r)&=1.98(4\pi r_D)^{-\frac{1}{2}}[e^{-\rho r}-e^{-\lambda r}],\\ \rho&=0.237\times10^{13}\text{ cm}^{-1},\quad \lambda=1.09\times10^{13}\text{ cm}^{-1}.\end{aligned}\tag{2.9}$$

Equation (2.9) is an exact solution to Schrödinger's equation using the Hulthén potential[8]

$$U(r)=\frac{-V_0\exp(-r/a)}{1-\exp(-r/a)},\quad \begin{aligned}V_0&=46.6\text{ Mev},\\ a&=1.17\times10^{-13}\text{ cm}.\end{aligned}\tag{2.10}$$

For small $r$, (2.10) is almost identical to (2.7), and a comparison of (2.9) with $u(r)$ obtained from (2.7) shows that the wave functions are identical to within a few percent.

In evaluating the integrals given by (2.5), we keep in mind their relative importance. $S_0$ and $D_2$ are evaluated exactly at the three energies mentioned earlier. This gives us a clear picture of the energy dependence of $\sigma(\pi^+,d)$ for $\beta=0$. We investigate the effect of the $\beta$ terms on the angular distribution at 340 Mev only, and

[8] Gunn, Powers, and Touschek, Phil. Mag. **42**, 523 (1951).

TABLE I. Radial integrals for the singlet and triplet transitions and the singlet and triplet phase shifts.

| Proton energy in Mev (lab system) | ${}^1S_0\to{}^3S_1$ $S_0$ | ${}^1D_2\to{}^3S_1$ $S_2$ | ${}^1S_0\to{}^3D_1$ $D_0$ | ${}^1D_2\to{}^3D_1$ $D_2$ | ${}^3P_1\to{}^3S_1$ $P_1$ | Phase shifts $\delta_0({}^1S_0)$ | $\delta_1({}^3P_1)$ | $\delta_2({}^1D_2)$ |
|---|---|---|---|---|---|---|---|---|
| 340 | 0.423 | 2.26 | 0.075 | 0.394 | −0.371 | −0.537 | 0.30 | 0.413 |
| 372 | 0.442 | | | 0.366 | | −0.630 | | 0.424 |
| 437 | 0.458 | | | 0.338 | | −0.734 | | 0.398 |

for that purpose we evaluate $S_2$ graphically and estimate $D_0$ rather roughly. The latter is justified since $D_0$ is very small. Finally, $P_1$ can be obtained to reasonable accuracy at 340 Mev and, because it is evaluated analytically we can express $P_1$ rather simply (to about 10% accuracy) as a function of energy,

$$P_1=-\frac{3.76}{\kappa_0^2+2.36},\quad \begin{array}{l}\kappa_0=\hbar k_0/\mu c,\\ 2.36=(\hbar\lambda/\mu c)^2,\end{array} \tag{2.11}$$

where $\mu$ is the meson mass and $\lambda$ is defined earlier in (2.9). Table I summarizes the results of our calculations.

Consider first the case where $\beta=0$. The differential cross section becomes

$$\frac{d\sigma}{d\Omega}=4(M/\mu)^2(\pi\rho_D)^{-1}g_a{}^2\eta^3\kappa_0{}^{-5}\frac{E}{E+2}(E_i/2Mc^2)\times[A+3B\cos^2\theta](\hbar/\mu c)^2, \tag{2.12}$$

$$A=(S_0)^2+1/2(D_2)^2-\sqrt{2}(S_0D_2)\cos(\delta_2-\delta_0),$$

$$B=\tfrac{1}{2}(D_2)^2+\sqrt{2}(S_0D_2)\cos(\delta_2-\delta_0),$$

$$\rho_D=\mu c r_D/\hbar=3.06,\quad \eta=\hbar q/\mu c.$$

$$E=E_q/\mu c^2,$$

where $E_q$ is the total, relativistic energy of the meson, $E_i$ is the total relativistic energy of the system, and $M$ is a nucleon mass. Table II shows the values of $A$, $B$, $\xi=3B/A$ at various energies. The value of $\xi$ at 340 Mev is consistent with experiments, though the experimental errors are too large to make a definite comparison. The values of $\xi$ at higher energies are less satisfactory, with definite disagreement with experiment at 437 Mev where we have the measured value of $\xi=5\pm0.5$ obtained by Fields *et al.*[9] The total cross section, for $\beta=0$, becomes

$$\sigma(\pi^+,d)=16\rho_D{}^{-1}(M/\mu)^2g_a{}^2\eta^3\kappa_0{}^{-5}\times\left(\frac{E}{E+2}\right)\left(\frac{E_i}{2Mc^2}\right)[(S_0)^2+(D_2)^2]\left(\frac{\hbar}{\mu c}\right)^2.$$

As seen from Table II, $(S_0)^2+(D_2)^2$ is practically constant, so that we write

$$(S_0)^2+(D_2)^2=0.323[1+\delta(E_i)],\quad \delta\ll 1. \tag{2.13}$$

[9] Fields, Fox, Kane, Stallwood, and Sutton, Phys. Rev. **95**, 638(A) (1954).

We obtain $\delta(E_i)$ as a function of energy by fitting the values of $\delta$ obtained from Table II with a smooth curve. With this substitution, $\sigma(\pi^+,d)$ becomes, with $\beta=0$,

$$\sigma(\pi^+,d)=73.3g_a{}^2\left(\frac{E}{2+E}\right)\left(\frac{E_i}{2Mc^2}\right)[1+\delta(E_i)]\times\left(\frac{\eta^3}{\kappa_0{}^5}\right)\left(\frac{\hbar}{\mu c}\right)^2 \tag{2.14}$$

In comparing (2.14) with experiment, we assume various energy dependences for $g_a{}^2$, i.e., we assume $g_a{}^2$, $g_a{}^2E$, or $g_a{}^2E^{-1}$ is constant. The value of $g_a{}^2$ is fixed in each case by requiring $\sigma(\pi^+,d)=0.269$ mb at 338 Mev. It is found that agreement with experiment over a wide range of energies is possible only for $g_a{}^2$ constant,

$$g_a{}^2=2.04. \tag{2.15}$$

The curve of (2.14) is given in Fig. 1. The low theoretical values of $\sigma(\pi^+,d)$ near threshold are to be expected since the $S$-wave meson emission has been neglected. An important factor in the comparatively good fit at the higher energies is the strong $\kappa_0{}^{-5}$ dependence on the proton momentum. While $\kappa_0$ does not vary much over this energy range, $\kappa_0{}^{-5}$ does. It is interesting that the use of the Born approximation for the proton wave functions would have resulted in a $\kappa_0$ dependence more like $\kappa_0{}^{-9}$, which would predict values of $\sigma(\pi^+,d)$ that are too low at the higher energies. As in the case of the angular distribution, (2.14) becomes less satisfactory at increasing energies.

Since the existence of an $S$ state interaction, as proven by reaction (1.3), requires that $\beta\neq 0$, let us consider briefly the effect of the $\beta$ terms on our results. The effect of $\beta$ on the angular distribution was investigated at 340 Mev. Using the values of the radial integrals given in Table I, (2.4) becomes, at 340 Mev,

$$\tfrac{1}{4}\textstyle\sum_m|\mathfrak{M}_s|^2+\tfrac{1}{4}\sum_{m,m'}|\mathfrak{M}_t|^2=(16\pi/r_D)k_0{}^{-2}\{(q/k_0)^2[0.118a^2-0.52ab+0.20b^2+(0.214a^2+0.52ab+0.30b^2)3\cos^2\theta]+0.256b^2\}. \tag{2.16}$$

The last term, proportional to $b^2$ in (2.16), comes from the $S$-wave emission in the transition from the proton triplet state and consequently lacks the factor of $(q/k_0)^2$. Since $(q/k_0)^2\approx 1/25$, the last term in (2.16) would certainly dominate unless $b\ll a$. This term is isotropic and would lead to too small values for $\xi$. It is found that for values of $b/a$ such that $-0.10 \leqq b/a \leqq 0.20$, $\xi$ ranges from 6.5 to 2.1. The slight increase in $\xi$ for very small values of $b/a$ is not enough to account for the large discrepancy with the experimental value of $\xi$ at 437 Mev. If we require $\xi > 3$ at 340 Mev, then we must restrict $b/a$ to the range,

$$-0.05 \leq b/a \leq 0.20. \tag{2.17}$$

We shall see now that a consideration of the energy dependence of $\sigma(\pi^+,d)$ for $\beta \neq 0$ will lead to a more stringent condition on $b/a$ than (2.17). After integrating the differential cross section over angles, the cross terms in $ab$ cancel out so that $\sigma(\pi^+,d)$ depends only on $a^2$ and $b^2$. Since the major contribution to the term proportional to $b^2$ comes from the triplet term $(P_1)^2$ (about 80%), we can approximate $\sigma(\pi^+,d)$ fairly accurately by the expression

$$\begin{aligned}\sigma(\pi^+,d) = 73.3 g_a^2 [E/(2+E)](E_i/2Mc^2)\kappa_0^{-3}\eta \\ \times [(\eta/\kappa_0)^2[1+\delta(E_i)] + 82.3(b/a)^2 \\ \times [\kappa_0^2 + 2.36]^{-2}](\hbar/\mu c)^2.\end{aligned} \tag{2.18}$$

We have used (2.11) for $P_1$. Again, we fix $g_a$ for a given value of $b/a$ by requiring $\sigma(\pi^+,d) = 0.269$ mb at 338 Mev. The term in $b/a$ will lower the curve at increasing energies and raise the curve at the low energies. Thus we can obtain an upper limit on $b/a$ if we compare (2.18) with experiment. Very roughly,

$$|b/a| < 0.13. \tag{2.19}$$

Smaller values of $b/a$ can result in a better fit with experiment at the lower energies. Figure 1 shows (2.18) plotted as a function of energy for $b/a = 0.054$, chosen so that $\sigma(\pi^+,d)$ falls exactly on the center of the lowest energy measurement at 311 Mev.

Before going on to discuss the case when the final nucleons are unbound, let us say a word or two about the importance of the potential well in obtaining these results. As has been mentioned earlier, (2.19) makes the transition from the proton singlet state the only important one, and, consequently, the model is sensitive only to the singlet well, at least for the case of the Jastrow well. The singlet $S$ and $D$ phase shifts and the important radial integrals, $S_0$ and $D_2$, were evaluated for various square wells with or without repulsive cores. For no well chosen was there any chance of obtaining a reasonably acceptable angular distribution at 340 Mev, nor did it seem possible for any other square well to give agreement. By examining the various effects of different square wells and the core, we concluded that only a singlet well of the general features of the Jastrow well, i.e., hard, repulsive core, and outside a deep, narrow well with a long tail, could give the proper angular distribution over any reasonable range of energies, i.e., 100 Mev. The hard core seemed to be essential. Also, the proton singlet well leads to a weaker dependence of the total cross section on the incident proton momentum, i.e., $k_0^{-5}$ rather than something like $k_0^{-9}$ as predicted by the Born approximation.

TABLE II. Values of $A$, $B$, and $\xi = 3B/A$ as a function of energy.

| Proton energy in Mev lab. frame | c.m. frame | $A$ | $B$ | $\xi = 3B/A$ | $A+B = (S_0)^2 + (D_2)^2$ |
|---|---|---|---|---|---|
| 340 | 163 | 0.118 | 0.214 | 5.44 | 0.332 |
| 372 | 182 | 0.154 | 0.173 | 3.37 | 0.327 |
| 437 | 207 | 0.180 | 0.143 | 2.41 | 0.323 |

### 3. $p+p \rightarrow \pi^+ + n + p$

We estimate the order of magnitude of the total cross section, $\sigma(\pi^+,np)$, at a lab energy of 340 Mev. Since the final neutron-proton state is now unbound, a few modifications are necessary. The matrix element for the transition should be written[10]

$$\mathfrak{M} = \langle \psi_f^{(-)} | T | \psi_i^{(+)} \rangle, \tag{3.1}$$

where $\psi_f^{(-)}$ denotes the time reversed final state, i.e., the scattered state with ingoing waves. We make use of the relation

$$\psi_{np}^{(-)*}(r,\theta) = \psi_{np}^{(+)}(r, \pi-\theta). \tag{3.2}$$

Including isotopic spin eigenfunctions in $\psi_{np}^{(-)}$ necessitates splitting up $\psi_{np}$ according to even and odd angular momenta since $\psi_{np}$ must be completely antisymmetric. For example, for the triplet spin state, $\chi_1^m$, we must write $\psi_{np}$ as

$$\psi_{np} = \psi_e(\mathbf{r})\chi_1^m \Lambda_0 + \psi_o(\mathbf{r})\chi_1^m \Lambda_1^0, \tag{3.3}$$

with $\psi_e(-\mathbf{r}) = \psi_e(\mathbf{r})$, $\psi_o(-\mathbf{r}) = -\psi_o(\mathbf{r})$; $\Lambda_0$, and $\Lambda_1^0$ are the antisymmetric and symmetric neutron-proton isotopic spin states. For simplicity, we have neglected tensor mixing of states and approximate $\mathfrak{M}$ by including only $S$ to $S$ state transitions. Since the singlet to singlet transition is forbidden, this leaves only one transition. The final state is now a three-body state so that a continuum of values of meson momentum, $\eta$, is possible. Consequently, we find for the total cross section per unit meson momentum,

$$\frac{d\sigma}{d\eta} \gtrapprox \frac{8}{\pi}\left(\frac{M}{\mu}\right)^2 g_a^2 \frac{\eta^4}{\kappa\kappa_0^5}\left(\frac{E_i}{2Mc^2}\right)|B_{00}|^2\left(\frac{\hbar}{\mu c}\right)^2. \tag{3.4}$$

Here $\kappa$ is the relative momenta of the final neutron or proton. $B_{00}$ is given by

$$B_{00} = \int_0^\infty w_0(x) u_0(x) j_0(\eta x/2\kappa_0) dx,$$

with $u_0$ the singlet $S$ proton radial wave function and $w_0(x)$ the triplet $S$ neutron proton radial wave function. We have assumed $\beta = 0$. $B_{00}$ is approximated by using

[10] B. A. Lippman and J. Schwinger, Phys. Rev. 79, 469 (1950).

the low-energy approximation for $w_0$, relating $w_0$ to the scattering length and effective range,

$$w_0=\sin(kr+\delta)-\sin\delta e^{-\beta r}. \tag{3.5}$$

This is reasonable since the maximum relative energy is only 20 Mev and, more significantly, the strong $\eta$ dependence favors transitions to final states with low neutron-proton relative energy. We apply a similar approximation to the incident proton state, writing $u_0$ in the same form as (3.5). We choose $\beta$ by requiring $u_0$ to give as close a solution to Schrödinger's equation as possible. This is done by substituting $u_0$ in the form of (3.5) in Schrödinger's equation, leaving us with

$$\Delta=-U(r-r_0)u_0-(k_0^2+\beta_0^2)\sin(\delta_0+k_0r_0)e^{-\beta_0(r-r_0)}=0.$$

In general, $\Delta$ is not zero, so that we find the "best" value of $\beta_0$ by requiring

$$\int_{r_0}^{\infty}|\Delta|^2dr$$

be a minimum. $r_0$ is the Jastrow core radius. This approximation, when compared with the exact numerical solution, is not too good and leads to a value of $B_{00}$ which is about 20% too high.

In evaluating $B_{00}$, it is necessary to redefine it, since, as the integral now stands, the integrand is not defined as $x$ approaches infinity. This is remedied by writing

$$w_0=\bar{w}_0+w_0(\infty),\quad u_0=\bar{u}_0+u_0(\infty),$$

where

$$\lim_{x\to\infty}w_0=w_0(\infty),$$

and similarly for $u_0$. $B_{00}$ is now defined by setting the nonconvergent part

$$\int w_0(\infty)u_0(\infty)j_0dx=0.$$

This follows since $w_0(\infty)$ and $u_0(\infty)$ are oscillating functions with radically different wavelengths. This shows how the nuclear interaction is necessary in order to obtain a nonzero result for the transition. Furthermore, neglecting terms in $\bar{u}_0$, i.e., neglecting the interaction in the initial state, would reduce $d\sigma/d\eta$ by a factor of two.

Evaluating $B_{00}$ for various values of $\eta$, at 340 Mev, we obtain the result

$$\sigma(\pi^+,np)\gtrsim 0.25\text{ mb}. \tag{3.6}$$

The results obtained for deuteron formation indicate that including tensor forces will make the $D$ state transition appreciable. Hence, it is more reasonable to expect $\sigma$ to be roughly equal to

$$\sigma(\pi^+,n,p)\approx 0.4\text{ mb}. \tag{3.6$'$}$$

Compared with $\sigma(\pi^+,d)\approx 0.3$ mb at this energy, we see that the theory predicts roughly the same order of magnitude for the two processes.

## 4. K-SHELL CAPTURE OF $\pi^-$ BY THE DEUTERON

The existence of the reaction,[11,12]

$$\pi^-+d\to n+n, \tag{1.3}$$

with absorption of the meson from the $K$-shell orbit about the deuteron, proves that meson absorption or production processes also takes place when the meson is in an $S$ state. In terms of our model, $\beta\neq 0$. It is found, however, that the alternate reaction,

$$\pi^-+d\to 2n+\gamma, \tag{4.1}$$

competes with (1.3). The ratio of the two reaction rates has been measured by Panofsky, Aamodt, and Hadley,[11] and by Steinberger and Chinowsky[12]; they obtain the values

$$R(2n)/R(2n,\gamma)=2.37\pm0.4,\quad 1.5\pm0.8, \tag{4.2}$$

respectively. Since one would expect the reaction with the emission of a gamma ray to be much less probable than (1.3) because of the weakness of the electromagnetic interaction, (4.2) shows that absorption of a meson from an $S$ state must be partially forbidden, or in terms of our model, $|\beta|\ll|\alpha|$. We shall show now that the value of $\beta$ required to satisfy (4.2) is consistent with the restrictions on $\beta$ deduced earlier. Our procedure will be to calculate $R(2n)$ with our model and estimate the value of $R(2n,\gamma)$ from the arguments of Brueckner, Serber and Watson.[13]

Reaction (1.3) is very similar to (1.1), and we assume charge symmetry so that the potential well defining the neutron $^3P_1$ state will be (2.3b). The only real modification of the earlier calculation is to replace the plane wave for the meson function by the $K$-orbit wave function

$$\phi(r)=(4\pi)^{-\frac{1}{2}}2a_0^{-\frac{3}{2}}e^{-r/a_0},$$
$$a_0=\hbar^2(\lambda e^2)^{-1}=2.05\times10^{-11}\text{ cm},\quad \lambda=2M\mu(2M+\mu)^{-1}. \tag{4.3}$$

Since the variation of $\phi(r)$ over the size of the deuteron is negligible, we can replace $\phi(i)$ in the interaction by $\phi(0)$. Using the approximation (2.8) for $u_1(k_0r)$, the Hulthén wave function, (2.9), for the deuteron $S$ state, and including the deuteron $D$ state by assuming $w(r)\approx\frac{1}{5}u(r)$, we obtain for the transition rate of (1.3),

$$\begin{aligned}R(2n)&=(0.946)(M/\mu)g_b^2[|\phi(0)|^2(\hbar/\mu c)^3]\kappa_0^{-1}(c/r_D)\\&=(0.946)(M/\mu)[2M/(2M+\mu)]^3(\pi)^{-1}\\&\qquad\times\alpha^3g_b^2\kappa_0^{-1}(c/r_D),\end{aligned} \tag{4.4}$$

where $\kappa_0$ and $r_D$ were defined earlier, $c$ is the velocity of light, and $\alpha$ here is the fine structure constant coming

[11] Panofsky, Aamodt, and Hadley, Phys. Rev. **81**, 565 (1951).
[12] W. Chinowsky and J. Steinberger, Phys. Rev. **95**, 1561 (1954).
[13] Brueckner, Serber, and Watson, Phys. Rev. **81**, 575 (1951).

from the factor $|\phi(0)|^2$. From the work of Brueckner, Serber, and Watson, we have

$$R(2n,\gamma)=(4/3)f'(1-\mu/M)|\phi(0)|^2 c\sigma(\gamma+n\rightarrow\pi^-+p)/\eta,$$

with $f'$ a factor introduced to allow for modifications introduced by the interaction of the two neutrons in the final state. We shall take $f'=1$, and using the latest measurements,[1]

$$\sigma(\gamma+n\rightarrow\pi^-+p)=(0.196\pm0.03)\eta \text{ mb},$$

we have

$$R(2n,\gamma)=2.5\times10^{14}\text{ sec}^{-1}. \tag{4.5}$$

Using the larger of the values given by (4.2), we have, from (4.4) and (4.5),

$$g_b^2=0.0329;\quad |b/a|=|g_b/g_a|=0.128. \tag{4.6}$$

Using the lower value of $R(2n)/R(2n,\gamma)$ will reduce $|b/a|$ to 0.10. While we might have hoped for a lower value of $|b/a|$, 0.128 falls within the limits deduced earlier. Considering the uncertainties in the values of $R(2n)/R(2n,\gamma)$ and $R(2n,\gamma)$, we cannot take the value of $g_b^2$ too seriously.

## V. CONCLUSIONS AND SUMMARY

The simple, nonrelativistic model of meson production used here is consistent with present experiments. In particular, the large anisotropy and energy dependence of the reaction, $p+p\rightarrow\pi^++d$, could be described over a reasonably wide energy range. The fact that this agreement is obtained with energy-independent parameters is certainly surprising. The model becomes less satisfactory at increasing energies, especially in the case of the angular distribution. It is possible, however, that a better choice of potential might improve the results. The total cross section for the alternate reaction, $p+p\rightarrow\pi^++n+p$, roughly calculated, is found to be of the same order of magnitude as the cross section with deuteron formation. Finally, the calculation of $\pi^-+d\rightarrow 2n$, where the $\pi^-$ is in a $K$ orbit about the deuteron, predicts an $S$-wave interaction sufficiently weak to be consistent with results of the calculation of $p+p\rightarrow\pi^++d$.

While it is difficult to draw any definite conclusions from the success of this model about the properties of a fundamental theory of meson production, the model does contain a number of interesting points which might be kept in mind when discussing meson production from a phenomenological point of view. The interaction between the nucleons in both the initial and final states is of major importance for the success of the model. In particular, the general features of the Jastrow singlet well, i.e., hard core, narrow deep well with a long tail, are necessary for agreement with experiment. The effect of the $D$ state in the deuteron (or of the unbound $n$-$p$ system) is not necessarily small since the $D$ to $D$ transition can be the same order of magnitude as the corresponding $S$ to $S$ transition. A discussion of the energy dependence of these reactions in terms of meson momentum alone can be misleading since a strong dependence of the cross section on the incident proton momentum is possible. It is not necessary to assume a $T=\frac{3}{2}$, $J=\frac{3}{2}$ meson-nucleon resonance in order to account for the anisotropy in the production of mesons. Finally, the model shows that the nuclear wave function can be important over the entire region of interaction rather than just at close distances.

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