Behavior of a Bose System

G. V. CHESTER
*Yale University, New Haven, Connecticut**
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It is shown that a weakly interacting system of Bose particles undergoes a second-order transition both under conditions of constant volume and under conditions of constant pressure. This type of behavior is qualitatively identical with that exhibited by liquid helium.

IT is now well established[1,2] that a system composed of noninteracting particles obeying Bose-Einstein statistics undergoes a phase transition. If the density of the system is taken to be equal to that of liquid helium and if the mass of the particles is taken to be that of a helium atom, then the transition takes place at about 3°K. This fact has led to many attempts to relate the λ transition that occurs in liquid helium, at about 2°K, to the transition exhibited by a system of noninteracting Bose particles. While it is quite clear that such a system cannot accurately account for the properties of a liquid, it is, nevertheless, true that theories based on it have had some success in correlating the experimental properties of liquid helium. In this note we shall compare the transition in the liquid with the transition that occurs in a system of *weakly interacting* Bose particles. It must be emphasized that in this type of system we can only take into account weak attractive and repulsive forces between the particles; we are quite unable to deal with the very strong repulsive forces that ultimately come into play when the helium atoms approach sufficiently close to one another. However, this type of system has the advantage that it can be given a fairly rigorous treatment by means of perturbation theory.[3]

The transition in liquid helium appears to be a second-order transition whether we study it under conditions of constant volume or under conditions of constant pressure. For instance, the specific heat curves at constant pressure and constant volume are very similar in shape and both exhibit the same type of discontinuity at the λ point. The transition that occurs in a system of noninteracting Bose particles is, however, a first-order transition if we study the system under conditions of constant pressure, and a third-order transition if we study it under conditions of constant volume. Thus, although such a system does exhibit a transition, the details of it are very different from those of the λ transition in the liquid. This fact has been emphasized by Rice[4] and London.[5]

In I the author discussed some of the properties of a system of weakly interacting Bose particles. It was shown that such a system undergoes a second-order transition under conditions of constant volume. We shall now show that it undergoes a second-order transition under conditions of constant pressure. The free energy of the system is, to the accuracy of first-order perturbation theory,

$$F=F_0+g(F_1+F_1'), \tag{1}$$

where F_0 is the free energy of the noninteracting system, g is the coupling constant between the particles, F_1 and F_1' are given by[3]

$$\left.\begin{aligned} F_1&=\tfrac{1}{2}\sum_{\mathbf{k}\neq\mathbf{k}'}\sum \bar{n}_{\mathbf{k}}\bar{n}_{\mathbf{k}'}K_{\mathbf{k},\mathbf{k}'}+\bar{n}_0\sum_{\mathbf{k}\neq 0}\bar{n}_{\mathbf{k}}K_{\mathbf{k},0},\\ F_1'&=\tfrac{1}{2}N(N-1)K_{0,0}.\end{aligned}\right\} \tag{2}$$

These equations follow at once from Eqs. (2.15) and (1.2) of I. It is easily seen from Eq. (2.14) of I that $K_{\mathbf{k},\mathbf{k}'}$ is proportional to V^{-1}, for all $\mathbf{k}$ and $\mathbf{k}'$. Since each summation over the wave vectors $\mathbf{k}$ introduces a factor proportional to V the first term in F_1 is proportional to V. On the other hand, $\bar{n}_0$ is equal to $N[1-C(T)V]$, where $C(T)$ is a function of T alone. The second term in F_1 therefore consists of a term that is proportional to V and a term that is independent of V. Equation (2.27) of I shows in fact that F_1 is of the form

$$F_1=N[a(T)\rho^{-1}+b(T)]; \tag{3}$$

while from Eq. (2.14) of I we see that F_1' is given by

$$F_1'=2\pi N\rho\int_{2r_0}^{\infty}w(r)r^2dr, \tag{4}$$

where $a(T)$ and $b(T)$ are functions of the temperature alone, $w(r)$ is the interaction potential between the particles and r_0 is a cut-off parameter. We first note that $(\partial^2F_0/\partial V^2)_T\equiv 0$, $T\leqslant T_\lambda$; next, from Eqs. (3) and (4) we see that $(\partial^2F_1/\partial V^2)_T\equiv 0$, while $(\partial^2F_1'/\partial V^2)_T>0$ provided that

$$\int_{2r_0}^{\infty}w(r)r^2dr>0. \tag{5}$$

If we take the same Lennard-Jones (6, 12) potential that we used in I, *viz.* Eq. (4.1) of I, then we find that the inequality (5) will be satisfied as long as $r_0<1.14$ A.

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The value of r_0 used in I was 1.09 A, so the inequality is easily satisfied. Consequently $(\partial p/\partial V)_T<0$, for $T<T_\lambda$, and the system is stable below the λ temperature. If we bear in mind that $(\partial p/\partial V)_T$ is discontinuous at T_λ then this implies that the system undergoes a second-order transition under conditions of constant pressure. The inequality (5) has a simple physical interpretation. It can only be satisfied if the potential is positive for some range of values of r. This simply means that it is the positive or "repulsive" part of the potential that prevents the particles from "condensing" into zero volume and exhibiting a first-order transition.

We can therefore conclude that a system of weakly interacting Bose particles undergoes a second-order transition both under conditions of constant pressure and constant volume. This behavior is qualitatively identical with that shown by liquid helium. These conclusions are, however, based on a treatment of a weakly interacting system and it remains to be shown that they carry over to the physically interesting case of a strongly interacting one.

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Quantum Effects in the Interaction between Free Electrons and Electromagnetic Fields

P. S. Faragó, *Department of Electromagnetic Waves, Central Research Institute for Physics, Budapest, Hungary*

AND

G. Marx, *Department of Theoretical Physics, Physical Institute of the Eötvös University, Budapest, Hungary*

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An electron beam shot through a transverse rf field may suffer a directional spread. If experimental conditions are suitably chosen, the directional spread may be due only to the quantum dispersion of energy exchange between free electrons and rf field. A simple collector electrode system might allow not only the detection of the directional spread of the electrons, but the presence of a quantum effect might be checked by plotting the collector current *versus* rf field amplitude, the plot for the quantum effect being different from those for classical effects. The results of various theoretical treatments of the effect are briefly compared, both from the point of view of their principal foundations and of the possibility of their experimental verification.

In recent years several papers[1–4] have been published dealing with that part of the dispersion in energy exchange between free electrons and rf fields which the classical theory cannot predict. The problem has been treated from various points of view and differing results were obtained for the energy spread due to the quantum nature of the interaction.

One method[1,3] consists essentially in the assumption of the independence of the elementary emission and absorption processes, so that statistical methods can be applied in treating them. In this case the magnitude of quantum dispersion of the velocity of the electrons can be estimated by taking the classical energy gain W_+ and energy loss W_- during the accelerating and decelerating periods and interpreting them as the average absorption and emission of $N_+=W_+/\hbar\omega$ and $N_-=W_-/\hbar\omega$ photons, respectively, with a standard deviation of

$$\Delta W=\hbar\omega(N_++N_-)^{\frac{1}{2}}. \quad (1)$$

Shulman[3] reports that he has found this effect by measuring the energy distribution of electrons shot through a longitudinal rf field in a wave guide. The difficulty of any experiment of this kind lies in the fact that the expected effect of quantum-mechanical origin and the classical energy spread resulting from the emission velocity distribution and from the dependence of transit time on the entrance phase angle of the electrons are of the same order of magnitude. The classical effects, however, can be made negligible in comparison with the quantum effect, if the electrons are shot through a transverse rf field and the transit angle of the electrons is properly chosen. In this case there should be a dispersion in the direction of the electron velocities caused only by the quantum nature of the energy exchange.

For the sake of simplicity we considered[5] a rectangular cavity excited in its TE_{012} mode. The only component of the electric field is, in this case:

$$E_z=E_0\sin(2\pi x/a)\cdot\sin(2\pi y/b)\cdot\sin\omega t,$$

the symbols having their usual meaning. For a given resonance frequency one can always find such a cavity that an electron entering along the x-direction with a given velocity v_0 spends an integral number of periods

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