Interactions between Configurations of Three Nucleons in the $1d$, $2s$ Shell*

MARTIN G. REDLICH
Department of Physics,† University of Wisconsin, Madison, Wisconsin
(Received May 5, 1955)

Calculations for the nuclei with mass 19 were based on a shell model with harmonic oscillator wave functions, central Gaussian interactions between the three outer nucleons, and a single-particle operator which leads to the energy differences between $1d_{\frac{3}{2}}$, $2s_{\frac{1}{2}}$, and $1d_{\frac{5}{2}}$ levels observed in O^{17}. Energy matrices for all configurations of the $1d$, $2s$ shell were calculated. Wave functions of four nuclear states were obtained. They lead to the observed ft value for the allowed unfavored O^{19} β^- transition, as well as for the favored Ne^{19} β^+ transition. Energy differences between $5/2+$ and $\frac{1}{2}+$ states of F^{19} and ground states of O^{19} and F^{19} are correctly given. Disagreement with the measured energy of the $\frac{1}{2}+$ state of O^{19} and the magnetic moment of F^{19}, however, indicates that at least modifications of this model are necessary to obtain general agreement.

1. INTRODUCTION

IN the preceding paper,[1] which will be referred to as I, a variation method for obtaining nuclear wave functions was described. The notation of I and the assumptions for $V(i,j)$ and $V'(r_i)$ introduced there will be used in this paper. Results[2] for mass numbers $A=17$ and 18 will be reviewed, and methods of calculation for $A=19$ as well as their numerical results will be presented. The set of ψ_α for each (T,J) will be taken to consist of the wave functions for all states of the configurations

$$(1s)^4(1p)^{12}(1d)^k(2s)^{A-16-k} \tag{1a}$$

with all possible k. Neither configurations with higher states of the harmonic oscillator potential $V'(r_i)$, nor those without completely closed $1s$ and $1p$ shells will be included. The considerations of I for the wave function of O^{17} lead us to expect that admixtures of these higher states are small. The present assumption for the set of ψ_α evidently permits only the description of states of positive parity.

In the light of Sec. 4 of I, we may substitute the configurations

$$(1d)^k(2s)^{A-16-k} \tag{1b}$$

for an $(A-16)$-particle system for (1a), provided that terms in $C(2s,1d)$, which accounts for the variation with nl of the interaction between an nl state and the double closed shells, are added to the diagonal elements of the energy matrix. A single-particle operator which can be written

$$\Delta(i)=-0.302(8.75-j_i^2)^2+2.525(8.75-j_i^2)\ \text{Mev}, \tag{2}$$

is introduced in order to account empirically both for $C(2s,1d)$, which follows from the model of I, and for the splitting between $\frac{3}{2}+$ and $5/2+$ levels of O^{17}, which does not. $\Delta(i)$ is analogous to $\Delta(j_i)$ of reference 2; it equals 0 for $j_i=5/2$, 0.875 Mev for $j_i=\frac{1}{2}$, and 5.08 Mev for $j_i=\frac{3}{2}$. It is diagonal in any j-j coupling representation. While the calculations which follow will be made in such a representation, their results will show that neither j-j nor L-S coupling is accurate in this mass region. We shall henceforth speak of one-, two-, and three-particle configurations, in the sense of (1b).

The Parameters of the Calculation

The Gaussian internucleon potential, $V_X(i,j)$ of I (13), which accounts for all the properties of the neutron-proton system in the triplet state at low energies except for the deuteron's quadrupole moment, will be used here. The depth of this potential is $V_0=70.8$ Mev. The harmonic oscillator potential $V'(r_i)$ of I (14) with parameter ν_0 was used in the calculations for $A=18$ and will be used for $A=19$. From Fig. 1 of I, however, it seems that a parameter as large as ν_1, corresponding to a smaller expectation value of r_i^2, would be more suitable.

The numerical calculations of the present paper were made for two values of the depth of the two-nucleon interaction. It turns out that an arbitrary increase in the depth to about cV_0, with $c=1.46$, leads to much better agreement with experiment for $A=19$ than does V_0. It can readily be seen that a happy correspondence exists between this change in depth and a change in ν. The matrix elements of $V_X(i,j)$ are linear combinations

TABLE I. Some diagonal matrix elements,

$$\langle a^2,T,J|\tfrac{1}{2}(V_W+V_M)|a^2,T,J\rangle,$$

in Mev, $\times(-1)$, evaluated for two sets of parameters. V_W and V_M are defined by Eq. (13) of paper I.

Config. a^2	T,J	ν Depth	ν_0 cV_0	ν_1 V_0
$(1d_{\frac{5}{2}})^2$	1,0		5.43	4.97
	1,2		1.59	1.62
	1,4		0.83	0.86
	0,1		3.06	2.86
	0,3		2.15	2.19
	0,5		4.16	4.31
$(2s_{\frac{1}{2}})^2$	1,0 or 0,1		4.68	4.60

* A preliminary report of this work has appeared in Phys. Rev. **98**, 199 (1955).

† This investigation was started at Palmer Physical Laboratory, Princeton University. It was supported by the U. S. Atomic Energy Commission.

[1] M. G. Redlich, preceding paper [Phys. Rev. **99**, 1421 (1955)].

[2] M. G. Redlich, Phys. Rev. **95**, 448 (1954).

of certain radial integrals, which vary roughly as $\nu^{\frac{3}{2}}$. Consequently an increase in depth corresponds approximately to an increase in ν. This is seen in Table I, where diagonal matrix elements for the configurations $(1d_{\frac{5}{2}})^2$ and $(2s_{\frac{1}{2}})^2$ are given for parameters (ν_0, cV_0) and (ν_1, V_0). Similar results are to be expected for other configurations and also for interconfiguration matrix elements, since the changes in all radial integrals save one range from -10 to $+13\%$ in going from (ν_0, cV_0) to (ν_1, V_0). The exception, $F_4(1d,1d)$ in the customary notation, changes by -30%. Its coefficients are, however, generally small, and its effect may be gauged from Table I, since it appears in all $(1d_{\frac{5}{2}})^2$ matrix elements. The numerical calculations throughout the text and in all except three of the tables in the following sections will refer to the parameters (ν_0, cV_0). In Tables III, XI, and XIII, results of calculations for both (ν_0, V_0) and (ν_0, cV_0) will be presented.

2. THREE-PARTICLE WAVE FUNCTIONS

Wave functions for states of a configuration $C=j_1j_2j_3$ of three particles, each with isotopic spin $t=\frac{1}{2}$, can readily be obtained. Let us consider the complete set of wave functions of j_1j_2, antisymmetric in coordinates, spins, and isotopic spins of particles 1 and 2, with total isotopic spin and angular momentum $[T',J']$. There exists a state of $j_1j_2j_3$ with (T,J) which corresponds to the following vector additions:

$$\mathbf{T'}+\mathbf{t}=\mathbf{T}; \quad \mathbf{J'}+\mathbf{j}_3=\mathbf{J}. \tag{3}$$

By use of the vector addition coefficients one can readily obtain a wave function $\psi(1,2;3)$ which has these (T,J) and $[T',J']$, and is antisymmetric in 1 and 2. A completely antisymmetric wave function is then

$$\Phi(\alpha)_a=N[\psi(1,2;3)-\psi(1,3;2)-\psi(3,2;1)], \tag{4}$$

where N is a normalization factor. The subscript a on Φ stands for antisymmetric. The state is described by

$$\alpha=(j_1j_2)j_3\,[T',J'],\ (T,J),\ T_\zeta J_z.$$

The two-particle state determined by $j_1j_2\,[T',J']$ will be called the parent of the three-particle state.

Fractional Parentage Coefficients

The fractional parentage coefficient[3] is defined by

$$\Phi(\alpha)_a=\sum_k{}' \sum_{T''J''} \varphi[(C_k,T''J'')_a j_k(3),TT_\zeta,JJ_z]\times(C_k,T''J''; j_k]T'J',TJ). \tag{5}$$

The symbol C_k stands for Cj_k^{-1}. The function φ describes the configuration C_k of particles 1 and 2 in an antisymmetric state $[T'',J'']$, with $T''J''$ added to t and j_k, which describe particle 3, to give $TT_\zeta JJ_z$. The factor of φ is a fractional parentage coefficient. The summation over k runs only over distinct j_k. The relation between expansions (4) and (5) can be seen most clearly in a representation given by

$$\beta=\beta_1\beta_2\beta_3, \quad \text{with} \quad \beta_i=j_iz_it_i\zeta_i. \tag{6}$$

Then we can write

$$\Phi(\beta)_a=\frac{1}{\sqrt{6}}\begin{vmatrix}\beta_1(1) & \beta_2(1) & \beta_3(1)\\ \beta_1(2) & \beta_2(2) & \beta_3(2)\\ \beta_1(3) & \beta_2(3) & \beta_3(3)\end{vmatrix}, \tag{7}$$

where $\beta_i(k)$ stands for the wave function of particle k specified by the quantum numbers β_i. The determinant $\Phi(\beta)_a$ can be expanded in terms of cofactors B_{ik} in two ways:

$$\Phi(\beta)_a=\sum_{k=1}^{3} B_{ik}\beta_i(k)=\sum_{i=1}^{3} B_{i3}\beta_i(3). \tag{8}$$

The first expansion corresponds to (4), the second to (5). It should be noted that the set of antisymmetric wave functions for all α is a complete set for the configuration $j_1j_2j_3$ of identical particles. This can be demonstrated by comparing the total number of linearly independent wave functions corresponding to all $[T',J']$ and the vector additions (3) with that obtained by the usual method of counting, made in the representation β.

Calculations in the representation α will be described. If we assume that ψ's and φ's are both normalized, then

$$\psi(1,2;3)=\varphi[(j_1j_2,T'J')_a j_3(3),TT_\zeta,JJ_z].$$

[3] G. Racah, Phys. Rev. **62**, 438 (1942). This is Racah's paper III.

The connection between the other terms of (4) and (5) can be established by application of a formula which follows directly from (4) of reference 3:

$$\chi(l_1l_2L';\,l_3LL_z)=\sum_{L''}h(l_1l_2l_3,L'L''L)\cdot\chi(l_2l_3L'';\,l_1LL_z), \tag{9a}$$

where

$$h(l_1l_2l_3,L'L''L)=[(2L'+1)(2L''+1)]^{\frac{1}{2}}(-)^{2l_1+l_2+l_3+L''}\begin{Bmatrix} l_1\, l_2\, L' \\ l_3\, L\, L'' \end{Bmatrix} \tag{9b}$$

$$=[(2L'+1)(2L''+1)]^{\frac{1}{2}}(-)^{l_1+L''-L}\,W(l_1l_2Ll_3;\,L'L''). \tag{9c}$$

(9b) is expressed in terms of the six-j-symbol[4] and (9c) in terms of the Racah coefficient.[3] The quantity l can stand equally well for total angular momentum j or isotopic spin t. The subscripts serve not only to differentiate angular momenta, but also to indicate coordinates in χ; i.e., $l_i=l_i(i)$ on both sides of (9a). The ψ of (4) can be expressed as a product of a space and spin function ψ_1 and an isotopic spin function ψ_2, each having the property (9a):

$$\psi(1,2;3)=\psi_1(j_1j_2J';\,j_3JJ_z)\times\psi_2(\tfrac{1}{2}\,\tfrac{1}{2}\,T';\,\tfrac{1}{2}\,T\,T_z). \tag{10}$$

If $j_1=j_2=j\neq j_3$, then (4), (5), (9), and (10) lead to

$$(j^2,T'J';\,j_3]\!]T'J',TJ)=N=3^{-\frac{1}{2}}, \tag{11a}$$

$$(jj_3,T''J'';\,j]\!]T'J',TJ)=(2/3)^{\frac{1}{2}}h(jjj_3,J'J''J)\times h(\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2},\,T'T''T). \tag{11b}$$

The details of this calculation, for a specific example, may be found elsewhere.[5]

If j_3 also equals j, the normalization N of (4) is changed, since the three ψ's are, in general, no longer orthogonal. The fractional parentage coefficient becomes

$$(j^2,T''J'';\,j]\!]T'J',TJ)=N'[\delta(J',J'')+2h(jjj,J'J''J)\times h(\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2},\,T'T''T)], \tag{12}$$

where N' is obtained from

$$\sum_{T''J''}|\,(j^2,T''J'';\,j]\!]T'J',TJ)\,|^2=1. \tag{13}$$

The wave functions of j^3 for the same (T,J), specified by different $[T',J']$ are not necessarily orthogonal. In particular, for $j=5/2$, we shall be concerned with two states of $(1d_{\frac{5}{2}})^3$ with $(1/2,\,5/2)$. The two wave functions with this (T,J) whose parents are $[0,1]$ and $[1,0]$ form a complete set, but are not orthogonal. We shall choose instead the wave function determined by $[1,0]$ and that other one which is orthogonal to it (T_z and J_z will sometimes be left off from now on):

$$\begin{aligned}&\Phi(d_{\frac{5}{2}}^3\,[1,0];\,1/2,\,5/2)_a;\\ &\Phi(d_{\frac{5}{2}}^3\perp;\,1/2,\,5/2)_a=-8^{-\frac{1}{2}}\{13^{\frac{1}{2}}\Phi(d_{\frac{5}{2}}^3\,[0,1];\,1/2,\,5/2)_a+5^{\frac{1}{2}}\Phi(d_{\frac{5}{2}}^3\,[1,0];\,1/2,\,5/2)_a\}.\end{aligned} \tag{14}$$

The fractional parentage coefficients for $(1d_{\frac{5}{2}})^3$ have also been calculated by another method.[6]

If all three j_i are different,

$$(j_1j_2,T'J';\,j_3]\!]T'J',TJ)=3^{-\frac{1}{2}}, \tag{15a}$$

$$(j_2j_3,T''J'';\,j_1]\!]T'J',TJ)=3^{-\frac{1}{2}}h(j_1j_2j_3,J'J''J)\times h(\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2},T'T''T), \tag{15b}$$

$$(j_1j_3,T''J'';\,j_2]\!]T'J',TJ)=3^{-\frac{1}{2}}(-)^{T'+J'-j_1-j_2}\,h(j_2j_1j_3,J'J''J)\times h(\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2},T'T''T). \tag{15c}$$

Formulas for L-S coupled wave functions can, of course, be obtained in an entirely analogous way with (9). The parent state is then described by

$$l_1l_2\,[^{(2T'+1)\,(2S'+1)}L'].$$

All matrix elements needed in the present paper can easily be calculated with the fractional parentage coefficients. The matrix element of an operator

$$F^{(3)}=\sum_{i=1}^{3}f(i), \tag{16}$$

[4] E. P. Wigner, "On the Matrices which Reduce the Kronecker Products of Representations of Simply Reducible Groups," *circa* 1942 (unpublished).
[5] M. G. Redlich, Princeton University dissertation, January, 1954 (unpublished).
[6] A. R. Edmonds and B. H. Flowers, Proc. Roy. Soc. (London) **A214**, 515 (1952).

where $f(i)$ operates only on particle i is[3]

$$\langle C,T'J',TT_\zeta JJ_z|F^{(3)}|\bar{C},\bar{T}'\bar{J}',\bar{T}\bar{T}_\zeta\bar{J}\bar{J}_z\rangle=3\sum_k{}'\sum_{T''J''}(C_k,T''J'';\ j_k][T'J',TJ)^*$$
$$\times\langle C_k,T''J'',j_kTT_\zeta JJ_z|f(3)|C_k,T''J'',\bar{j}_k\bar{T}\bar{T}_\zeta\bar{J}\bar{J}_z\rangle\cdot\delta(C_k,\bar{C}_k)\cdot(\bar{C}_k,T''J'';\ \bar{j}_k][\bar{T}'\bar{J}',\bar{T}\bar{J}). \quad (17)$$

The star denotes a complex conjugate. The function δ equals 1 if C_k is the same configuration as $\bar{C}_k$ and 0 otherwise.

The matrix element of

$$V^{(3)}=V(1,2)+V(1,3)+V(2,3), \quad (18)$$

where $V(i,j)$ is a scalar operator on particles i and j only, has also been given by Racah:

$$\langle C,T'J',TJ|V^{(3)}|\bar{C},\bar{T}'\bar{J}',TJ\rangle=3\sum_k{}'\sum_{T''J''}(C_k,T''J'';\ j_k][T'J',TJ)^*$$
$$\times\langle C_k,T''J''|V(1,2)|\bar{C}_k,T''J''\rangle\cdot\delta(j_k,\bar{j}_k)\cdot(\bar{C}_k,T''J'';\ j_k][\bar{T}'\bar{J}',TJ). \quad (19)$$

This formula shows that matrix elements of $V^{(3)}$ can be obtained directly from those of $V(1,2)$ for two-particle configurations by use of the fractional parentage coefficients.

3. RESULTS FOR ONE- AND TWO-PARTICLE CONFIGURATIONS

$A=17$. No interaction between outer states appears here. In Table II, experimental results are compared with (i) the model of I, and (ii) a model with two bodies, a core and an outer nucleon. In the second model center-of-mass effects are taken into account. The experimental magnetic moments in Tables II and XI are taken from a current table[7]; all other data in all tables as well as in the text of the present paper, are taken from a recent summary.[8]

The magnetic moment, μ, of O^{17} is given rather accurately by either model; this agreement is most remarkable. Model (ii) leads to roughly the correct value of the quadrupole moment, q, of O^{17}. The next line of Table II, however, contains a sharp disagreement with experiment. The theoretical value for the half-life of the $E2$ γ transition from the 875-kev state to the ground state of O^{17} is about 220 times the experimenta one.

TABLE II. Comparison of theory with experiment for $A=17$. Column (i) refers to the variation method of I, with one wave function for the configuration $(1s)^4(1p)^{12}(1d)^1$. Column (ii) refers to a model for the system: Core+outer nucleon. $E_J(X^{17})$ is the energy (in Mev) due to nuclear forces only of the lowest $J+$ state of X^{17}. The ft value for $F^{17}(\beta^+)O^{17}$ is not included here, since it was used to determine the Gamow-Teller coupling constant.

Datum	Theory (i)	Theory (ii)	Experiment	
$\mu(O^{17})$	−1.9130	−1.8542	−1.8937 ±0.0001	nm
$q(O^{17})$	0	−0.0020	−0.005 ±0.002	$\times10^{-24}$ cm^2
$T_{\frac{1}{2}}(O^{17},s_{\frac{1}{2}})$	∞	380	1.7 ±0.7	$\times10^{-10}$ sec
$E_{\frac{5}{2}}(F^{17})-E_{\frac{5}{2}}(O^{17})$	0	0	−0.3[a]	Mev

[a] This number $=K_{\beta^+}(F^{17})+R_e+D_{np}-\Delta E_C(9)$. [For definitions, see Eq. (21) ff.]

In the last line of the table there appears the contribution of nuclear forces to the energy difference between the $(T,J)=(1/2, 5/2)$ states with $T_\zeta=-\frac{1}{2}$ and $\frac{1}{2}$. The theoretical value is 0. The value in the column marked "Experiment" was obtained by using the following rough formula for the Coulomb energy E_C:

$$E_C(Z)=\frac{3}{5}\times\frac{Z(Z-1)e^2}{1.4A^{\frac{1}{3}}\times10^{-13}}=\frac{Z(Z-1)}{A^{\frac{1}{3}}}\times0.61\text{ Mev}. \quad (20)$$

Coulomb energy differences are defined as

$$\Delta E_C(Z)=E_C(Z)-E_C(Z-1), \quad (21)$$

and therefore always positive. The details of the energy difference due to nuclear forces as deduced from experiment are given in footnote a of Table II. The notation, here and in Tables III and XI, is: K_β=maximum kinetic energy of the β-particles, R_e=rest energy of the electron=0.511 Mev, and D_{np}=neutron-proton rest energy difference=1.293 Mev.

A calculation of the ratio $R^{\frac{1}{2}}$ of Gamow-Teller (G-T) to Fermi coupling constants in β decay can be based on the assumption that the G-T matrix element for the transition $F^{17}(\beta^+)O^{17}$ is of pure single-particle type. Actually, we expect that there will be admixtures of wave functions of other configurations, with small amplitudes a_α. Their effects upon the G-T matrix element, however, will be of order a_α^2, since there are no cross terms between $(1s)^4(1p)^{12}(1d)^1$ and any other configuration. In the formula

$$ft=\bar{A}[\textstyle\sum_f F^2+R\sum_f \mathbf{G}\cdot\mathbf{G}]^{-1}, \quad (22)$$

F and $\mathbf{G}$ are Fermi and G-T matrix elements, and the summation is over final states. The constants are

$$\bar{A}=6550\pm150\text{ sec},\quad R=1.22\pm0.11. \quad (23)$$

$\bar{A}$ has been determined[9] from a $0+\rightarrow0+$ transition,

[7] H. E. Walchli, Oak Ridge National Laboratory Report 1469, 1953 (unpublished).

[8] F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **27**, 77 (1955).

[9] J. B. Gerhart, Phys. Rev. **95**, 288 (1954).

R from the F^{17} decay. These constants will be used in later calculations.

$A=18$. Calculations had been made[2] for these nuclei with an interaction potential

$$V(r_{12})=\tfrac{1}{2}[V_W(r_{12})+V_M(r_{12})], \tag{24}$$

with V_W and V_M the Gaussian potentials of I, Sec. 2. The reason for this mixture of ordinary and exchange forces will appear in the following section. Matrices of

$$V(r_{12})+\Delta(1)+\Delta(2) \tag{25}$$

were diagonalized. The calculations have been repeated with cV_0, and applied also to the recently investigated transition $Ne^{18}(\beta^+)F^{18}$. The wave functions for cV_0 are similar to (8) and (9) of reference 2; however, the amplitudes of the states with $s_{\frac{1}{2}}$ or $d_{\frac{3}{2}}$ particles are somewhat larger than for V_0, and those of $(d_{\frac{5}{2}})^2$ are slightly smaller. This is to be expected, since an increase in V_0 to cV_0 has the same effect upon the wave functions as a decrease in $\Delta(i)$ to $c^{-1}\Delta(i)$.

The calculations are compared with experiment in Table III. There is definite disagreement for the contribution of nuclear forces to the difference in energy between the ground states of F^{18} and O^{18}. The theoretical value lies 2 Mev above the one deduced from experiment, as has recently been emphasized.[10] The inaccuracy in the calculated Coulomb energy difference is unlikely to be more than 0.5 Mev, which is the discrepancy between theory and experiment for the $T_\zeta=-1$ and $T_\zeta=1$ components of the (1,0) charge multiplet. A third energy difference given in Table III is that between the lowest 2+ and 0+ states of O^{18}. It is not known whether the only excited state yet observed for O^{18}, which has energy 1.99 Mev, is 2+. The theoretical ft value for $F^{18}(\beta^+)O^{18}$ is too small. The possible error due to uncertainties in the constants (23) amounts to about 11 percent, and is not sufficient to increase the calculated value to within experimental uncertainty of the observed one. For the Ne^{18} decay, on the other hand, there is agreement with the observed value, whose uncertainty is, however, extremely large.

From charge independence we expect that the ground states of Ne^{18} and O^{18} differ only in T_ζ. The double-bar or reduced matrix elements (reference 2, Sec. VI) for the transitions $Ne^{18}(\beta^+)F^{18}$ and $F^{18}(\beta^+)O^{18}$ are therefore equal. The theoretical ft value for the Ne^{18} transition, however, equals $\frac{1}{3}$ that for the F^{18} transition (see Table III), since there are three final states for F^{18}, but only one for O^{18}.

4. THREE-PARTICLE CONFIGURATIONS: RESULTS OF SOME CALCULATIONS

Energies

Let us write

$$V_X^{(3)}=V_X(r_{12})+V_X(r_{13})+V_X(r_{23}), \tag{26}$$

[10] J. P. Elliott and B. H. Flowers, Proc. Roy. Soc. (London) **A229**, 536 (1955).

TABLE III. Comparison of theory with experiment for $A=18$. The potential energy operator is

$$\tfrac{1}{2}[V_W(r_{12})+V_M(r_{12})]+\Delta(1)+\Delta(2).$$

Here $V_W(r_{12})$ and $V_M(r_{12})$ are the Gaussian potentials of I, Eq. (13); results are given for two depths, $V_0=70.8$ Mev, and $cV_0=103.4$ Mev. The operator $\Delta(i)$ is defined by Eq. (2). In this table, $E_J(X^{18})$ is the energy (in Mev) of the lowest state of X^{18} with $J+$, due to nuclear forces only.

Datum	Theory, with V_0	Theory, with cV_0	Experiment
$E_1(F^{18})-E_0(O^{18})$	1.02	0.73	−1.3[a]
$E_0(Ne^{18})-E_0(O^{18})$	0	0	−0.5[b]
$E_2(O^{18})-E_0(O^{18})$	2.95	4.40	(1.99)[c]
ft value for $F^{18}\rightarrow O^{18}$	3110	2890	4170 ±330
ft value for $Ne^{18}\rightarrow F^{18}$	1037	963	800 ±340

[a] This number $=K\beta^+(F^{18})+R_e+D_{np}-\Delta E_C(9)$. [For definitions of symbols, see Eq. (21) ff.]
[b] This number $=K\beta^+(Ne^{18})+K\beta^+(F^{18})+2R_e+2D_{np}-\Delta E_C(10)-\Delta E_C(9)$.
[c] The only excited state of O^{18} yet known has this energy: however, its spin has not been determined.

and

$$\Delta^{(3)}=\Delta(1)+\Delta(2)+\Delta(3). \tag{27}$$

We shall denote the expectation values of (26)+(27) by

$$\langle V_X^{(3)}+\Delta^{(3)}\rangle; \tag{28}$$

they were calculated for all states of three-particle configurations with $d_{\frac{5}{2}}$ and $s_{\frac{1}{2}}$ particles for the four types of exchange forces. This is readily done with the fractional parentage coefficients (Sec. 2), the two-particle matrix elements (reference 2), and formula (19). The results are given in Table IV. The various configurations can be divided into three types: The ones with no d particles, which lie lowest, are of Type 0; those with 1 or ≥ 2 particles in $d_{\frac{5}{2}}$ states are of Type 1 or 2, respectively. The operator $\Delta^{(3)}$ raises Type 1 configurations by 5 Mev and Type 2 configurations by 10 or 15 Mev above the Type 0 ones; we may therefore expect that the Type 0 configurations alone will give a rough indication of the order of the lowest levels.

We notice from Table IV first of all that $T=\frac{3}{2}$ states lie generally above those with $T=\frac{1}{2}$. This is the result of the greater space-symmetry permitted for lower T. Since the interaction is attractive and has short range, states with a large number of space-symmetric couplings between pairs of nucleons will lie lowest. This situation is, of course, just the reverse of that in the atom, where the interelectron forces are repulsive, and the states of minimum space-symmetry lie lowest. It is also plain from the table that for ordinary (W) and space-exchange (M) forces, states with small J are the lowest (for either $T=\frac{3}{2}$ or $\frac{1}{2}$), while for spin-exchange (B) and space-and-spin-exchange (H) forces, the states with maximum spin are lowest. This is an indication that the total space-symmetry of a three-particle configuration is very

TABLE IV. Expectation values of $V_X^{(3)}+\Delta^{(3)}=V_X(r_{12})+V_X(r_{13})+V_X(r_{23})+\Delta(1)+\Delta(2)+\Delta(3)$, in Mev, $\times(-1)$, for all states of all three-particle configurations of Type 0 (i.e., those which have no $d_{\frac{5}{2}}$ particles). Here $V_X(r_{ik})=-cV_0P_X\exp(r_{ik}^2/2.245)$, where r_{ik} is the internucleon coordinate in 10^{-13} cm, and $cV_0=103.4$ Mev. $P_W=1$, P_M=space-exchange operator, P_B=spin-exchange operator, and $P_H=P_MP_B$. The single-particle operator $\Delta(i)$ is diagonal in j-j coupling and equals 0 if $j_i=5/2$, 0.875 Mev if $j_i=\frac{1}{2}$, and 5.08 Mev if $j_i=\frac{3}{2}$. Whenever a state is not determined by (T,J) alone, the parent $[T',J']$ or other specification is given in a footnote.

T	J	$(d_{\frac{5}{2}})^3$	$\langle V_W^{(3)}+\Delta^{(3)}\rangle$ $(d_{\frac{5}{2}})^2s_{\frac{1}{2}}$	$\langle V_W^{(3)}+\Delta^{(3)}\rangle$ $(s_{\frac{1}{2}})^2d_{\frac{5}{2}}$	$(s_{\frac{1}{2}})^3$	$(d_{\frac{5}{2}})^3$	$\langle V_M^{(3)}+\Delta^{(3)}\rangle$ $(d_{\frac{5}{2}})^2s_{\frac{1}{2}}$	$\langle V_M^{(3)}+\Delta^{(3)}\rangle$ $(s_{\frac{1}{2}})^2d_{\frac{5}{2}}$	$(s_{\frac{1}{2}})^3$	$(d_{\frac{5}{2}})^3$	$\langle V_H^{(3)}+\Delta^{(3)}\rangle$ $(d_{\frac{5}{2}})^2s_{\frac{1}{2}}$	$\langle V_H^{(3)}+\Delta^{(3)}\rangle$ $(s_{\frac{1}{2}})^2d_{\frac{5}{2}}$	$(s_{\frac{1}{2}})^3$	T	J
3/2	9/2	6.25	4.70			−0.28	−4.03			−6.25	−6.45			0.28	2.28			3/2	9/2
	7/2		6.53				0.99				−8.27				−2.74				7/2
	5/2	9.69	6.32	7.50		2.71	−2.61	2.18		−9.69	−8.07	−11.00		−2.71	0.86	−5.69			5/2
	3/2	8.07	7.35			0.18	0.18			−8.07	−9.10			−0.18	−1.93				3/2
	1/2		10.32				2.59				−12.07				−4.34				
1/2	13/2	8.96				6.00				3.33				6.49				1/2	13/2
	11/2	7.85	7.85			4.22	2.51			2.37	2.51			4.70	7.84				11/2
	9/2	9.75	8.96[a]			5.33	5.58[a]			1.42	0.28[a]			5.08	1.70[a]				9/2
			7.45[b]				3.50[b]				−3.66[b]				3.29[b]				
	7/2	10.30[c]	6.48[d]	7.50		5.04[e]	0.61[d]	2.18		1.19[e]	0.54[d]	2.18		4.56[e]	4.28[d]	7.50			7/2
		8.15[b]	6.53[b]			2.19[b]	0.99[b]			−2.68[b]	−5.48[b]			2.67[b]	−1.73[b]				
	5/2	8.88[e]	7.20[d]	8.25[f]		3.02[e]	2.58[d]	4.10[f]		1.06[e]	−0.88[d]	0.80[f]		2.91[e]	0.36[d]	3.55[f]			5/2
		14.09[g]	8.45[e]	9.05[g]		9.31[g]	3.26[e]	6.35[g]		2.93[g]	−5.29[e]	−2.63[g]		−0.51[g]	1.87[e]	−2.63[g]			
	3/2	9.76	8.86[f]	8.74		2.05	0.85[f]	5.54		3.24	2.31[f]	−0.26		2.27	2.83[f]	0.80			3/2
			7.94[g]				1.86[g]				−6.30[g]				−0.92[g]				
	1/2	8.45	9.17[f]		11.44	2.78	1.69[f]		11.44	−0.67	1.70[f]		0	2.33	1.15[f]		0		1/2
			11.86[g]				6.78[g]				−9.30[g]				−3.33[g]				

[a] [0,5]. [b] [1,4]. [c] [1,2]. [d] [0,3]. [e] ⊥ (see formula 14). [f] [0,1]. [g] [1,0].

largely determined by that of its parent two-particle configuration. Specifically, the lowest state of $(d_{\frac{5}{2}})^3$ for W and M forces has $J=5/2$; the state $(d_{\frac{5}{2}})^2$ [1,0], which is predominantly $d^2\,{}^1S$, may be considered its parent. (This two-particle state evidently cannot be taken as parent of any state of $(d_{\frac{5}{2}})^3$ that does not have $J=5/2$.) We recall that of the states of $(d_{\frac{5}{2}})^2$ for W and M forces, the [1,0] state is by far the lowest.[2] The situation for $(d_{\frac{5}{2}})^2s_{\frac{1}{2}}$, with $J=\frac{1}{2}$ for the lowest state, is entirely similar. For H and B forces, the lowest $T=\frac{1}{2}$ state has $J=13/2$ for $(d_{\frac{5}{2}})^3$ and $J=11/2$ for $(d_{\frac{5}{2}})^2s_{\frac{1}{2}}$. The parent of each of these is $(d_{\frac{5}{2}})^2$ [0,5], which is just $d^2\,{}^3G$, and this is the lowest state of $(d_{\frac{5}{2}})^2$ for B forces and near the lowest for H forces.

In Fig. 1, all states of the nuclei with $A=19$ for which there is experimental evidence of parity + are given. We note that there are in fact two low states of F^{19} with $J=\frac{1}{2}$ and 5/2. There is a state with $J=\frac{1}{2}$ and negative parity just between them. If this state could be accounted for by the present method, it would require either a breakup of the O^{16} double closed shell or moving one of the outer nucleons into the $1f$ shell. Either process would be expected to lead to a state with much higher energy. On the basis of these offhand considerations, it is not clear how the present model could account for such a low $\frac{1}{2}-$ state. At all events, in the present calculations all outer nucleons are in the $1d$, $2s$ shell, and only states with parity + can be accounted for.

It is apparent, then, that on this model the dominant role in accounting for the observed low lying parity + levels of F^{19} is played by W and M forces. The potentials $V_B{}^{(3)}$ and $V_H{}^{(3)}$ lead to low states with very high spins. If these potentials are multiplied by -1, then Table IV shows that the $T=\frac{3}{2}$ states lie far below those with $T=\frac{1}{2}$, contrary to experiment. It is for that reason that the equal mixture of W and M potentials denoted $V(r_{ij})$ and given in (24) was chosen for these calculations. $V^{(3)}$ is defined by (26), with the subscript X left off. Expectation values of $V^{(3)}+\Delta^{(3)}$ are given in Table V, together with the lowest characteristic values of the matrices of states of all configurations in this shell, for some (T,J).

A remark about the pairing energies for $1d_{\frac{5}{2}}$ and $2s_{\frac{1}{2}}$ particles: The expectation value of the interaction $V(r_{12})$ between two particles in a (1,0) state amounts to −4.69 Mev for $(2s_{\frac{1}{2}})^2$; −5.43 Mev for $(1d_{\frac{5}{2}})^2$. Is it to be expected then that the addition of two $1d_{\frac{5}{2}}$ particles to, specifically, a one-particle configuration $(nl_j)^1$, coupled to yield a three-particle state with $J=j$, will result in a lower energy than the similar addition of two $2s_{\frac{1}{2}}$ particles? If $nl_j=1d_{\frac{5}{2}}$, the answer is yes: From Table V, the expectation values of $V^{(3)}$ for the lowest $J=5/2$ states are

$$-11.70 \text{ Mev for } d_{\frac{5}{2}}(d_{\frac{5}{2}})^2,$$
$$-9.45 \text{ Mev for } d_{\frac{5}{2}}(s_{\frac{1}{2}})^2.$$

The answer is no for $nl_j=2s_{\frac{1}{2}}$. Now $\langle V^{(3)}\rangle$ is

$$-10.20 \text{ Mev for } s_{\frac{1}{2}}(d_{\frac{5}{2}})^2,$$
$$-14.06 \text{ Mev for } s_{\frac{1}{2}}(s_{\frac{1}{2}})^2,$$

for the lowest $J=\frac{1}{2}$ state of the first and the only state of the second configuration. (These states are higher in Table V because of the effect of $\Delta^{(3)}$.) The reason for the dependence of the pairing energy upon nl_j follows: For s particles the dependence of the energy $E(ns_{\frac{1}{2}}^m)$ of the configuration $(ns_{\frac{1}{2}})^m$ upon the number m of particles does not vary with the range of the forces. It is just

$$E(ns_{\frac{1}{2}}^m)=\tfrac{1}{2}m(m-1)F^0(ns,ns), \tag{29}$$

where F^0 is the usual Slater integral. Thus even in the present case, with forces much closer to the 0-range than to the ∞-range limit, the energy of $(2s_{\frac{1}{2}})^3$ is 3 times that of $(2s_{\frac{1}{2}})^2$. For configurations involving other particles, a formula like (29) holds only at ∞ range. At lower ranges, the energy varies much more slowly with m. The reversal in sign of the pairing energy is therefore not surprising.

Wave Functions

It was plain from Table V that for each of $T=\frac{3}{2}$ and $\frac{1}{2}$, the lowest two states have $J=\frac{1}{2}$ and 5/2. The matrices of

$$M=V^{(3)}+\Delta^{(3)} \tag{30}$$

were calculated for all configurations of the $1d$, $2s$ shells and these four (T,J) values. The submatrix of the Type 0 and 1 configurations was diagonalized by a card-programmed electronic computer, and the characteristic vector for the lowest characteristic (i.e., energy) value was obtained. All states of Type 2 configurations were taken into account by further subdiagonalization and first-order perturbation theory. Subdiagonalization between two terms α and β is generally necessary when

$$|M_{\alpha\alpha}-M_{\beta\beta}|<|M_{\alpha\beta}|. \tag{31}$$

For $(\frac{3}{2},\frac{1}{2})$, there is only one Type 0 configuration and Type 1 configurations are almost negligible. There is, however, a significant higher configuration, $(d_{\frac{3}{2}})^2s_{\frac{1}{2}}$, which is of Type 2. This configuration was included in the original submatrix. The contributions of Type 2 configurations due to second-order perturbation theory are generally of the order of 0.01 in amplitude. This is small compared to the large first-order amplitudes, which are around 0.30. The second-order contribution entails a substantial change for the very small amplitudes; however, this has been neglected, since their effect upon matrix elements is generally also very small. The total energy is denoted $E(T,J)$ and was given in Table V. We note that the (1/2, 5/2) state lies 0.17 Mev above that for $(\frac{1}{2},\frac{1}{2})$. This is good agreement with the experimental difference of 0.20 Mev; differences in Coulomb energies, as well as uncertainties in the numerical calculation are considerably larger than 0.03 Mev. This calculation was first made for V_0, and led to -0.08 instead of 0.17 Mev (see Table XI); we see that with increasing depth, that is, in effect, decreased nuclear radius, the $J=\frac{1}{2}$ and 5/2 levels cross over.

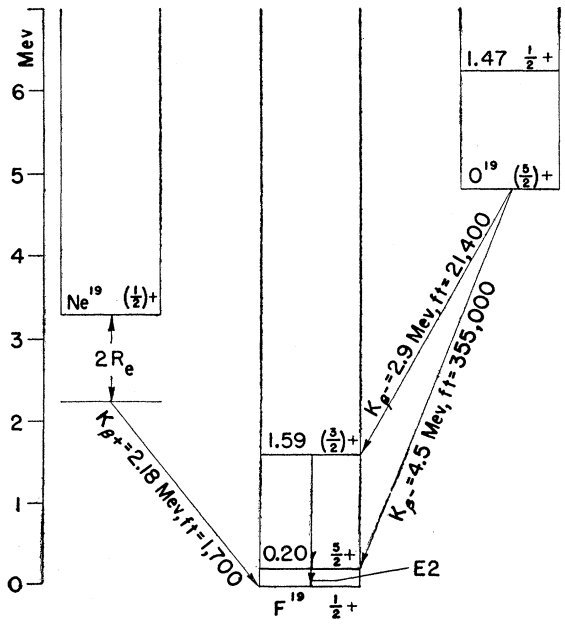

FIG. 1. All states of the nuclei with mass 19 for which there is experimental evidence of parity +. A number of spins are uncertain and are enclosed in parentheses. R_e = rest energy of the electron. K_β = maximum kinetic energy of a β transition. The multipole order of the 0.20 Mev γ transition in F^{19} is $E2$.

TABLE V. Expectation values of

$$V^{(3)}+\Delta^{(3)}=\tfrac{1}{2}[V_W^{(3)}+V_M^{(3)}]+\Delta^{(3)}, \tag{*}$$

for the states of the three-particle configurations of Type 0. The symbols are defined in the caption for Table IV. Whenever a state is not determined by (T,J) alone, the parent $[T',J']$ or other specification is given in a footnote. Also given is $E(T,J)$ for the two lowest states with each T. This quantity is the lowest characteristic value of the matrix of (*) for all states of all configurations of the $1d$, $2s$ shell, not just of the Type 0 configurations listed here. All quantities are in Mev, $\times(-1)$.

		$\langle V^{(3)}+\Delta^{(3)}\rangle$				
T	J	$(d_{\frac{5}{2}})^3$	$(d_{\frac{5}{2}})^2s_{\frac{1}{2}}$	$(s_{\frac{1}{2}})^2d_{\frac{5}{2}}$	$(s_{\frac{1}{2}})^3$	$E(T,J)$
3/2	9/2	2.98	0.32			
	7/2		3.76			
	5/2	6.21	1.86	4.84		8.60
	3/2	4.13	3.76			
	1/2		6.45			8.02
1/2	13/2	7.48				
	11/2	6.03	5.18			
	9/2	7.54	7.27[a]			
			5.47[b]			
	7/2	7.67[c]	3.54[d]	4.84		
		5.18[b]	3.76[b]			
	5/2	5.95[e]	4.89[d]	6.17[f]		
		11.70[g]	5.86[c]	7.70[g]		16.89
	3/2	5.90	4.86[f]	7.14		
			4.90[c]			
	1/2	5.61	5.42[f]		11.44	
			9.32[g]			17.06

[a] [0,5]. [b] [1,4]. [c] [1,2]. [d] [0,3]. [e] $\perp$ (see Eq. 14). [f] [0,1]. [g] [1,0].

TABLE VI. Amplitudes of the wave functions of the states of the three-particle configurations of the $1d$, $2s$ shell in the wave function for the lowest state with (T,J). The quantum numbers $[T',J']$ of the parent state or the symbol $\perp$, defined by Eq. (14), are given where necessary to specify a state. The potential energy operator for the three-nucleon system is $V^{(3)}+\Delta^{(3)}$. The amplitudes of this table form just the characteristic vectors of the matrices of $V^{(3)}+\Delta^{(3)}$ corresponding to the characteristic values $E(T,J)$ of Table V. (The symbols are defined in the captions of Tables IV and V.)

		Type 0				Type 1					Type 2				
(T,J)	$\leftarrow(d_{\frac{5}{2}})^2\rightarrow$	$\leftarrow(d_{\frac{5}{2}})^2s_{\frac{1}{2}}\rightarrow$	$\leftarrow(s_{\frac{1}{2}})^2d_{\frac{5}{2}}\rightarrow$	$(s_{\frac{1}{2}})^3$	$\leftarrow(d_{\frac{5}{2}})^2d_{\frac{3}{2}}\rightarrow$	$\leftarrow(d_{\frac{5}{2}})^2s_{\frac{1}{2}}\rightarrow$	$\leftarrow(d_{\frac{5}{2}}d_{\frac{3}{2}})s_{\frac{1}{2}}\rightarrow$	$(s_{\frac{1}{2}})^2d_{\frac{3}{2}}$	$\leftarrow(d_{\frac{3}{2}})^2d_{\frac{5}{2}}\rightarrow$	$\leftarrow(d_{\frac{3}{2}})^2s_{\frac{1}{2}}\rightarrow$	$(d_{\frac{3}{2}})^3$
(1/2, 5/2)	$\begin{cases}\perp \\ 0.15\end{cases}$	[1,0] 0.69	[0,3] −0.22	[1,2] 0.23	[0,1] −0.25	[1,0] 0.36	[1,2] 0.13	[1,4] 0.20	[0,1] 0.13	[0,3] 0.10	[1,2] −0.08	[1,3] 0.01	[0,2] −0.07	[0,3] −0.09	0.08	[1,0] 0.29	[1,2] 0.07	[0,1] 0.11	[0,3] 0.04	[1,2] 0.05	[0,3] 0.01	0.02
(1/2, 1/2)	$\{0.30$		[0,1] −0.37	[1,0] 0.52				[1,2] 0.17	[0,1] −0.08			[1,1] 0.01	[0,1] −0.33		−0.00		[1,2] 0.12	[0,3] 0.01		[1,0] 0.22	[0,1] 0.09	0.03
(3/2, 5/2)	$\{0.82$		−0.04		−0.44			[1,2] −0.12	[1,4] −0.09			[1,2] 0.04	[1,3] 0.05				[1,0] −0.30	[1,2] 0.07		0.03		
(3/2, 1/2)			0.95					0.06				−0.06					−0.05			0.30		

It is plain from Table V that the combined effects of interactions among the Type 0 configurations, which are listed there, and the Type 1 and 2 configurations, which are not, are very substantial. $E(\frac{1}{2},\frac{1}{2})$, for example, is 5.6 Mev below the lowest of the energies of individual states.

Table VI contains the detailed wave functions. The predominant configurations of $(\frac{1}{2},\frac{1}{2})$, which corresponds to the ground states of F^{19} and Ne^{19}, are $(s_{\frac{1}{2}})^3$ and $(d_{\frac{5}{2}})^2s_{\frac{1}{2}}$. The main configuration of (3/2, 5/2), which is theoretically the ground state of O^{19}, is $(d_{\frac{5}{2}})^3$. The weights (quantum mechanical probabilities) of the individual Type 0 configurations, together with the total weights of all three types of configurations are given in Table VII. Type 0 configurations are by far predominant. None the less the admixtures of the higher configurations are very important; without them it would be impossible, for instance, to account for the high unfavored ft value of an allowed β transition from O^{19} to F^{19} (see Sec. 5).

It may seem surprising at first sight that admixtures of Type 2 configurations are nearly as large as (for $T=\frac{1}{2}$) or even larger than (for $T=\frac{3}{2}$) those of Type 1 configurations. The Type 2 configurations are around 5 Mev above those of Type 1; however, the largest interconfiguration matrix elements connect Type 0 and Type 2 configurations. Let us consider this first for the two-particle case. The interconfiguration matrix elements between various j-j coupling configurations of $(1d)^2$ are given in Table VIII. By far the largest element connects $(d_{\frac{5}{2}})^2$ and $(d_{\frac{3}{2}})^2$ in the (1,0) state. This is easy to understand, since each of these states contains a large amount of $(1d)^2\,{}^1S$, and that state is by far the lowest of $(1d)^2$, since (1) it is space-symmetric and (2) the two d particles are at or near coincidence more often in an S state than a D or G state. The largest admixture of $d^2\,{}^3S$ appears in the (0,1) states of $(d_{\frac{5}{2}})^2$, $(d_{\frac{3}{2}})^2$, and $d_{\frac{5}{2}}d_{\frac{3}{2}}$, so it is not surprising that two matrix elements between these configurations are large. (In the third case the ^3S and ^1P contributions are of opposite sign and the matrix element is small.) The other interconfiguration elements, involving $2s$ particles, are generally much smaller because of less overlapping of the radial wave functions. The one with largest magnitude equals 1.81 Mev. In this framework, it is easy to picture the situation for three particles. For each (T,J)

TABLE VII. Weights (i.e., quantum mechanical probabilities) of the individual Type 0 configurations, and total weights of the different types of configurations, for the wave functions of Table VI. All entries are in percent.

(T,J)	Type Config.	0 $(d_{\frac{5}{2}})^3$	0 $(d_{\frac{5}{2}})^2s_{\frac{1}{2}}$	0 $(s_{\frac{1}{2}})^2d_{\frac{5}{2}}$	0 $(s_{\frac{1}{2}})^3$	Type 0 Total	Type 1 Total	Type 2 Total
(1/2, 5/2)		50	10	19	···	79	11	10
(1/2, 1/2)		9	40	···	30	79	14	7
(3/2, 5/2)		68	0	19	···	87	3	10
(3/2, 1/2)		···	90	···	···	90	1	9

of Table VI, the state with largest amplitude of all Type 0 states except $(s_{\frac{1}{2}})^3$ is listed, together with its amplitude, in Table IX. Similar data are given for Type 2 configurations. The Type 2 state in each case has a $(d_{\frac{5}{2}})^2$ [1,0] parent; similarly, each Type 0 state has a $(d_{\frac{5}{2}})^2$ [1,0] parent. From formula (19), the matrix element between them will equal that between $(d_{\frac{5}{2}})^2$ and $(d_{\frac{3}{2}})^2$, which amounts to -4.44 Mev, multiplied by 3 times a product of fractional parentage coefficients, which altogether amounts to 1.08, 1, -0.82, and 1, for the states listed above. No other two-particle matrix element can enter, since the two configurations differ by two single-particle states.

j-j vs LS Coupling

The two-particle wave functions for (0,1) and (1,0) were obtained in reference 2 for parameter V_0. They and also those for (1,2) and (0,3) have been calculated for cV_0. The wave functions for these four states of two-particle configurations, as well as for the four states of three-particle configurations in Table VI,

TABLE VIII. The interconfiguration matrix elements,

$$\langle C_i,T,J|\,V(r_{12})\,|C_f,T,J\rangle,$$

in Mev, for the various j-j coupling configurations C_i and C_f of $(1d)^2$. The potential $V(r_{12})$ is defined by Eq. (24).

T	J	C_i / C_f	$(d_{\frac{5}{2}})^2$ / $d_{\frac{5}{2}}d_{\frac{3}{2}}$	$(d_{\frac{5}{2}})^2$ / $(d_{\frac{3}{2}})^2$	$d_{\frac{5}{2}}d_{\frac{3}{2}}$ / $(d_{\frac{3}{2}})^2$
0	1		−3.14	2.35	0.74
	3		−1.33	0.91	−1.22
1	0		...	−4.44	...
	2		1.12	−1.21	0.86
	4		1.67	...	...

were analyzed in order to determine whether j-j or L-S coupling is a better approximation. For the configuration ll' or $ll'l''$, the weight of the dominant j-j coupling configuration, $l_jl_{j'}'$ or $l_jl_{j'}'l_{j''}''$, and the weight of the dominant L-S coupling state, $^{(2S+1)}L$, are given in Table X. For two particles, the wave functions are easily transformed from one type of coupling to another, and there is exactly one state specified by $l_jl_{j'}'$ and one by $^{(2S+1)}L$. For three particles, there are in general several states in the specified groups. Their weights have been added, since we could have altered the remaining specifications of the states in each group, for instance, by adding wave functions of several different $[T',J']$, and obtained a single state state of $l_jl_{j'}'l_{j''}''$ or with $^{(2S+1)}L$ whose weight is the sum of the weights of the individual states thus specified. For the three-particle configurations, the j-j coupling wave functions of the present paper were used. Those for L-S coupling, however, were taken from the paper of Elliott and Flowers.[10] Their calculations are based on a Yukawa interaction and a Rosenfeld mixture of forces. None the less, the wave functions which they obtain appear to be very similar to those of the present paper. This is definitely true for two particles. For three particles the total weights of various configurations, which are, of course, independent of coupling, are usually about equal for their wave functions and the ones calculated here, as is seen in Table X.

TABLE IX. The states of Type 0 and Type 2 configurations except $(s_{\frac{1}{2}})^3$ with maximum amplitudes in the wave function for the lowest state with (T,J), together with these amplitudes. The data are taken directly from Table VI.

(T,J)	Type 0	Amplitude	Type 2	Amplitude
(1/2, 5/2)	$(d_{\frac{5}{2}})^3$ [1,0]	0.69	$(d_{\frac{3}{2}})^2\,d_{\frac{5}{2}}$ [1,0]	0.29
(1/2, 1/2)	$(d_{\frac{5}{2}})^2\,s_{\frac{1}{2}}$ [1,0]	0.52	$(d_{\frac{3}{2}})^2\,s_{\frac{1}{2}}$ [1,0]	0.22
(3/2, 5/2)	$(d_{\frac{5}{2}})^3$	0.82	$(d_{\frac{3}{2}})^2\,d_{\frac{5}{2}}$ [1,0]	−0.30
(3/2, 1/2)	$(d_{\frac{5}{2}})^2\,s_{\frac{1}{2}}$	0.95	$(d_{\frac{3}{2}})^2\,s_{\frac{1}{2}}$	0.30

TABLE X. Comparison of weights of the dominant states, specified in the third column, of the main configurations for several two- and three-particle wave functions. The weights are in percent. Wave functions for two different potentials, described in the footnote, were used.

(T,J)	Configuration	j-j coupling configuration or $^{(2S+1)}L$	Weight of specified state(s)	Weight of other state(s)	Total weight of configuration	Source[a]
(1,0)	d^2	$(d_{\frac{5}{2}})^2$	75	9	84	a
		^1S	74	10	84	a
(1,2)	d^2	$(d_{\frac{5}{2}})^2$	50	5	55	a
		^1D	42	13	55	a
	ds	$d_{\frac{5}{2}}s_{\frac{1}{2}}$	42	3	45	a
		^1D	38	7	45	a
(3/2, 1/2)	d^2s	$(d_{\frac{5}{2}})^2\,s_{\frac{1}{2}}$	90	9	99	a
		^2S	74	25	99	b
(3/2, 5/2)	d^3	$(d_{\frac{5}{2}})^3$	68	12	80	a
		^2D	63	27	90	b
	ds^2	$d_{\frac{5}{2}}(s_{\frac{1}{2}})^2$	19	...	19	a
		^2D	10	...	10	b
(0,1)	d^2	$(d_{\frac{5}{2}})^2$	42	35	77	a
		^3S	71	6	77	a
(0,3)	d^2	$(d_{\frac{5}{2}})^2$	37	6	43	a
		^3D	40	3	43	a
	ds	$d_{\frac{5}{2}}s_{\frac{1}{2}}$	57	...	57	a
		^3D	57	...	57	a
(1/2, 1/2)	d^2s	$(d_{\frac{5}{2}})^2\,s_{\frac{1}{2}}$	40	16	56	a
		^2S	54	6	60	b
	d^3	$(d_{\frac{5}{2}})^3$	9	5	14	a
		^2S	10	2	12	b
(1/2, 5/2)	d^3	$(d_{\frac{5}{2}})^3$	50	18	68	a
		^2D	57	9	66	b
	ds^2	$d_{\frac{5}{2}}(s_{\frac{1}{2}})^2$	19	1	20	a
		^2D	20	1	21	b
	d^2s	$(d_{\frac{5}{2}})^2\,s_{\frac{1}{2}}$	10	2	12	a
		^2D	12	2	14	b

[a] The sources are:

a ≡ The present investigation. The two-nucleon inter-action has Gaussian shape with depth cV_0 and equal ordinary and space-exchange forces. The potential energy is given by Eq. (25) for two-particle configurations, and by $V^{(3)}+\Delta^{(3)}$, defined in the caption of Table V, for three-particle configurations. Two-particle wave functions have not been given explicitly here, but the three-particle wave functions were taken from Table VI.

b ≡ J. P. Elliott and B. H. Flowers, reference 10. The two-nucleon interaction has Yukawa shape and a Rosenfeld mixture of forces.

The table indicates that, in general, neither j-j nor L-S coupling is a good approximation. This is not surprising in view of the fact that the matrix elements of the interaction $V^{(3)}$, which is diagonal in L-S coupling, are of about the same order of magnitude as those of the operator $\Delta^{(3)}$, which is diagonal in j-j coupling. The present calculations may therefore be considered to be of "intermediate coupling" type.

It is plain, however, that j-j coupling is a better approximation than L-S coupling for the larger T values, here for $T=\frac{3}{2}$ and 1. Just the reverse is true for $T=\frac{1}{2}$ and 0. Again, this is easily understood in terms of the greater space-symmetry of the states with lower T, which has as its consequence larger expectation values of the two-nucleon interaction. For higher excited states, the matrix elements of $V^{(3)}$ decrease, while those of $\Delta^{(3)}$ do not change. The j-j coupling approximation therefore improves with increasing energy of excitation.

5. THREE-PARTICLE CONFIGURATIONS: COMPARISON OF THEORY WITH EXPERIMENT

Differences in Energy Between Various States

In Table XI the values of several quantities which have been measured or deduced from experiment are compared with those calculated from theory for both V_0 and cV_0. The energy differences between various states due to nuclear forces only are readily obtained from the quantities $E(T,J)$ of Table V. The energy difference between the 5/2+ and $\frac{1}{2}$+ states of F^{19} has already been discussed (Sec. 4). The energy difference due to nuclear forces between the ground states of the mirror pair Ne^{19} and F^{19} is 0. This is in good agreement with the total mass difference and the Coulomb energy difference calculated by (20).

The lowest state with parity + of O^{19} is predicted to have $J=5/2$. It is assumed in Fig. 1 that this is in fact the ground state of this radioactive nucleus. (Its half-life is 29 seconds.) It is then possible to compare the observed energy difference between the ground states of O^{19} and F^{19} plus the Coulomb energy difference with the calculated difference between the nuclear (3/2, 5/2) and $(\frac{1}{2},\frac{1}{2})$ states. The value deduced from experiment lies between the calculated ones for V_0 and cV_0, but much closer to the latter. This is, of course, better evidence that cV_0 is a more accurate depth than the agreement for the 200-kev energy difference in F^{19}.

For either parameter, however, the energy difference between the lowest $(\frac{3}{2},\frac{1}{2})$ state of O^{19}, which presumably should correspond to the observed 1.47-Mev state with $\frac{1}{2}$+, and the (3/2, 5/2) state is only about 0.5 Mev. This discrepancy of 1 Mev is of similar magnitude to one for $A=18$ given in Table III. The inaccuracy of this calculated energy difference casts some doubt upon the predicted spin of the ground state of O^{19}. Indeed, there seems to be an empirical indication that the spin of the ground state of an odd nucleus is determined by the odd nucleons. Such a rule would lead us to expect $J=\frac{3}{2}$ for the ground state of O^{19}, since Na^{23} with three odd protons has that spin. The present calculations certainly do not confirm this: From Table V, we would expect the $(\frac{3}{2},\frac{3}{2})$ state to lie higher than either the (3/2, 5/2) or the $(\frac{3}{2},\frac{1}{2})$ state. Further calculations have been based on a (3/2, 5/2) ground state for O^{19}, in order to ascertain the consequences of a definite assumption.

TABLE XI. Comparison of theory with experiment for $A=19$. The potential energy operator is

$$\tfrac{1}{2}[V_W{}^{(3)}+V_M{}^{(3)}]+\Delta^{(3)}.$$

These symbols are defined in the caption of Table IV. Results of calculations are given for two depths of the Gaussian two-nucleon interaction, V_0 and cV_0. The wave functions for V_0 were not given explicitly, but the ones for cV_0 were given in Table VI. Here $E_J(X^{19})$ is the energy of the lowest state of X^{19} with spin J and parity +, due to nuclear forces only. $T_{\frac{1}{2}}$=half-life. μ=magnetic moment.

Datum	Theory, with V_0	Theory, with cV_0	Experiment[a]	
$E_{\frac{5}{2}}(F^{19})-E_{\frac{1}{2}}(F^{19})$	−0.08	0.17	0.200 ±0.002	Mev
$E_{\frac{1}{2}}(Ne^{19})-E_{\frac{1}{2}}(F^{19})$	0	0	−0.1[b]	Mev
$E_{\frac{5}{2}}(O^{19})-E_{\frac{1}{2}}(F^{19})$	5.39	8.47	7.7[c]	Mev
$E_{\frac{1}{2}}(O^{19})-E_{\frac{5}{2}}(O^{19})$	0.58	0.59	1.47 ±0.02	Mev
ft value for $Ne^{19}\rightarrow F^{19}$	1640	1590	1700 ±170	
ft value for $O^{19}\rightarrow F^{19}(5/2+)$	129 000	373 000	335 000 ±100 000	
$\mu(F^{19},\frac{1}{2}+)$	2.94	2.94	2.628	nm
$T_{\frac{1}{2}}(F^{19},5/2+)$	2.7×10^{-7}	2.7×10^{-7}	1.0×10^{-7} ±0.2	sec

[a] I am indebted to Dr. D. H. Wilkinson for communicating to me a number of these results prior to publication.
[b] This number $=K_{\beta^+}(Ne^{19})+R_e+D_{np}-\Delta E_C(10)$. [These symbols were defined by Eq. (21) ff.]
[c] This number $=K_{\beta^-}(O^{19})+(200\text{-kev }\gamma\text{-energy})+R_e-D_{np}+\Delta E_C(9)$.

ft Values

ft values for two allowed β transitions at $A=19$ have been calculated, and are given in Table XI. The method of calculating matrix elements between states of single configurations given by Talmi[11] was used. It is based on formula (17). Some details of calculations involving configuration interaction may be found in reference 2.

$Ne^{19}(\beta^+)F^{19}$. This is a favored transition. The calculated value is slightly low, as is that for $F^{18}(\beta^+)O^{18}$, but it is just within present experimental uncertainty of the measured one.

$O^{19}(\beta^-)F^{19}$, 5/2+. This is an unfavored transition, with $ft=355\,000$. Calculations based upon single configurations in j-j coupling lead to a favored ft value.[11,12]

[11] I. Talmi, Phys. Rev. **91**, 122 (1953).
[12] E. P. Wigner, Proceedings of the Harwell Nuclear Physics Conference (Ministry of Supply, Harwell, 1950).

A somewhat larger, but still not unfavored, ft value is obtained if all Type 0 configurations are taken into account. It is because of this result of earlier calculations, that it was decided to take the Type 1 and 2 configurations into account. The contributions of the various types of configurations to the total double-bar (or reduced) matrix element $\mathfrak{A}$ of Sec. VI, reference 2, are given in Table XII for cV_0. The large contribution of Type 1 configurations, which have rather small amplitudes, is due to cross terms between them and terms of Type 0. In particular, the largest such contribution is that of

$$(d_{\frac{5}{2}})^3 \rightarrow (d_{\frac{5}{2}})^2 d_{\frac{3}{2}} \; [1,4]$$

and is

$$0.824 \times 0.199 \times (-2.343) = -0.384.$$

The first two numbers in the product are the initial and final amplitudes, the third is the value of $\mathfrak{A}$ for this transition. For comparison, the contributions to $\mathfrak{A}$ of the various types of configurations for the favored transition are also given in Table XII. The contributions of Type 1 and 2 configurations now amount to only 38 percent of those of Type 0 configurations, and are of the same sign, as against 63 percent and opposite sign for the unfavored transition. It is thus possible to account in a rather unambiguous way for one allowed unfavored ft value. Only further calculations will tell whether the present model can account for some of the many other unfavored transitions.

If the basic wave functions of the L-S coupling scheme are used, the reason for the small matrix element of $O^{19}(\beta^-)F^{19}$, $5/2+$ is seen more directly. A state $d^3\,{}^2D$ has the largest weight[10] in both the initial and final wave functions. However, the initial and final ^2D states differ in orbital symmetry, i.e., belong to different partitions, and the β-decay matrix element between them is 0. The smallness of the matrix element of this transition is therefore due to breakdown of j-j coupling; on the other hand, the fact that it is considerably larger than 0 is due to breakdown of L-S coupling.

There is also a β^- transition of lower energy, from the ground state of O^{19} to a state of F^{19} at 1.59 Mev. The ft-value of this transition is 21 400; it is therefore allowed and we expect the 1.59-Mev state to have parity $+$. However, neither its spin nor the spin of the initial state of O^{19} has been definitely established by experiment. For this reason the ft value has not been calculated.

Magnetic Moment of F^{19}

This quantity was calculated by means of formula (17) and the fractional parentage coefficients. The theoretical value is too large by 0.31 nm (Table XI). It varies only very slowly with a change in the depth of the potential. At first, the submatrix of $(\frac{1}{2},\frac{1}{2})$ for Type 0 configurations only was diagonalized. The magnetic moment for a wave function of these configurations alone was just 2.63 nm, the observed value. We see that the effect of the higher configurations here, too, is rather substantial. We must conclude that the $(\frac{1}{2},\frac{1}{2})$ wave function of the present paper is not accurate; however, it has been pointed out that small admixtures of higher configurations can affect nuclear moments substantially.[13]

TABLE XII. The contributions of the various types of configurations to the total double-bar or reduced Gamow-Teller matrix element, $\mathfrak{A}$, for two β transitions. Each line includes the contributions of all direct and cross terms between configurations of the listed type. The second line contains in addition the cross terms between Type 1 and Type 0 configurations, and the third, the cross terms between Type 2 and Type 1 configurations.

	$O^{19}(\beta^-)F^{19}$, $5/2+$	$Ne^{19}(\beta^+)F^{19}$
Type 0	0.803	−1.644
Type 1	−0.421	−0.310
Type 2	−0.088	−0.311
Total $\mathfrak{A}$	0.294	−2.265

Half-Life of the $5/2+$ State of F^{19}

The 200 kev, $5/2+$ level of F^{19} decays to the ground state by a γ transition of order $E2$. With the initial and final wave functions for $(1/2, 5/2)$ and $(\frac{1}{2},\frac{1}{2})$, the half-life for this transition can be calculated.[14] The operator is of type (16) with

$$f(i) = (\tfrac{1}{2} - T_{\zeta i}) r_i^2 (5/4\pi)^{\frac{1}{2}} C_q^{(2)}(i), \qquad (32)$$

in the notation of Racah.[3] The calculated half-life (Table XI) is 2.7 times larger than the measured one. For a single-particle $E2$ transition, of type $1d_{\frac{5}{2}} \rightarrow 2s_{\frac{1}{2}}$, with the same energy, the result would have been

$$T_{\frac{1}{2}} = 1.4 \times 10^{-7} \text{ sec}.$$

For a three-particle configuration with only one proton, the matrix element is necessarily smaller and $T_{\frac{1}{2}}$ larger.

We saw from Sec. 1 and Table I that matrix elements of the two-nucleon interaction do not vary much for the change from (ν_0, cV_0) to (ν_1, V_0). Consequently, the wave functions and energies are about the same. This is true also for matrix elements of all operators (16) which, like the magnetic moment operator, do not contain r_i. It is not true for (32). In order to obtain the half-life for (ν_1, V_0) from that for (ν_0, cV_0) we must multiply the latter by a factor

$$(\nu_1/\nu_0)^2 = 2.02,$$

since the matrix element of (32) varies as ν^{-1}. This demonstrates simply that a decrease in nuclear radius increases the half-life.

[13] A. Arima and H. Horie, Progr. Theoret. Phys. (Japan) **12**, 623 (1954). Also, a paper on quadrupole moments, to be published.

[14] R. G. Sachs, *Nuclear Theory* (Addison-Wesley Press, Cambridge, 1953), pp. 232–241; J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), p. 595.

TABLE XIII. The interaction energy between states of B outer neutrons and states of the double closed shell, obtained from the experimental binding energy, $\mathfrak{B}_B$, of B neutrons, and the calculated interaction energy E_B of outer states. E_B is defined by Eq. (34), and given for two values, V_0 and cV_0, of the depth of the Gaussian two-nucleon potential. All energies are in Mev.

			$-E_B$		$\epsilon(B)=(\mathfrak{B}_B+E_B)/B$	
B	Nucleus	$\mathfrak{B}_B$	V_0	cV_0	V_0	cV_0
1	O^{17}	4.14	...	...	4.14	4.14
2	O^{18}	12.21	5.05	7.81	3.58	2.20
3	O^{19}	16.16	5.54	8.59	3.54	2.52

It seems possible that better agreement between theory and experiment for the half-lives of the states of both F^{19} and O^{17} which decay by $E2$ transitions could be obtained if higher configurations are taken into account. In particular, configurations in which the core is broken up may change the radiative transition matrix elements substantially, even though their amplitudes in the wave functions are very small. In transitions between such configurations more than one proton may be involved, in contrast to the transitions considered here. The contributions of higher configurations to the diagonal matrix elements of operator (32), that is, to the quadrupole moments, have been calculated by Horie and Arima.[13] Their work shows that these contributions generally lead to better agreement with experimental quadrupole moments.

Interaction Energy between the Outer States and the Double Closed Shell

According to the model of I, there should be a term in the potential energy due to interaction between each outer state and the double closed shell. For central forces this term depends only upon the orbital angular momentum of the outer state; furthermore, a harmonic oscillator potential leads to a kinetic energy term which has the same value for each state of the same main shell. Since both $s_{\frac{1}{2}}-d_{\frac{5}{2}}$ and $d_{\frac{3}{2}}-d_{\frac{5}{2}}$ energy differences have been taken into account empirically, we would expect to find a contribution $\epsilon(B)$ to the energy which is just proportional to the number of outer nucleons. [Compare, I (41).] This contribution

$$\epsilon(B)=(\mathfrak{B}_B+E_B)/B, \tag{33}$$

is given in Table XIII. $\mathfrak{B}_B$ is the experimental binding energy of B outer neutrons in an oxygen isotope. E_B is the calculated total interaction between outer states plus the empirical contributions just mentioned. Thus

$$E_2=\langle V(r_{12})+\Delta(1)+\Delta(2)\rangle_0 \text{ and } E_3=\langle V^{(3)}+\Delta^{(3)}\rangle_0, \tag{34}$$

where the brackets denote the expectation value for the ground state. It is plain from the table that $\epsilon(B)$ is not constant for cV_0. A similar discrepancy appears for the F and Ne isotopes, since the observed mass differences for each A are roughly accounted for by the present model. The situation is improved somewhat for the smaller depth V_0; however, we recall that for this parameter much of the agreement of Table XI is spoiled. We may put this result in another way: The present model does not account for the joint binding energy of two or three outer nucleons.

6. CONCLUSION

In I a variation method is used to obtain the best shell model wave function for a system of A nucleons under two-nucleon interactions. In reference 2 and the present paper calculations have been made for $A=17$, 18, and 19 which assume that the two-nucleon interaction is just an equivalent central potential which accounts for part of what is known about the neutron-proton system. The energy differences between single-particle levels in the $1d$, $2s$ shell observed for O^{17} have been introduced in an empirical way.

The ft value for the unfavored O^{19} β^- transition, and the energy difference due to nuclear forces between the ground states of O^{19} and F^{19}, as well as that between the $5/2+$ and $\frac{1}{2}+$ states of F^{19}, change rather rapidly with the depth. The remaining quantities in Table XI are seen to change slowly. It is plain that generally better agreement is obtained for the set of parameters (ν_0, cV_0) than for (ν_0, V_0); however, it was shown in Sec. 1 that these results are roughly equivalent to others with exactly the deuteron depth V_0, but a new well parameter ν_1, corresponding to a somewhat smaller nuclear radius. Actually a parameter intermediate between ν_0 and ν_1, but very close to ν_1, would seem best. The effects of a change in $\Delta(i)$ can also be related to a change in depth (see Sec. 3), and hence to the nuclear radius. The same holds for changes in the range ρ_0 of the two-nucleon interaction, since its matrix elements depend only upon $\rho_0^{-2}\nu^{-1}$ and V_0 (see references 2 and 5).

The calculations for cV_0 lead to agreement between theory and experiment for the following quantities (neglecting very small discrepancies, such as that of 0.02 nm in the magnetic moment of O^{17}, and taking account of present experimental uncertainties):

(1) The magnetic moment of O^{17}.
(2) The ft value for $Ne^{18}(\beta^+)F^{18}$ (which, however, is very inaccurately known).
(3) The energy difference between $5/2+$ and $\frac{1}{2}+$ states of F^{19}.
(4) The energy difference between ground states of O^{19} and F^{19}.
(5) The ft value for $Ne^{19}(\beta^+)F^{19}$.
(6) The ft value for $O^{19}(\beta^-)F^{19}$, $5/2+$.
[(7) The quadrupole moment of O^{17}.]

But there is at least quantitative disagreement for:

(−1) The energy difference between the ground states of F^{18} and O^{18}.
(−2) The ft value of $F^{18}(\beta^+)O^{18}$.
(−3) The energy of the $\frac{1}{2}+$ state of O^{19}.

(−4) The magnetic moment of F^{19}.
(−5) The half-life of the 5/2+ state of F^{19}.
(−6) The binding energy of the two outer neutrons of O^{18}.
(−7) The binding energy of the three outer neutrons of O^{19}.
[(−8) The half-life of the $\frac{1}{2}$+ state of O^{17}.]

Two quantities are enclosed in brackets. They are the ones which depend most substantially upon a center-of-mass effect. This has been taken account of by a core + single-particle model, which destroys the full antisymmetry of the A-particle wave function, and is therefore not consistent with the present model. Nonetheless this assumption might have some validity, and for that reason $q(O^{17})$ is included in the first list. We recall from paper I that the present model also accounts for the binding energy of one neutron in O^{17}, but not for the total binding energy of O^{16}.

There are two conclusions:

(1) The assumptions of the present model together with the approximations used here fail to account in a general way for the properties of states with parity + in the region of $A=17$ to 19. The very similar results obtained[10] for a Yukawa central interaction and the Rosenfeld mixture of forces make it appear unlikely that still another central potential or mixture of central forces could substantially alter this situation.

The possibility that contributions of configurations with $1f$ and $2p$ and even higher states may lead to better agreement with experiment should, however, be mentioned. Indeed, some calculations[13] based on a model in which neutrons and protons couple separately have indicated that this is generally true for the magnetic and quadrupole moments. Furthermore, one would expect that noncentral forces which are surely present in the interaction between two free nucleons would also prevail between bound nucleons. Their effect upon already measured quantities in the oxygen region has not yet been calculated and might sometimes be substantial.

(2) The agreement with experiment which was sometimes obtained may well be accidental. Nevertheless, in view of the possibilities mentioned under (1) it is conceivable that an accurate picture does contain some features of the present model. One might venture to surmise that such a picture would include: (a) A Hamiltonian containing a term

$$\sum_{i<j=2}^{A} V(i,j),$$

where $V(1,2)$ is an interaction which accounts accurately for all that is known about the two-nucleon system at low energies. (b) A totally antisymmetric A-nucleon wave function which, for nuclei in the oxygen region, can be approximated by a linear combination of a limited number of products of A single-particle wave functions.

ACKNOWLEDGMENTS

It is with much pleasure that I thank Professor Eugene P. Wigner for suggesting this investigation, and for his advice, interest, and encouragement. The benefit of Dr. Igal Talmi's knowledge of the new methods of spectroscopy which was obtained through his advice and valuable discussions is gratefully acknowledged. I am indebted to Mr. William H. Peirce of the Numerical Analysis Laboratory, University of Wisconsin, for diagonalizing many matrices.