# Radiative Correction to High-Energy Electron Scattering*

HIROSHI SUURA†
*Department of Physics, Stanford University, Stanford, California*
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The one-photon radiative correction to the high-energy electron scattering by a nuclear field is analyzed to all orders of Born approximation for the nuclear potential. The leading term of the fractional decrease of the elastic scattering cross section is shown to be exactly the same as that given by the first Born approximation. A detailed analysis of the effect of the long Coulomb tail on the infrared divergence is made. Also, an analysis is made on the origin of the $\log m$ term which appears in the radiative correction.

## I. INTRODUCTION AND SUMMARY

THE radiative correction to electron scattering by a nuclear field was calculated by Schwinger[1] in the first Born approximation for the nuclear potential, and to the lowest order in the coupling between the electron and the radiation field. (The phrase "Born approximation" will refer here to the interaction between the electron and the nuclear potential.) In view of recent experiments[2] on the scattering of high-energy electrons by heavy nuclei, it is the purpose of this paper to investigate higher Born approximations of the radiative correction, still including only one-photon processes. According to A, the contribution to the cross section from the radiative correction, $\sigma_{\rm rad}$, is proportional to the first Born elastic scattering cross section, $\sigma_{\rm el}$, so that to order $(Z\alpha)^2$ we have

$$\sigma_{\rm rad}=-\delta\sigma_{\rm el}, \tag{1}$$

with the fractional decrease $\delta$ being given asymptotically in the high-energy limit by

$$\delta\sim(4\alpha/\pi)\log[(2p_0\sin\theta/2)/m]\log(p_0/\Delta E), \tag{2}$$

Here, the notation is as follows: $\alpha$ is the fine structure constant, $m$ the electron mass, $p_0$ the electron energy, $\theta$ the scattering angle, and $\Delta E$ is the energy resolution associated with the experiment. We use units such that $\hbar=c=1$ throughout. The expression (2) is the leading term of $\delta$ in the sense that it is the product of the two logarithms with the large arguments $p_0/m$ and $p_0/\Delta E$, the neglected terms being either linear in these logarithms or constants of order unity depending only on the scattering angle $\theta$. We shall use the symbol $\sim$ for this meaning. We show in this paper that, in the same sense as above, the expression (2) represents the leading term of $\delta$ to *all* orders of the Born approximation, so that $\sigma_{\rm el}$ in Eq. (1) now stands for the exact elastic scattering cross section. Since the expression (2) is apparently connected with the infrared divergence, we shall prove the above statement in two steps. First we shall show that the coefficient of the infrared divergence, namely, the coefficient of $\log(\Delta E)$, is exactly given by the same expression as (2) to all orders of Born approximation in the high-energy limit. The second step is to show that no $(\log m)^n$ with $n\geqq 2$ appears in the radiative correction. This will exclude such large terms as $[\log(p_0/m)]^2$ and at the same time will fix the argument of the second logarithm in (2) as $p_0/\Delta E$, since $\delta$ should be a function of the three parameters $m/p_0$, $\Delta E/p_0$ and $\Delta E/m$ in the case of the point Coulomb potential. (We consider large angle scattering, so that $\sin\theta/2$ is of order unity, and we can omit it from consideration in our discussion of the leading term.) If the nuclear charge has an extension of order $1/a$, Eq. (2) will be still true. This is because no term such as $\log(a/p)$ appears in the cross section. If it did, the differential cross section would diverge in the point charge limit $a\rightarrow\infty$.

Concerning the first step, the same conclusion has been reached by Newton[3] and also by Jauch and Rohrlich.[4] The essential point of the argument of these authors is that in the inelastic scattering of an electron with emission of a soft photon, the infrared divergence occurs only when the photon is emitted either before or after the potential scatterings (see diagrams A and C of Fig. 1). This follows from a simple inspection of the energy denominators appearing in the integrand of the various processes. It is also physically acceptable, since the electron confined in the scattering region would be more likely to emit hard photons of wavelength shorter than the atomic dimension, rather than soft ones.

However, such an argument applies only for low-energy electrons in a screened potential.[5] For electrons of energy as high as a few hundred Mev the large-angle scattering takes place almost inside the nucleus, and since the energy resolution also increases, we have to deal with relatively high energy "soft photons." In such a case the screening plays no role at all. In the unscreened potential, the electron could emit soft photons in the course of the gradual deflection by the long tail of the Coulomb potential. Therefore, we should expect an infrared divergence also in the processes omitted in the

* Supported by the Office of Scientific Research, Air Research and Development Command.
† On leave of absence from Hiroshima University, Hiroshima, Japan. Now at Cornell University, Ithaca, New York.
[1] J. Schwinger, Phys. Rev. **76**, 790 (1949), referred to as A.
[2] Hofstadter, Fechter, and McIntyre, Phys. Rev. **92**, 978 (1953); Hofstadter, Hahn, Knudsen, and McIntyre, Phys. Rev. **95**, 512 (1954).
[3] R. G. Newton, Phys. Rev. **94**, 1773 (1954).
[4] J. M. Jauch and F. Rohrlich, Helv. Phys. Acta **27**, 613 (1954).
[5] The importance of the screening was noticed by Mittleman in his paper which also reached to Eq. (1) for the nonrelativistic electron in a screened potential. M. H. Mittleman, Phys. Rev. **93**, 453 (1954).

foregoing discussion. Mathematically, this is reflected in the fact that the elastic scattering matrix element between the two virtual states (off-energy-shell matrix element) has a discontinuity on the energy shell, as was noticed by Dalitz.[6] This means that we cannot always neglect the photon momentum in the internal electron propagators between the successive potential scatterings, as was done in references 3 and 4.

In Sec. II, we shall perform the potential integrations of the second Born approximation in the soft photon limit, and separate out the discontinuity term which contributes to the additional infrared divergences. It will be shown that this discontinuity term actually comes from the small momentum transfer by the electron to the external field, either just before or after the emission of the photon, which indicates the effect of the long Coulomb tail. However, these additional infrared divergences cancel out completely, when the contributions from all the possible processes of Bremsstrahlung are added. The same cancellation occurs also for the virtual radiative processes. This cancellation of the additional infrared divergences resulting from the Coulomb tail is quite general, and can be shown to hold to any order of Born approximation and also for any number of soft photons emitted.[7] Thus, we are justified to take only such diagrams as (A) and (C) of Fig. 1 and (A) of Fig. 3, and to set $k=0$ in the internal electron line $AB$ in the consideration of the infrared divergence.

The second step, namely to prove the nonexistence of a term $(\log m)^2$, needs an analysis of the origin of $\log m$,[8] which appears in the first Born approximation. This will be done in Sec. III by a careful examination of the photon momentum integration. It develops that $\log m$ originates from each of the "outer" electron propagators such as $XA$ and $BY$ of diagram (A) of Fig. 3, when the photon momentum is parallel to the incident or outgoing electron momentum. Since for a large-angle scattering the incident and scattered electron momenta are not parallel, and since the poles of the internal electron lines between the potential scatterings do not generally coincide with those of the outer electron lines, we can infer the nonexistence of the higher powers of $\log m$.[9]

A remark will be necessary about the fact that some methods for treating the infrared divergence, for instance, the method employed in A and the method of assuming a small photon mass, actually give $(\log m)^2$ in some integrals even in the calculation of the first Born approximation. However, such $(\log m)^2$ terms are of a quite fictitious character, since they are shown to originate from the contribution of the soft photon, and hence are bound to cancel in the final expression. The method employed in the present paper, namely, to restrict the photon momentum space by the condition $K \geqq K_{\min}$, has the advantage of giving no fictitious $(\log m)^2$ of the above kind[10] and is suited for the present purpose. The noncovariant character of the method leads to no trouble at all as is seen by the facts that all the methods give the same value for the noninfrared divergent integrals and that we get a unique answer for the radiative correction in the first Born approximation by any method.

A more important point to examine is the effect of the long tail of the Coulomb potential, which has the possibility of giving rise to higher powers of $\log m$ by overlapping the poles of the internal electron lines with those of the outer. This is also checked and the answer is negative.

## II. INFRARED DIVERGENCE IN A COULOMB FIELD

For the three possible diagrams of the inelastic scattering in the second order (Fig. 1) we have the following matrix elements in momentum representation:

$$M_A=[ieF/(2K)^{\frac{1}{2}}]\int d\mathbf{l}(\mathbf{l}^2\mathbf{l}'^2)^{-1}\bar{u}_q\gamma_0[i\gamma\cdot(p-k-l)+m]^{-1} \times\gamma_0[i\gamma\cdot(p-k)+m]^{-1}i\gamma\cdot\epsilon u_p, \quad (3)$$

$$M_B=[ieF/(2K)^{\frac{1}{2}}]\int d\mathbf{l}(\mathbf{l}^2\mathbf{l}'^2)^{-1}\bar{u}_q\gamma_0[i\gamma\cdot(p-k-l)+m]^{-1} \times i\gamma\cdot\epsilon[i\gamma\cdot(p-l)+m]^{-1}\gamma_0 u_p, \quad (4)$$

where the momenta $l$, $l'$, and $k$ satisfy the conditions

$$\mathbf{q}-\mathbf{p}+\mathbf{l}+\mathbf{l}'+\mathbf{k}=0, \quad q_0-p_0+k_0=0, \quad \text{and} \quad k^2=0.$$

$M_C$ is similar to $M_A$ and will not be mentioned here. Here, light-face letters $p$, $q$, and $k$ denote the four momenta of the incident electron, the scattered electron

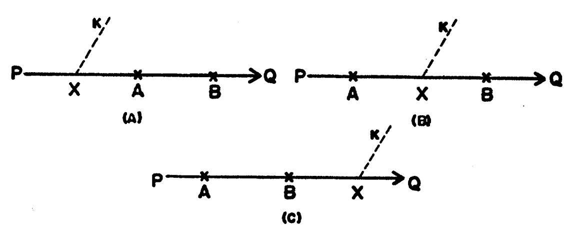


FIG. 1. Feynman diagrams for second order inelastic processes. ———, electron line; - - - - - - -, photon line; ×, external field.

[6] R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).

[7] Such a Coulomb anomaly has been investigated independently by Newton in his paper on the second Born calculation of the radiative correction to the electron scattering. R. G. Newton, Phys. Rev. **97**, 1162 (1955). His result is essentially the same as that of the present paper, although his calculation is limited to the second order, and the anomaly is treated not as an additional infrared divergence but as a Coulomb divergence.

[8] The pure elastic scattering cross section can be expanded in powers of $m$. Yennie, Ravenhall, and Wilson, Phys. Rev. **95**, 500 (1954).

[9] The aforementioned paper of Newton gave $(\log m)^2$ in the second Born approximation. However, after the completion of the present work, his result was corrected and it no longer gives a $(\log m)^2$ term. Hence, his result confirms the validity of the general argument presented in this paper. Roger G. Newton, Phys. Rev. **98**, 1514 (1955). See also, Max Chrétien, Phys. Rev. **98**, 1515 (1955).

[10] The invariant minimum momentum method employed in A, has some mathematical difficulty as pointed out by Elton and Robertson. A close analysis shows that the method in A is almost equivalent to assuming a small photon mass. L. R. B. Elton and H. H. Robertson, Proc. Phys. Soc. (London) **A65**, 144 (1952).

and the emitted photon, respectively. $l$ and $l'$ are the momentum transfers at $A$ and $B$, respectively; bold-face letters will be used for the spatial vectors, and capital letters for their magnitude. Thus, for instance,

$$p=(\mathbf{p};p_0),\quad |\mathbf{p}|=P;\quad l=(\mathbf{l};0),\quad |\mathbf{l}|=L.$$

$\epsilon$ denotes the polarization vector of the photon. $F$ represents $F=(2\pi)^4(-Ze^2/2\pi^2)^2$. The other notations will be evident. The second Born elastic scattering matrix element is given by

$$M_{\mathrm{el}}=iF\int d\mathbf{l}(\mathbf{l}^2\mathbf{l}'^2)^{-1}\bar{u}_q\gamma_0[i\gamma\cdot(p-l)+m]^{-1}\gamma_0u_p, \quad (5)$$

with

$$\mathbf{q}-\mathbf{p}+\mathbf{l}+\mathbf{l}'=0,\quad\text{and}\quad q_0-p_0=0.$$

Rationalizing the denominators and neglecting $k$ in the numerator, we get

$$M_A=\frac{ieF}{(2K)^{\frac{1}{2}}}\left(-\frac{p\cdot\epsilon}{p\cdot k}\right)\int d\mathbf{l}\times\frac{\bar{u}_q(2\gamma_0p_0-i\gamma\cdot l)u_p}{[(k+l)^2-2p\cdot(k+l)]\mathbf{l}^2(\mathbf{l}-\Delta\mathbf{p}+\mathbf{k})^2}, \quad (3')$$

$$M_B=\frac{ieF}{(2K)^{\frac{1}{2}}}\int d\mathbf{l}\times\frac{[2\epsilon\cdot(p-l)]\bar{u}_q(2\gamma_0p_0-i\gamma\cdot l)u_p}{[(k+l)^2-2p\cdot(k+l)]\times(l^2-2l\cdot p)\mathbf{l}^2(\mathbf{l}-\Delta\mathbf{p}+\mathbf{k})^2}, \quad (4')$$

$$M_{\mathrm{el}}=iF\int d\mathbf{l}\frac{\bar{u}_q(2\gamma_0\cdot p_0-i\gamma\cdot l)u_p}{(l^2-2l\cdot p)\mathbf{l}^2(\mathbf{l}-\Delta\mathbf{p})^2}, \quad (5')$$

with

$$\Delta\mathbf{p}=\mathbf{p}-\mathbf{q},\quad |\Delta\mathbf{p}|=\Delta P=2P\sin(\theta/2)\quad\text{for } M_{\mathrm{el}}.$$

In Eq. (4′), we have neglected a noninfrared divergent term

$$\int d\mathbf{l}\frac{\bar{u}_q i\gamma\cdot\epsilon u_p}{[(k+l)^2-2p\cdot(k+l)]\mathbf{l}^2(\mathbf{l}-\Delta\mathbf{p}+\mathbf{k})^2},$$

which is discontinuous but finite in the limit $k\to 0$, as will be seen from Eqs. (6), (9), and (10) below. The potential integrations in (3′) and (5′) are all reduced to the first two of the following integrals by the change of the integration variable according to $\mathbf{p}-\mathbf{k}-\mathbf{l}\to\mathbf{l}$ for (3′) and $\mathbf{p}-\mathbf{l}\to\mathbf{l}$ in (5′)

$$(I_0;I_i;I_{ij})=\int d\mathbf{l}(1;l_i;l_il_j)\times[(\mathbf{l}^2-R^2)(\mathbf{l}-\mathbf{p}')^2(\mathbf{l}-\mathbf{q}')^2]^{-1}, \quad (6)$$

with

$$\mathbf{p}'=\mathbf{p}-\mathbf{k},\quad \mathbf{q}'=\mathbf{q},\quad R^2=Q^2=P^2-2p_0K+K^2\quad\text{for } M_A, \quad (7)$$

$$\mathbf{p}'=\mathbf{p},\quad \mathbf{q}'=\mathbf{q},\quad R^2=Q^2=P^2\quad\text{for } M_{\mathrm{el}}. \quad (8)$$

[In (5), the pole of the denominator $l^2-R^2$ is defined by adding a small negative imaginary part, according to Feynman's prescription.[11]] The integrals (6) can be integrated exactly. Although the quantities $\mathbf{p}'$, $\mathbf{q}'$ and $R$, defined by (7) for $M_A$, approach the corresponding quantities for $M_{\mathrm{el}}$, (8), in the soft photon limit $k\to 0$, the corresponding values of the integrals do not. In order to see that there is this discontinuity on the energy shell, the integration was made by employing the Yukawa potential $e^{-\lambda\gamma}/\gamma$ instead of the Coulomb potential. The value of the integral (6) as well as its value near the energy shell (8) in the limit $\lambda\to 0$ are listed in the Appendix. The integrals assume the following form near the energy shell:

$$\begin{aligned}I_0&=I_0^{\mathrm{el}}+\Delta(f)+\Delta(g),\\ I_i&=I_i^{\mathrm{el}}+p_i\Delta(f)+q_i\Delta(g),\\ I_{ij}&=I_{ij}^{\mathrm{el}}+p_ip_j\Delta(f)+q_iq_j\Delta(g),\end{aligned} \quad (9)$$

where $I_0^{\mathrm{el}}$, $I_i^{\mathrm{el}}$ and $I_{ij}^{\mathrm{el}}$ denote the value of the integrals (6) on the energy shell (8), while $\Delta(f)$ is the discontinuity term defined by

$$\Delta(f)=\lim_{\lambda\to 0}\frac{\pi^2}{P\Delta P^2}\left(\tan^{-1}\frac{f}{2\lambda P}+i\log\frac{2\lambda P}{(f^2+4\lambda^2P^2)^{\frac{1}{2}}}\right). \quad (10)$$

Here $f$ and $g$ represent the measure of the distance from the energy shell,

$$f=P'^2-R^2,\quad g=Q'^2-R^2. \quad (11)$$

It is seen from (10) that the value of $\Delta(f)$ depends critically on the relative magnitude of $f$, $g$, and $\lambda$, which is the origin of the discontinuity. The condition on the magnitude of $K$ for Eq. (9) to hold is

$$K\ll\Delta P, \quad (12)$$

which is necessary and sufficient to replace $\Delta P'=|\mathbf{p}'-\mathbf{q}'|$, $P'$, and $Q'$ by the corresponding quantities $\Delta P$, $P$, and $Q$, on the energy shell (8) respectively, everywhere except in $f$ and $g$.

From (7) and (11), and remembering $q_0+K=p_0$, we have

$$f=-2p\cdot k>0,\quad g=0\quad\text{for } M_A.$$

Using these values in (9) and (10), we get from (3′) and (5′)

$$M_A=\frac{ie}{(2K)^{\frac{1}{2}}}\left(-\frac{p\cdot\epsilon}{p\cdot k}\right)\times[-iM_{\mathrm{el}}+F(\bar{u}_q2p_0\gamma_0u_p)\Delta(-2p\cdot k)]. \quad (13)$$

In just the same way, we have also that

$$M_C=\frac{ie}{(2K)^{\frac{1}{2}}}\left(\frac{q\cdot\epsilon}{q\cdot k}\right)\times[-iM_{\mathrm{el}}+F(\bar{u}_q2p_0\gamma_0u_p)\Delta(2q\cdot k)]. \quad (14)$$

[11] R. P. Feynman, Phys. Rev. 76, 749 (1949).

Since the real parts of $\Delta(-2p\cdot k)$ and $\Delta(2q\cdot k)$ are $\pm\pi^3/2P\Delta P^2$, respectively, these terms do actually give the additional infrared divergence.

The evaluation of $M_B$ can be performed as follows. Introducing an auxiliary variable $x$, we combine the two propagators of (4′) into

$$\int_0^1 dx[l^2-2l\cdot(p-xk)-2xp\cdot k]^{-2}.$$

If we change the integration variable by $\mathbf{p}-\mathbf{l}-x\mathbf{k}\rightarrow\mathbf{l}$, we get integrals of the form

$$\int_0^1 dx(\partial/\partial R^2)I_i \quad\text{and}\quad \int_0^1 dx(\partial/\partial R^2)I_{ij},$$

with

$$R^2=P^2-2xp_0K+x^2K^2,$$
$$\mathbf{p}'=\mathbf{p}-x\mathbf{k} \quad\text{and}\quad \mathbf{q}'=\mathbf{q}+(1-x)\mathbf{k},$$

which give from (11)

$$f=-2xp\cdot k \quad\text{and}\quad g=2(1-x)q\cdot k. \tag{15}$$

At $x=0$ and $x=1$, we have $f=0$ and $g=0$, respectively, and the discontinuity at these points gives an infinite integrand when differentiated by $R^2$. Therefore, only the discontinuity factors $\Delta(f)$ and $\Delta(g)$ in $I_i$ and $I_{ij}$ contribute terms of order $1/K$. Thus, the foregoing integrals reduce to the evaluation of

$$\int_0^1 dx(\partial/\partial R^2)\Delta(f) \quad\text{and}\quad \int_0^1 dx(\partial/\partial R^2)\Delta(g).$$

From (11), (15), and (A-8), we have

$$\begin{aligned}\int_0^1 dx(\partial/\partial R^2)\Delta(f)&=-\int_0^1 dx(\partial/\partial f)\Delta(f)\\&=\int_0^1 dx(2p\cdot k)^{-1}(\partial/\partial x)\Delta(-2xp\cdot k)\\&=(2p\cdot k)^{-1}\Delta(-2p\cdot k).\end{aligned} \tag{16}$$

In the same way, we get

$$\int_0^1 dx(\partial/\partial R^2)\Delta(g)=-(2q\cdot k)^{-1}\Delta(2q\cdot k). \tag{17}$$

Equations (16) and (17), together with (6), enable us to write (4′) as

$$\begin{aligned}M_B&=[ieF/(2K)^{\frac{1}{2}}](\bar{u}_q 2p_0\gamma_0 u_p)\\&\quad\times[(p\cdot\epsilon/p\cdot k)\Delta(-2p\cdot k)-(q\cdot\epsilon/q\cdot k)\Delta(2q\cdot k)].\end{aligned} \tag{18}$$

Adding (13), (14), and (18), we get

$$\begin{aligned}M_{\text{inel}}&=M_A+M_B+M_C\\&=-[e/(2K)^{\frac{1}{2}}](p\cdot\epsilon/p\cdot k-q\cdot\epsilon/q\cdot k)M_{\text{el}},\end{aligned} \tag{19}$$

which is equivalent to what we would obtain if we were to take only $M_A$ and $M_C$ and neglect $k$ in the potential integration from the beginning. The same procedure applied to the higher Born approximations is readily shown to give the same expression as above, so that Eq. (18) is the exact expression, so long as the cancellation of the additional infrared divergences always occur, as shown later. The inelastic cross section for the emission of a photon of energy smaller than certain energy $\Delta E$ such that $\Delta E\ll\Delta P$ is then

$$\begin{aligned}\sigma_{\text{inel}}&=[e^2/(2\pi)^3]\sum_\epsilon\int_{K_{\min}}^{\Delta E} d\mathbf{k}(1/2K)\\&\quad\times(p\cdot\epsilon/p\cdot k-q\cdot\epsilon/q\cdot k)^2\cdot\sigma_{\text{el}}.\end{aligned} \tag{20}$$

Performing the polarization sum and rearranging the terms, we have

$$\begin{aligned}\sigma_{\text{inel}}&=(\alpha/2\pi^2)\int_{K_{\min}}^{\Delta E} d\mathbf{k}(1/K)\\&\quad\times[p^2/(p\cdot k)^2-p\cdot q/(p\cdot k)(q\cdot k)]\sigma_{\text{el}}.\end{aligned} \tag{21}$$

We shall next show that the second term of $M_A$, Eq. (13), stems actually from the small momentum transfer at the point $A$. This means that in (3′), small $\mathbf{l}$ contributes to $\Delta(-2p\cdot k)$ term, or that in $I_0$, $I_i$ and $I_{ij}$ defined by (6) the small region around $\mathbf{l}=\mathbf{p}$ gives $\Delta(f)$. In order to see this, we take the difference $M_A$, (3′), and $M_{\text{el}}$, (5′), and show that this gives $\Delta(-2p\cdot k)$ when integrated over a small region of $\mathbf{l}$ around the origin.

For small $\mathbf{l}$, we may approximate the integrands of (3′) and (5′) as follows:

$$\begin{aligned}\int d\mathbf{l}\rightarrow\int d\mathbf{l}2p_0\gamma_0/\{&[-2p\cdot(k+l)-i\epsilon]\\&\times(\mathbf{l}^2+\lambda^2)\Delta\mathbf{p}^2\},\quad(\lambda,\epsilon\rightarrow0),\end{aligned} \tag{22}$$

$$\begin{aligned}\int_{\text{el}} d\mathbf{l}\rightarrow\int d\mathbf{l}2p_0\gamma_0/[&(-2p\cdot l-i\epsilon)\\&\times(\mathbf{l}^2+\lambda^2)\Delta\mathbf{p}^2],\quad(\lambda,\epsilon\rightarrow0).\end{aligned} \tag{23}$$

Taking the difference and performing the angle integration first, we have

$$\begin{aligned}\int_A d\mathbf{l}-\int_{\text{el}} d\mathbf{l}&\\=2\gamma_0p_0(\pi/P\Delta P^2)&\\\times\Big\{&\int dL(1/L)\log(p\cdot k-PL)/(p\cdot k+PL)\\&+i\int dL[L/(L^2+\lambda^2)]\\&\qquad\times[\tan^{-1}(2p\cdot k+2PL)/\epsilon-\tan^{-1}2PL/\epsilon]\Big\}.\end{aligned}$$

But the difference of the two inverse tangents in the second term has the value $-\pi$ for $PL<-p\cdot k(>0)$, and vanishes for $PL>-p\cdot k$. Hence the second term gives, when integrated over $[0, -p\cdot k/P]$,

$$(i\pi^2/P\Delta P^2)\log[\lambda P/(-p\cdot k)],$$

which is just equal to the imaginary part of $\Delta(-2p\cdot k)$. The first integrand becomes infinite at $L=-p\cdot k/P$, and decreases as $1/L^2$ for large $L$. Therefore, the main contribution comes from $L=-p\cdot k/P$. Extending the integration limit to infinity, we get the real part of $\Delta(-2p\cdot k)$:

$$\pi^3/(2P\Delta P^2).$$

This result at the same time justifies the use of approximations (22) and (23). In the same way, we can show that the anomalous term of $M_C$ comes from the small momentum transfer $l'$ at $B$, and that those of $M_B$ come from small $\mathbf{l}$ at $A$ and small $\mathbf{l}'$ at $B$.

Once we know that the Coulomb anomalies come from the small momentum transfer just before or after the emission of the photon as in the foregoing, the cancellation can be shown quite generally as follows. For instance, suppose $\mathbf{l}$ is small. By the same approximation made in (22) and (23) the integrand of (3′) and (4′) become

$$M_A\colon\ 2p_0\gamma_0(2\epsilon\cdot p)/\{[-2p\cdot(k+l)](-2p\cdot k)\mathbf{l}^2\Delta\mathbf{p}^2\},$$
$$M_B\colon\ 2p_0\gamma_0(2\epsilon\cdot p)/\{[-2p\cdot(k+l)](-2p\cdot l)\mathbf{l}^2\Delta\mathbf{p}^2\}.$$

Combining these two, we have

$$2p_0\gamma_0(-\epsilon\cdot p/p\cdot k)[(-2p\cdot l)\mathbf{l}^2\Delta\mathbf{p}^2]^{-1},$$

which is simply equal to what is obtained by neglecting $k$ in the internal electron line $AB$ of $M_A$ and assuming $\mathbf{l}$ is small.

For the third Born approximation we have four diagrams shown in Fig. 2. Call the momentum transfer at $A$, $B$, and $C$ as $\mathbf{l}_1$, $\mathbf{l}_2$, and $\mathbf{l}_3$, respectively. The anomalous terms occur in the following way:

$\mathbf{l}_1$ is small: (A) and (B) have anomalous terms.
$\mathbf{l}_1$ and $\mathbf{l}_2$ is small: (A), (B), and (C) have anomalous terms.
$\mathbf{l}_2$ and $\mathbf{l}_3$ is small: (B), (C), and (D) have anomalous terms.
$\mathbf{l}_3$ is small: (C) and (D) have anomalous terms.

The first and the last cases are just the same as those of the second Born approximation discussed above. In the second case, after making the approximation analogous to (22) in the matrix elements, we combine (A) and (B). It is obvious that we get what is equivalent to neglect of $k$ in the electron line $AB$ of (A). Then, we combine the resulting term with (C). We then get a term which is equivalent to neglect of $k$ in the two electron lines $AB$ and $BC$ of (A).

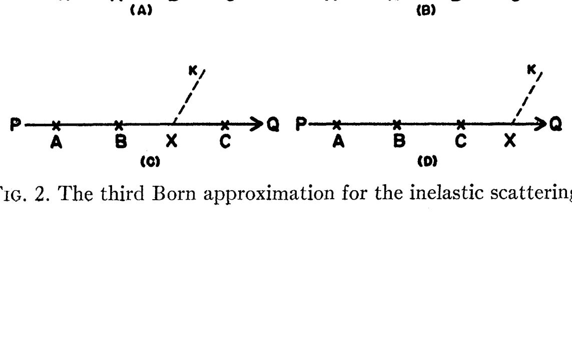

FIG. 2. The third Born approximation for the inelastic scattering.

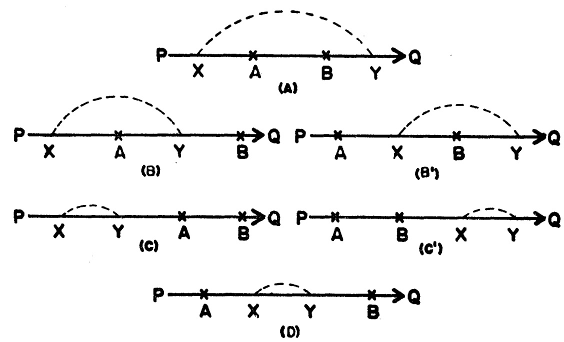

FIG. 3. Virtual photon processes for the second Born approximation.

A similar discussion can readily be made for many photon emissions as well. Thus, we conclude that the cancellation of the additional infrared divergences holds to any order of the Born approximation and also to any number of photons emitted.

Next we shall examine the virtual photon processes. Here we have such processes as are given in Fig. 3. We omit from our consideration the diagrams containing $\delta m$ and also the vacuum polarization diagrams, because they contain neither an infrared divergence nor $(\log m)^2$. The matrix elements are, for instance,

$$M_{(\mathrm{A})}=-[e^2F/(2\pi)^4]\int d^4k/k^2\int d\mathbf{l}(\mathbf{l}^2\mathbf{l}'^2)^{-1}$$
$$\times\bar{u}_q\gamma_\mu[i\gamma(q-k)+m]^{-1}\gamma_0[i\gamma(p-k-l)+m]^{-1}$$
$$\times\gamma_0[i\gamma(p-k)+m]^{-1}\gamma_\mu u_p;\quad(24)$$

$$M_{(\mathrm{B})}=-[e^2F/(2\pi)^4]\int d^4k/k^2\int d\mathbf{l}(\mathbf{l}^2\mathbf{l}'^2)^{-1}$$
$$\times\bar{u}_q\gamma_0[i\gamma(p-l)+m]^{-1}\gamma_\mu[i\gamma(p-k-l)+m]^{-1}$$
$$\times\gamma_0[i\gamma(p-k)+m]^{-1}\gamma_\mu u_p,\quad(25)$$

$$M_{(\mathrm{D})}=-[e^2F/(2\pi)^4]\int d^4k/k^2\int d\mathbf{l}(\mathbf{l}^2\mathbf{l}'^2)^{-1}$$
$$\times\bar{u}_q\gamma_0[i\gamma(p-l)+m]^{-1}\gamma_\mu$$
$$\times[i\gamma(p-k-l)+m]^{-1}\gamma_\mu[i\gamma(p-l)+m]^{-1}\gamma_0u_p,\quad(26)$$

with

$$q-p+l+l'=0.$$

We are interested at the moment only in the contribution of the soft photon, so that we integrate over the momentum space defined by

$$K_{\min}<K<\bar{K}(\ll\Delta P).\quad(27)$$

We shall examine these integrals without doing the separation of the ultraviolet divergent term, although they are infrared divergent at the same time.[12] This is justified because the ultraviolet divergent terms of the vertex and self-energy part in diagrams (B), (B′), (C), (C′) and (D) are known to cancel with each other from the renormalization theory as a result of $Z_1=Z_2$.[13,14] Then, diagrams (B), (B′) and (D), without separation of the ultraviolet divergence, contain no infrared divergence besides those coming from Coulomb anomalies. The exceptional case is diagrams (C) and (C′), where the finite part of the self-energy part drops as a result of the Dirac spinor operating from the left or right, leaving an ultraviolet as well as an infrared divergent term. The treatment of these diagrams is well known.[15] (C) and (C′) give

$$M_{(\mathrm{C})}+M_{(\mathrm{C}')}=iIM_{\mathrm{el}}, \tag{28}$$

where

$$I=-[e^2/(2\pi)^4]\int d^4k(1/k^2)\{\bar{u}_p\gamma_\mu[i\gamma(p-k)+m]^{-1} \times\gamma_\lambda[i\gamma(p-k)+m]^{-1}\gamma_\mu u_p\}/(\bar{u}_p\gamma_\lambda u_p).$$

Performing the $k_0$ integration and taking only the infrared divergent term we get

$$M_{(\mathrm{C})}+M_{(\mathrm{C}')}=-[\alpha/(4\pi^2)]\cdot\int_{K_{\min}}^{\bar{K}} d\mathbf{k}(1/K) \times[p^2/(p\cdot k)^2]M_{\mathrm{el}}. \quad (k^2=0) \tag{29}$$

This expression holds of course to any order of Born approximation for such diagrams as (C) and (C′), which have a self-energy part on the extreme right or left.

The potential integrations in $M_{(\mathrm{A})}$, $M_{(\mathrm{B})}$, $M_{(\mathrm{B}')}$ and $M_{(\mathrm{D})}$ can be reduced to (6) and the result again contains two discontinuity terms $\Delta(k^2-2k\cdot p)$ and $\Delta(k^2-2k\cdot q)$, which can, however, be shown to cancel out exactly when we add the four matrix elements. Here we shall show this cancellation by the general argument presented before. Assuming $k$ and $\mathbf{l}$ are small, the integrands of (24), (25), and (26) reduce to

$$M_{(\mathrm{A})}:\ -4(p\cdot q)\cdot 2\gamma_0p_0/\{(-2k\cdot q)(-2k\cdot p)\times[-2p\cdot(k+l)]\mathbf{l}^2\Delta\mathbf{p}^2\}, \tag{30}$$

$$M_{(\mathrm{B})}:\ 4m^2\cdot 2\gamma_0p_0/\{(-2p\cdot l)(-2k\cdot p)\times[-2p\cdot(k+l)]\mathbf{l}^2\Delta\mathbf{p}^2\}, \tag{31}$$

$$M_{(\mathrm{B}')}:\ -4(p\cdot q)2\gamma_0p_0/\{(-2k\cdot q)(-2p\cdot l)\times[-2p\cdot(k+l)]\mathbf{l}^2\Delta\mathbf{p}^2\}, \tag{32}$$

$$M_{(\mathrm{D})}:\ 4m^2\cdot 2\gamma_0p_0/\{(-2l\cdot p)^2[-2p\cdot(k+l)]\mathbf{l}^2\Delta\mathbf{p}^2\}. \tag{33}$$

Adding $M_{(\mathrm{B})}$ and $M_{(\mathrm{D})}$, we get

$$4m^2 2\gamma_0p_0/[(-2l\cdot p)^2(-2p\cdot k)\mathbf{l}^2\Delta\mathbf{p}^2],$$

which is no longer infrared divergent. Adding $M_{(\mathrm{A})}$ and $M_{(\mathrm{B}')}$, we have

$$-4(p\cdot q)2\gamma_0p_0/[(-2k\cdot p)(-2k\cdot q)(-2p\cdot l)\mathbf{l}^2\Delta\mathbf{p}^2],$$

which is equal to what we obtain by putting $k=0$ in the internal propagator of $M_{(\mathrm{A})}$. Thus, we are allowed to take only $M_{(\mathrm{A})}$ and to put $k=0$ always in the internal propagator $AB$ in the consideration of infrared divergence. Therefore, we have

$$M_{(\mathrm{A})}+M_{(\mathrm{B})}+M_{(\mathrm{B}')}+M_{(\mathrm{D})} =-ie^2/(2\pi)^4\int d^4k(4p\cdot q)/ [k^2\cdot(k^2-2k\cdot p)(k^2-2k\cdot q)]M_{\mathrm{el}}. \tag{34}$$

Performing the $k_0$ integration, we get as the infrared divergent term

$$M_{(\mathrm{A})}+M_{(\mathrm{B})}+M_{(\mathrm{B}')}+M_{(\mathrm{D})} =(\alpha/4\pi^2)(p\cdot q)\int_{K_{\min}}^{\bar{K}} d\mathbf{k}(1/K) \times[1/(k\cdot p)(k\cdot q)]M_{\mathrm{el}}. \tag{35}$$

From the interference with the pure potential scattering matrix element $M_{\mathrm{el}}$, (35) plus (29) give the following contribution of order $\alpha$ to the cross section,

$$\sigma_{\mathrm{virt}}=-(\alpha/2\pi^2)\int_{K_{\min}}^{\bar{K}} d\mathbf{k}(1/K) \times[p^2/(p\cdot k)^2-p\cdot q/(q\cdot k)(p\cdot k)]\sigma_{\mathrm{el}}. \tag{36}$$

Adding the inelastic cross section (21), the lower limit $K_{\min}$ cancels out and is replaced by the physically meaningful quantity $\Delta E$. The $\mathbf{k}$ integration can be done easily and gives as a leading term

$$\sigma_{\mathrm{rad}}=\sigma_{\mathrm{inel}}+\sigma_{\mathrm{virt}} \sim-(4\alpha/\pi)\log(\Delta P/m)\log(\bar{K}/\Delta E)\cdot\sigma_{\mathrm{el}}. \tag{37}$$

The coefficient of $\log\Delta E$ coincides with that of Eq. (2), which is the required result.

## III. EXAMINATION OF $(\log m)^2$

To understand the origin of $\log m$, which does not appear in the pure potential scattering, we shall examine the typical integral

$$\int d^4k[k^2(k^2-2k\cdot p)(k^2-2k\cdot q)]^{-1}, \tag{38}$$

which appeared in Eq. (34), and is also the most important integral in the first Born approximation. Since this is infrared divergent, we have either to restrict the photon momentum space by

$$K\geqq K_{\min} \tag{39}$$

[12] R. Karplus and N. Kroll, Phys. Rev. **77**, 536 (1950).
[13] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).
[14] J. C. Ward, Phys. Rev. **78**, 182 (1950).
[15] See, for instance, F. J. Dyson, "Advanced Quantum Mechanics," Lecture notes at Cornell University, 1951 (unpublished).

as before, or to assume a small photon mass $\mu$, and replace the photon propagator by $1/(k^2+\mu^2)$. Although the former method is noncovariant, it gives the correct answer, as explained later. For our purpose of examining the log$m$ term, it is convenient to employ it. We perform the $k_0$ integration first by contour integral. The pole of the photon propagator $1/k^2$ contributes a term

$$(\pi i/4)\int d\mathbf{k}(1/K^3)[1/(p_0-\mathbf{n}\cdot\mathbf{p})][1/(p_0-\mathbf{n}\cdot\mathbf{q})], \quad (40)$$

where $\mathbf{k}=\mathbf{n}K$. The angular integration can be readily done and gives

$$(16\pi/\Delta P^2)\log(\Delta P/m). \quad (p_0\gg m)$$

It is apparent from the form of the integrand that log$m$ is contributed from $\mathbf{n}\|\mathbf{p}$ or $\mathbf{n}\|\mathbf{q}$, where one of the denominators becomes

$$p_0-P\approx m^2/2p_0,$$

while the other becomes

$$p_0-\mathbf{p}\cdot\mathbf{q}/P\approx \Delta P^2/2P.$$

The pole of the factor $1/(k^2-2k\cdot p)$ of (38) gives the following contribution after a little manipulation:

$$\frac{\pi i}{4}\int d\mathbf{k}\frac{1}{\mathbf{k}\cdot\Delta\mathbf{p}}\cdot\frac{1}{[(\mathbf{k}-\mathbf{p})^2+m^2]^{\frac{1}{2}}} \cdot\frac{\{\mathbf{k}\cdot\mathbf{p}-p_0^2+p_0[(\mathbf{k}-\mathbf{p})^2+m^2]^{\frac{1}{2}}\}}{K^2[p_0^2-(\mathbf{n}\cdot\mathbf{p})^2]}. \quad (41)$$

Again, this has a denominator $(\mathbf{n}\cdot\mathbf{p})^2-p_0^2$, which becomes small when $\mathbf{k}\|\mathbf{p}$ or $\mathbf{k}\|-\mathbf{p}$, and gives log$m$ by angular integration. When especially $\mathbf{k}\approx\mathbf{p}$, the second denominator becomes $\sim m$, but at the same time the numerator of the last factor becomes $-m^2+mp_0$, so that no $(\log m)^2$ can be expected from this additional singularity. (We shall call such factors as $1/[(\mathbf{k}-\mathbf{p})^2+m^2]^{\frac{1}{2}}$ or $1/[K^2p_0^2-(\mathbf{k}\cdot\mathbf{p})^2]$, $m$-singular, because they become singular when $m\to 0$.) The first denominator never vanishes as long as $\mathbf{k}\|\mathbf{p}$ except when $\mathbf{k}\approx 0$, where the numerator also becomes $\sim 2k\cdot p$. The same analysis holds for the third factor $1/(k^2-2k\cdot q)$. Thus, we can expect no $(\log m)^2$ term from the integrals of the type (38). The actual evaluation of the integral (38) can be done more easily by combining the three denominators by the usual perscription, integrating first over $k_0$ and then integration over $k$ subject to (39). The result is

$$\sim(4i\pi^2/\Delta P^2)\log(\Delta P/m)\log(p_0/K_{\min}). \quad (42)$$

Thus we actually get no term $(\log m)^2$. It will be noticed also that we get exactly the term in Eq. (2) if we substitute the value (42) into Eq. (34).

On the other hand, if we assume the small photon mass $\mu$, the integral (38) becomes

$$\sim(2i\pi^2/\Delta P^2)\log(\Delta P/m) \times[2\log(\Delta P/\mu)-\log(\Delta P/m)], \quad (43)$$

which now contains a $(\log m)^2$ term. But this $(\log m)^2$ is cancelled in the calculation of the first Born approximation by another $(\log m)^2$ in the bremsstrahlung integral which is now, for instance [the second term of (21)],

$$\int_0^{\Delta E} d\mathbf{k}[1/(K^2+\mu^2)][1/(k\cdot p)(k\cdot q)] \sim(8\pi/\Delta P^2)\log(\Delta P/m)[2\log(\Delta E\Delta P/\mu p_0) -\log\Delta P/m], \quad (k^2+\mu^2=0), \quad (44)$$

instead of the noncovariant integral

$$\int_{K_{\min}}^{\Delta E} d\mathbf{k}(1/K)[1/(k\cdot p)(k\cdot q)]\sim(16\pi/\Delta P^2) \times\log(\Delta P/m)\log(\Delta E/K_{\min}), \quad (k^2=0) \quad (45)$$

employed in the evaluation of (37). That the $(\log m)^2$ terms in (43) and (44) come from the contribution of small $k$ will be evident by remarking that the integral

$$\int d^4k\cdot k_\lambda/[k^2\cdot(k^2-2k\cdot p)(k^2-2k\cdot q)]$$

no longer gives $(\log m)^2$ and leads to a unique value by any integration method. Since we have established in Sec. II the exact cancellation of the contribution from the soft photon $K<\Delta E\ll\Delta P$, we conclude that the difference of the two methods, that is solely associated with the soft photon, should cancel in the final answer completely. [Note that (21) and (36) have the same integrand. If we assume a small photon mass $\mu$, we have only to replace the factor $1/K$ by $1/(K^2+\mu^2)^{\frac{1}{2}}$, and $k_0=K$ by $k_0=(K^2+\mu^2)^{\frac{1}{2}}$, and set $K_{\min}=0$, in *both* integrands. This leads to no change in the above conclusion.] In fact, in the case of the first Born approximation, we can show that the two methods give exactly the same answer including the terms of order unity.

It remains to examine the effect of the long Coulomb tail. Let us take as a typical integral $M_{(A)}$, Eq. (24). The integrand is

$$N\{k^2(k^2-2k\cdot p)(k^2-2k\cdot q) \times[(k-p+l)^2+m^2]\mathbf{l}^2(\mathbf{l}-\Delta\mathbf{p})^2\}^{-1}, \quad (46)$$

where the numerator $N$ is given by

$$N=\bar{u}_q\gamma_\mu[-i\gamma(q-k)+m]\gamma_0[-i\gamma(p-k+l)+m]\gamma_0 \times[-i\gamma(p-k)+m]\gamma_\mu u_p. \quad (47)$$

In order to get $(\log m)^2$ in (46), it is clear from the foregoing analysis that the $\mathbf{l}$-integral,

$$\int d\mathbf{l}\{1/[l^2-2l\cdot(p-k)+k^2-2k\cdot p]\} \times(1/\mathbf{l}^2)[1/(\mathbf{l}-\Delta\mathbf{p})^2], \quad (48)$$

must be singular or $m$-singular at some point near $\mathbf{k}\|\mathbf{p}$ or $\mathbf{k}\|\mathbf{q}$. In the first denominator of (48) we can expect that the singularity will occur at that point which satisfies the following two conditions:

$$\mathbf{k}-\mathbf{p}\approx 0, \quad k^2-2k\cdot p\approx 0, \tag{49}$$

because there the first factor of (48) is of order $1/\mathbf{l}^2$, which gives a very high singularity at $\mathbf{l}\approx 0$, together with the potential factor $1/\mathbf{l}^2$ in (48). The condition (49) is equivalent to $k-p\approx 0$. (The symbol $\approx$ allows an error of order $\sim m$.) In the same way we know that when $\mathbf{k}-\mathbf{q}\approx 0$ and $k^2-2k\cdot q\approx 0$, the first factor of (48) becomes of the order $1/(\mathbf{l}-\Delta\mathbf{p})^2$, which again overlaps with the second potential factor of (48). Thus the higher $m$-singularity, if any, will occur at either

$$k\approx p, \quad \mathbf{l}\approx 0 \quad \text{or} \quad k\approx q, \quad \mathbf{l}\approx \Delta p, \tag{50}$$

where the singularity of the $\mathbf{l}$-integral overlaps with those of the second or third denominators of (46). The conclusion is readily confirmed by looking at the exact value of the $\mathbf{l}$-integral given by (A-1). For instance, assuming $f$, $g>0$, (A-1) gives for the real part of the integrals the value

$$\pi^3/[\Delta P(R^2\Delta P^2+f\cdot g)^{\frac{1}{2}}],$$

which is singular at $R^2\approx 0$ and $f\approx 0$ (or $g\approx 0$). Remembering that $R^2=(k_0-p_0)^2-m^2$, $f=k^2-2k\cdot p$ and $g=k^2-2k\cdot q$ for (48), the singular point is equivalent to $k\approx p$ (or $k\approx q$). On the other hand if the integral (48) has one $l_i$ in the numerator, the integral is no longer singular at $k\approx p$, as is readily seen from (A-1) and (A-9). This implies that the singularity at $k\approx p$ occurs because $\mathbf{l}\approx 0$.

However, at the points (50), the numerator (47) is of order $\sim m^3$, which is enough to cancel the $m$-singularity of the $l$-integration. Therefore, we get no $(\log m)^2$ from diagram $A$ of Fig. 3.

The integrand of $M_B$, Eq. (25), is

$$N\{k^2(k^2-2k\cdot p)[(k-p+l)^2+m^2] \times (l^2-2l\cdot p)\mathbf{l}^2(\mathbf{l}-\Delta\mathbf{p})^2\}^{-1}, \tag{51}$$

with

$$N=\bar{u}_q\gamma_0[-i\gamma(p-l)+m]\gamma_\mu[i\gamma(p-k-l)+m]\gamma_0 \times[-i\gamma(p-k)+m]\gamma_\mu u_p. \tag{52}$$

Comparing (51) with (38), it is apparent that (51) cannot give a higher $m$-singularity than (38), except when $\mathbf{l}\approx 0$. From $\mathbf{l}\approx 0$, the $\mathbf{l}$-integration gives rise to a singular term

$$-(\pi^3/2P\Delta P^2)[1/(k^2-2k\cdot p)].$$

(Incidentally, this was the origin of the additional infrared divergence appearing in $M_{(B)}$.) Inserting this into (51) we get the double pole $(k^2-2k\cdot p)^2=0$, which gives a term of the order $1/m^2$ after $k$-integration. However, the direct calculation shows that this $1/m^2$ is cancelled by the numerator $N$, as we should expect. [The matrix element should be expressed as a power series of $m/p_0$ plus some function of $\log(m/p_0)$.] The $\mathbf{l}$-integration of (51) gives another less singular term of the same order as (48), which is singular only at $k\approx p$. (This comes from an integral that has one power of $\mathbf{l}$ in the numerator.) However, again at $k\approx p$ the numerator (52) becomes of the order $\sim m$, which cancels the singularity.

The same kind of argument applies also for $M_{(B')}$, $M_{(C)}$, $M_{(C')}$, and $M_{(D)}$ and in any case we can exclude the possibility of the appearance of $(\log m)^2$. A more rigorous analysis has been done by integrating first by $k_0$, then performing the $\mathbf{l}$-integration and finally integrating over $\mathbf{k}$ in the small region around $\mathbf{k}=\mathbf{p}$. The result conforms with the foregoing discussion.

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## APPENDIX

The potential integrations were made following the method of Dalitz[6] and Lewis[16] employed for the calculation of the second Born elastic scattering matrix elements. The calculation was made for the Yukawa potential $e^{-\lambda r}/r$, in order to express the results for off-and-on-energy shell matrix elements compactly, and also to exhibit the discontinuity on the energy shell clearly. The necessary integrals for the evaluation of (6) are the following:

$$\begin{aligned} I_0(\mathbf{p}',\mathbf{q}',R^2) &= \int d\mathbf{l}\cdot 1/\{(l^2-R^2-i\epsilon)[(\mathbf{l}-\mathbf{p}')^2+\lambda^2] \times[(\mathbf{l}-\mathbf{q}')^2+\lambda^2]\}, \quad (\epsilon\to 0) \\ &= (i\pi^2/\sqrt{A})\log[(B-iC-iD)/(B-iC+iD)] \\ &= (\pi^2/\sqrt{A})[\tan^{-1}(D+C)/B+\tan^{-1}(D-C)/B] \\ &\quad +(i\pi^2/\sqrt{A})\log\{[B^2+(C+D)^2]/[B^2+(C-D)^2]\}^{\frac{1}{2}}, \end{aligned} \tag{A-1}$$

where

$$\begin{aligned} A&=R^2\Delta P'^2(\Delta P'^2+4\lambda^2)-\lambda^2(f-g)^2 +\Delta P'^2(\lambda^2+f)(\lambda^2+g), \\ B&=\lambda(2\lambda^2+f+g)\Delta P'^2; \quad C=R\Delta P'^2(\Delta P'^2+4\lambda^2), \\ D&=\Delta P'^2\sqrt{A}, \end{aligned} \tag{A-2}$$

[16] R. R. Lewis, Jr., thesis, University of Michigan, 1954 (unpublished).

with

$$\Delta P'=|\mathbf{p}'-\mathbf{q}'|,\quad f=P'^2-R^2,\quad \text{and}\quad g=Q'^2-R^2.$$

$$J(\mathbf{p}',R^2)=\int d\mathbf{l}\cdot 1/\{(l^2-R^2-i\epsilon)[(\mathbf{l}-\mathbf{p}')^2+\lambda^2]\}$$
$$=(i\pi^2/P')\log[(\lambda-iR+iP')/(\lambda-iR+iP')]$$
$$=(\pi^2/P')[\tan^{-1}(R+P')/\lambda+\tan^{-1}(P'-R)/\lambda]$$
$$+(\pi^2 i/P')\log\{[\lambda^2+(R+P')^2]/[\lambda^2+(R-P')^2]\}^{\frac{1}{2}}.\quad \text{(A-3)}$$

$$K(\mathbf{p}',\mathbf{q}')=\int d\mathbf{l}\cdot 1/\{[(\mathbf{l}-\mathbf{p}')^2+\lambda^2][(\mathbf{l}-\mathbf{q}')^2+\lambda^2]\}$$
$$=(\pi^2 i/\Delta P')\log[(2\lambda-i\Delta P')/(2\lambda+i\Delta P')]$$
$$=(2\pi^2/\Delta P'^2)\tan^{-1}\Delta P'/\lambda.\quad \text{(A-4)}$$

In all these expressions the argument of the numerator and the denominator of the logarithmic functions must be taken as $[-\pi/2,\pi/2]$.

The value of the integrals depends critically on the relative magnitude of $f$, $g$, and $\lambda^2$, in the limit $f, g, \lambda^2\to 0$. Keeping these quantities finite, whenever the relative magnitudes are relevant, the soft photon limit, as well as the unscreened limit, is obtained by replacing $P'$, $Q'$, $R^2$, and $\Delta P'$ by $P$, $Q$, $P^2=Q^2$ and $\Delta P$, respectively, and taking $\lambda\to 0$. The results are

$$I_0(\mathbf{p}',\mathbf{q}',R^2)\approx\frac{\pi^2}{P\Delta P^2}\left[\left(\tan^{-1}\frac{f}{2\lambda P}+\tan^{-1}\frac{g}{2\lambda P}\right)+i\log\frac{4P^2\Delta P^2}{[(f^2+4\lambda^2P^2)(g^2+4\lambda^2P^2)]^{\frac{1}{2}}}\right],\quad \text{(A-5)}$$

$$J(\mathbf{p}',R^2)\approx\frac{\pi^2}{P}\left[\frac{\pi}{2}+i\tan^{-1}\frac{f}{2\lambda P}+i\log\frac{4P^2}{f^2+4\lambda^2P^2}\right],$$
$$K(\mathbf{p}',\mathbf{q}')\approx\pi^3/\Delta P.$$

Separating the terms which remain when $f=g=0$ (on-energy-shell value) and denoting them as $I_0^{\mathrm{el}}$, $J^{\mathrm{el}}$ and $K^{\mathrm{el}}$, we get

$$\begin{aligned}I_0(\mathbf{p}',\mathbf{q}',R^2)&\approx I_0^{\mathrm{el}}+\Delta(f)+\Delta(g),\\ J(\mathbf{p}',R^2)&\approx J^{\mathrm{el}}+\Delta P^2\Delta(f),\\ J(\mathbf{q}',R^2)&\approx J^{\mathrm{el}}+\Delta P^2\Delta(g),\\ K(\mathbf{p}',\mathbf{q}')&\approx K^{\mathrm{el}},\end{aligned}\quad \text{(A-6)}$$

where

$$\Delta(f)=\lim_{\lambda\to 0}\frac{\pi^2}{P\Delta P^2}\left(\tan^{-1}\frac{f}{2\lambda P}+i\log\frac{2\lambda P}{(f^2+4\lambda^2P^2)^{\frac{1}{2}}}\right).\quad \text{(A-7)}$$

From this definition, we have, of course,

$$\Delta(0)=0.\quad \text{(A-8)}$$

$I_i$ defined in (6), Sec. II, can be reduced to the linear combination of $I_0$, $J$ and $K$, as follows:

$$I_i=\{p_i'/[P'^2Q'^2-(\mathbf{p}'\cdot\mathbf{q}')^2]\}\cdot\{(\mathbf{p}'\cdot\mathbf{q}')J(\mathbf{p}',R^2)-Q'^2J(\mathbf{q}',R^2)+(Q'^2-\mathbf{p}'\cdot\mathbf{q}')K+[R^2(Q'^2-\mathbf{p}'\cdot\mathbf{q}')+p'^2(P'^2-\mathbf{p}'\cdot\mathbf{q}')]I_0\}+(\mathbf{p}'\rightleftarrows\mathbf{q}').\quad \text{(A-9)}$$

From (A-6) we have, in the soft photon limit,

$$I_i\approx I_i^{\mathrm{el}}+p_i\Delta(f)+q_i\Delta(g).\quad \text{(A-10)}$$

In the same way, we get

$$I_{ij}\approx I_{ij}^{\mathrm{el}}+p_ip_j\Delta(f)+q_iq_j\Delta(g).\quad \text{(A-11)}$$