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Radiations from a Spinning Rod

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The angular distribution and total value of the power carried in the gravitational and electromagnetic radiations from a spinning rod are evaluated and compared.

RECENTLY in the course of a study of gravitational radiation we had occasion to make a comparison between the electromagnetic and the gravitational radiation of energy from a spinning rod—the only example which seems at all simple to work out in the gravitational case.[1] Since the results of the comparison are by no means obvious in advance, we shall give a brief account of them here.

1. Gravitational Theory

Let J_{ik} be the tensor which gives the moment of inertia of the rod,

$$J_{ik}=\int (3x_ix_k-\delta_{ik}r^2)\rho(\mathbf{r})dv \quad (x_i,x_k=x,y,z), \tag{1}$$

where $\rho(\mathbf{r})$ is the mass density. It has been shown from the general theory of relativity that if the rod is long and narrow, and its ends are moving at a speed much less than that of light, and if $\mathbf{n}$ is a unit vector in the direction of observation, then the power per unit solid angle radiated in this direction is[2]

$$P_{\mathrm{n}}(G/32\pi c^5)[\tfrac{1}{4}(n_in_kA_{ik})^2+\tfrac{1}{2}A_{ik}{}^2-n_in_kA_{ij}A_{jk}]_{\mathrm{ret}}, \tag{2}$$

where

$$A_{ik}=d^3J_{ik}/dt^3. \tag{3}$$

To evaluate this we can choose coordinates such that the rod spins about the z-axis and at the retarded time lies instantaneously along the x-axis. At this instant only one independent component of J survives, and

$$A_{xy}=A_{yx}=-12I\omega^3, \tag{4}$$

where I is the rod's moment of inertia about the z-axis, or $\int \rho r^2dv$. With this, (2) is

$$P_{\mathrm{n}}=(4GI^2\omega^6/\pi c^5)[(n_xn_y)^2+n_z{}^2].$$

If the direction of observation has the polar angles θ and φ, with θ measured from the z-axis and φ vanishing on the x-axis, this becomes[3]

$$P_{\mathrm{n}}=(4GI^2\omega^6/\pi c^5)[\tfrac{1}{4}\sin^4\theta\,\sin^2 2\varphi+\cos^2\theta].$$

The angle φ gives the difference in azimuth between the rod and the observer. Averaging over φ gives

$$P_{\mathrm{n}}=(4GI^2\omega^6/\pi c^5)[\cos^2\theta+\tfrac{1}{8}\sin^4\theta], \tag{5}$$

and this is the radiation pattern, as illustrated in Fig. 1. The total radiated power is

$$P_T=2\pi\int_0^\pi P_{\mathrm{n}}\sin\theta d\theta=32GI^2\omega^6/5c^5 \tag{6}$$

in agreement with Eddington's result.[1] As a result of this energy loss the rod will spin more and more slowly, and the radiated energy, like that in Maxwell's theory, will either impart motion to surrounding objects or be lost in outer space.

2. Electromagnetic Theory

The radiation in the electromagnetic case is somewhat arbitrary, since the rod's center of charge does not necessarily correspond with its center of mass. But in order to get results comparable with those just found, let us suppose that it does, so that the electric

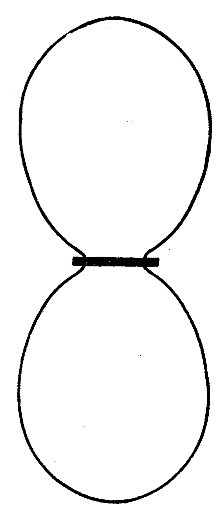

FIG. 1. Angular distribution of the intensity of the gravitational power radiated by a rod spinning about a vertical axis through its center of mass.

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[1] The problem of determining the total power radiated was first studied by Einstein [A. Einstein, Sitzber. preuss. Akad. Wiss., Physik-math. Kl. 1916, p. 688]. A computational error led to a second attack (*ibid.*, 1918, p. 154). That here also there is a mistake [it is a factor of 2 in Eq. (16) of Einstein's second paper] was pointed out by Eddington in his discussion [Proc. Roy. Soc. (London) **A102**, 268 (1922); *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, 1924), second edition, Note 8]. The question of the radiation pattern does not seem to have been discussed before.

[2] For convenience, we use the form given by Landau and Lifschitz *The Classical Theory of Fields* (Addison-Wesley Press, Cambridge, 1951), p. 331.

[3] This formula follows also from Eq. (18) of Eddington's paper cited in footnote 1.

dipole moment of the rod is zero and the magnetic dipole moment is constant. The radiation is therefore electric quadrupole, and using the notation of (1) and (3), but with ρ now denoting the charge density, we have for the Poynting flux per unit solid angle in the direction $\mathbf{n}$,[4]

$$P_{\mathbf{n}}=(1/144\pi c^5)[n_i n_k A_{ij}A_{jk}-(n_i n_k A_{ik})^2]. \tag{7}$$

If we introduce polar angles as before and average over all orientations of the rod, we find

$$P_{\mathbf{n}}=(I^2\omega^6/2\pi c^5)(1-\cos^4\theta), \tag{8}$$

and this pattern is plotted in Fig. 2. The total power given out is

$$P_T=8I^2\omega^6/5c^5, \tag{9}$$

where I is, of course, the electrostatic analog of the quantity which appears in (5).

In order to understand better the striking dissimilarity between the two patterns, let us consider the radiation along the rod's axis of rotation. In the electromagnetic case, symmetry excludes any transverse electric or magnetic field along the axis, but for the gravitational potentials the situation is quite different. Writing g_{ik} as $\eta_{ik}+h_{ik}$, where η_{ik} are the Galilean values, we find that only the transverse components h_{xx} h_{yy}, and h_{xy} do not vanish on the axis, and it is well known that these components carry energy. If one evaluates them in a new coordinate system whose z-axis coincides with the old one and whose x- and y-axes are rotated so as to make h_{xy} vanish, we find that at any point z this system is rotating at a rate 2ω, its angular position being given by $2\omega(t-z/c)$. The gravitational field spirals out along the axis, so to speak, with the speed of light.

In order to clarify further the question of the angular distribution of these two sorts of quadrupole radiation, let us consider the following idealized situation. There are two masses, M and m, with $M\ll m$, whose center of gravity is at rest at the origin. By means of an essentially massless arrangement of rods and ropes some people in the large mass impart to the small one a velocity $\mathbf{v}(\ll c)$ and a uniform acceleration $\mathbf{a}$. We shall find the quadrupole gravitational radiation due to this motion and, assuming that the charges are in the same ratio as the masses, the quadrupole electromagnetic radiation also. The inertia tensor is

$$J_{ik}=m(3x_ix_k-\delta_{ik}r^2)$$

plus a corresponding term contributed by M. But since x_i and its derivatives vary inversely with the

[4] See reference 2, p. 206.

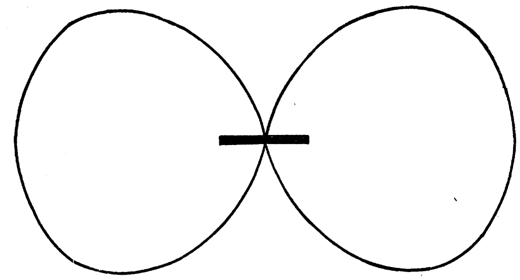

FIG. 2. Angular distribution of the intensity of the electromagnetic power radiated by a charged rod spinning about a vertical axis through its center of charge.

masses, and since they appear quadratically in J_{ik}, the latter contribution can be neglected. Evaluation of (2) and (7) now gives

$$P_{\mathbf{n}}=(81Gm^2/32\pi c^5)[\mathbf{a}\times\mathbf{n}]^2[\mathbf{v}\times\mathbf{n}]^2, \tag{10}$$

with

$$P_T=(27Gm^2/20c^5)[3a^2v^2+(\mathbf{a}\cdot\mathbf{v})^2], \tag{11}$$

and

$$P_{\mathbf{n}}=(9e^2/16\pi c^5)[(\mathbf{a}\cdot\mathbf{n})[\mathbf{v}\times\mathbf{n}]+(\mathbf{v}\cdot\mathbf{n})[\mathbf{a}\times\mathbf{n}]]^2, \tag{12}$$

with

$$P_T=(6e^2/5c^5)[2a^2v^2-(\mathbf{a}\cdot\mathbf{v})^2], \tag{13}$$

respectively, from which one sees that if $\mathbf{n}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{v}$, the first expression for $P_{\mathbf{n}}$ is a maximum and the second is zero, whereas if $\mathbf{n}$ is parallel to $\mathbf{a}$ or $\mathbf{v}$, the reverse is true.

This difference in patterns is characteristic of the nature of the field quantities. It does not reflect the nonlocalizability of energy in a gravitational field, for though the density of this energy is indeed not an observable quantity, the flux which we have calculated can in principle be measured. (We have verified this conclusion by computing the energy-flux terms in Pauli's modified expression for the gravitational energy[5] and showing that they reduce to the expressions given above.) And whereas it is natural to attempt to account for the difference in the numerical coefficients in (6) and (9) by the fact that the gravitational field has the more components, it can unfortunately be shown that the number of independent components is only two in each case.[6]

This investigation was carried out in a seminar in cooperation with L. Scarfone and W. Williams.

[5] W. Pauli, Physik. Z. **20**, 25 (1919).
[6] M. Fierz, Helv. Phys. Acta **12**, 3 (1939); **13**, 45 (1940); J. S. de Wet, Phys. Rev. **58**, 236 (1940).