the role of indicies while the $\phi_A(x)$ are the independent variables of our theory. The chief advantage to using functional derivatives as defined in (A) instead of the variational derivative is that functional differentiation is commutative, while variational differentiation is not.

By keeping in mind the idea that the space variables are really indices, we can, by analogy, define a Poisson bracket between two functionals F and G as

$$(F,G)\equiv\int\left\{\frac{\delta F}{\delta\phi_A(x)}\frac{\delta G}{\delta\pi^A(x)}-\frac{\delta G}{\delta\phi_A(x)}\frac{\delta F}{\delta\pi^A(x)}\right\}d^3x. \quad \text{(B)}$$

APPENDIX II

A canonical transformation from variables ϕ_A, π^A to $\bar{\phi}_A$, $\bar{\pi}^A$ can be generated by the generating functional $C(\phi_A,\bar{\pi}^A)$ by requiring that

$$\pi^A(x)=\delta C/\delta\phi_A(x),\quad \bar{\phi}_A(x)=\delta C/\delta\bar{\pi}^A(x),\quad H'=H+\partial C/\partial t. \quad \text{(C)}$$

It can be shown by direct calculation that the transformation (C) preserves Poisson bracket relations and the canonical form of the equations of motion.

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Multiple Bremsstrahlung

SURAJ N. GUPTA
Department of Physics, Purdue University, Lafayette, Indiana
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The cross section for the multiple production of photons in bremsstrahlung at high energies is calculated, and it is shown that the probability for multiple bremsstrahlung is quite small compared with that for the ordinary bremsstrahlung even in the very high energy region of cosmic rays.

The general problem of multiple photon production is further discussed, and it appears that even at high energies the probability for multiple photon production is appreciable only in the following two cases: (1) an energetic charged particle is deflected through an angle, which is large compared with the ratio of its rest energy and its total energy; (2) an energetic charged particle is annihilated, captured or converted into a neutral particle.

1. INTRODUCTION

THE Bethe-Heitler[1,2] formula for bremsstrahlung is one of the most widely used results in the study of cosmic rays. Therefore, various types of corrections to this formula have been investigated from time to time.[3] In this paper, we shall consider the possibility of multiple photon production in bremsstrahlung to see how far the Bethe-Heitler formula is adequate to describe the emission of photons and energy loss of electrons at very high energies. It would also be interesting to see whether there is a reasonable possibility of directly observing the multiple bremsstrahlung in cosmic rays.

Recently, the author[4] has investigated the multiple production of photons in the electron-positron annihilation, and it has been found that the probability for the production of several photons in this process becomes appreciable at high energies, which are available in cosmic rays. But, we shall see that the probability for multiple bremsstrahlung is quite small at all energies of experimental interest. The reason for this difference in the two cases will be discussed in this paper. We shall also further discuss the general problem of multiple photon production to see under what conditions such processes are most likely to occur.

It seems to us convenient to represent the ordinary bremsstrahlung process as

$$e^- \overset{N}{\longrightarrow} e^- + \gamma,$$

where $\overset{N}{\longrightarrow}$ indicates that the above process can take place only in the presence of a nucleus. We can then represent a bremsstrahlung process involving the multiple production of n photons as

$$e^- \overset{N}{\longrightarrow} e^- + n\gamma.$$

2. MATRIX ELEMENT FOR DOUBLE BREMSSTRAHLUNG

In the presence of an external electromagnetic field, we can write Dyson's S matrix[5] as

$$S_n=(1/n!)(-i/c\hbar)^n\int dx'\cdots\int dx^{(n)} \times P[H(x'),\cdots,H(x^{(n)})], \quad (1)$$

with

$$H=-ie\bar{\psi}\gamma_\mu\psi(A_\mu+A_{\mu,\,\text{ex}}), \quad (2)$$

[1] H. A. Bethe and W. Heitler, Proc. Roy. Soc. (London) **A146**, 83 (1934).
[2] H. A. Bethe, Proc. Cambridge Phil. Soc. **30**, 524 (1934).
[3] For the latest work on this subject, see H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954), and Davies, Bethe, and Maximon, Phys. Rev. **93**, 788 (1954).
[4] S. N. Gupta, Phys. Rev. **98**, 1502 (1955).
[5] F. J. Dyson, Phys. Rev. **75**, 486, 1736 (1949).

where $A_{\mu,\,\mathrm{ex}}$ denotes the external field. If this external field is due to a heavy nucleus of charge Ze at the origin, then we have

$$A_{i,\,\mathrm{ex}}(x)=0,\qquad A_{0,\,\mathrm{ex}}(x)=(Ze/V)\textstyle\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}(1/\mathbf{q}^2), \tag{3}$$

where V is the volume of a large cubical box within which the interacting fields have been confined.

Following Dyson's treatment, we can write the S matrix element for double bremsstrahlung as

$$\begin{aligned}S_3=(e/c\hbar)^3\int dx\int dx'\int dx'' A_{\mu,\,\mathrm{ex}}(x)A_\nu(x')A_\lambda(x'')\\ \times[\bar{\psi}(x)\gamma_\mu S_F(x-x')\gamma_\nu S_F(x-x'')\gamma_\lambda\psi(x'')\\ +\bar{\psi}(x')\gamma_\nu S_F(x'-x)\gamma_\mu S_F(x-x'')\gamma_\lambda\psi(x'')\\ +\bar{\psi}(x')\gamma_\nu S_F(x'-x'')\gamma_\lambda S_F(x''-x)\gamma_\mu\psi(x)],\end{aligned} \tag{4}$$

where the various terms in (4) represent the contribution of the Feynman diagrams shown in Fig. 1. We also put

$$\begin{aligned}\psi(x)&=V^{-\frac{1}{2}}a_r(\mathbf{k})u_r(\mathbf{k})e^{ikx},\\ \bar{\psi}(x')&=\sum_{\mathbf{k}',s}V^{-\frac{1}{2}}a_s^*(\mathbf{k}')\bar{u}_s(\mathbf{k}')e^{-ik'x'},\\ \mathbf{A}(x')&=\sum_{\mathbf{q}',\mathbf{e}'}V^{-\frac{1}{2}}(c\hbar/2q_0')^{\frac{1}{2}}\mathbf{e}'a_{\mathbf{e}'}^*(\mathbf{q}')e^{-iq'x'},\\ \mathbf{A}(x'')&=\sum_{\mathbf{q}'',\mathbf{e}''}V^{-\frac{1}{2}}(c\hbar/2q_0'')^{\frac{1}{2}}\mathbf{e}''a_{\mathbf{e}''}^*(\mathbf{q}'')e^{-iq''x''},\\ S_F(x-x')&=\lim_{\epsilon\to+0}(2\pi)^{-4}\int dp\,e^{ip(x-x')}\frac{ip\gamma-\kappa}{p^2+\kappa^2-i\epsilon},\end{aligned} \tag{5}$$

where k and k' are the propagation four vectors for the initial and the final electrons respectively, q' and q'' are the propagation four vectors for the two photons, and other quantities have the usual meaning. Substituting (3) and (5) in (4), we obtain

$$S_3=V^{-3}\sum_{\mathbf{q}}\sum_{\mathbf{q}',\mathbf{e}'}\sum_{\mathbf{q}'',\mathbf{e}''}\sum_{\mathbf{k}',s}\int dx\,e^{i\mathbf{q}\cdot\mathbf{x}}e^{ix(k-k'-q'-q'')}K, \tag{6}$$

where

$$\begin{aligned}K=\frac{iZe^4}{(c\hbar)^2}\frac{1}{(4q_0'q_0'')^{\frac{1}{2}}}\frac{1}{|\mathbf{k}-\mathbf{k}'-\mathbf{q}'-\mathbf{q}''|^2}\\ \times\Bigg[\bar{u}_s(\mathbf{k}')\gamma_4\frac{i(k-q'-q'')\gamma-\kappa}{(k-q'-q'')^2+\kappa^2}(\boldsymbol{\gamma}\cdot\mathbf{e}')\\ \times\frac{i(k-q'')\gamma-\kappa}{(k-q'')^2+\kappa^2}(\boldsymbol{\gamma}\cdot\mathbf{e}'')u_r(\mathbf{k})+\bar{u}_s(\mathbf{k}')(\boldsymbol{\gamma}\cdot\mathbf{e}')\\ \times\frac{i(k'+q')\gamma-\kappa}{(k'+q')^2+\kappa^2}\gamma_4\frac{i(k-q'')\gamma-\kappa}{(k-q'')^2+\kappa^2}(\boldsymbol{\gamma}\cdot\mathbf{e}'')u_r(\mathbf{k})\\ +\bar{u}_s(\mathbf{k}')(\boldsymbol{\gamma}\cdot\mathbf{e}')\frac{i(k'+q')\gamma-\kappa}{(k'+q')^2+\kappa^2}(\boldsymbol{\gamma}\cdot\mathbf{e}'')\\ \times\frac{i(k'+q'+q'')\gamma-\kappa}{(k'+q'+q'')^2+\kappa^2}\gamma_4u_r(\mathbf{k})\Bigg].\end{aligned} \tag{7}$$

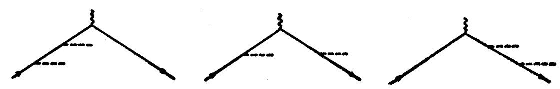

FIG. 1. Feynmann diagrams for double bremsstrahlung.

We shall now consider the extreme relativistic case, i.e., we shall take $k_0\gg\kappa$. We shall also assume that the major contribution to the cross section for the process under consideration arises from those values of q_0' and q_0'', which are small compared with k_0. We can then simplify the denominators and the numerators in (7) as

$$\begin{aligned}(k'+q'+q'')^2+\kappa^2&=2k'q'+2k'q'',\\ (k-q'-q'')^2+\kappa^2&=-2kq'-2kq'';\end{aligned} \tag{8}$$

$$\begin{aligned}[i(k-q')\gamma-\kappa](\boldsymbol{\gamma}\cdot\mathbf{e}')u_r(\mathbf{k})&=2i(\mathbf{k}\cdot\mathbf{e}')u_r(\mathbf{k}),\\ \bar{u}_s(\mathbf{k}')(\boldsymbol{\gamma}\cdot\mathbf{e}')[i(k'+q')\gamma-\kappa]&=\bar{u}_s(\mathbf{k}')2i(\mathbf{k}'\cdot\mathbf{e}'),\quad\text{etc.}\end{aligned} \tag{9}$$

Therefore, we can express (7) as

$$\begin{aligned}K=\frac{-iZe^4}{2(c\hbar)^2}\frac{1}{(q_0'q_0'')^{\frac{1}{2}}}\frac{1}{|\mathbf{k}-\mathbf{k}'-\mathbf{q}'-\mathbf{q}''|^2}[\bar{u}_s(\mathbf{k}')\gamma_4u_r(\mathbf{k})]\\ \times\Bigg[\frac{(\mathbf{k}\cdot\mathbf{e}')(\mathbf{k}\cdot\mathbf{e}'')}{(kq'+kq'')(kq'')}-\frac{(\mathbf{k}'\cdot\mathbf{e}')(\mathbf{k}\cdot\mathbf{e}'')}{(k'q')(kq'')}\\ +\frac{(\mathbf{k}'\cdot\mathbf{e}')(\mathbf{k}'\cdot\mathbf{e}'')}{(k'q')(k'q'+k'q'')}\Bigg].\end{aligned} \tag{10}$$

But, due to the indistinguishability of photons in the same state, we have to write (6) as

$$S_3=V^{-3}\sum_{\mathbf{q}}\sum_{\mathbf{q}',\mathbf{q}''}{}'\sum_{\mathbf{e}',\mathbf{e}''}\sum_{\mathbf{k}',s}\int dx\,e^{i\mathbf{q}\cdot\mathbf{x}}\times e^{ix(k-k'-q'-q'')}(K+K'), \tag{11}$$

where K' is obtained from K by interchanging the roles of the photons $\mathbf{q}'$ and $\mathbf{q}''$, and $\sum'_{\mathbf{q}',\mathbf{q}''}$ denotes summation over all values of $\mathbf{q}'$ and $\mathbf{q}''$ such that each physically different state occurs only once. It then follows from (10) that

$$\begin{aligned}K+K'=\frac{-iZe^4}{2(c\hbar)^2}\frac{1}{(q_0'q_0'')^{\frac{1}{2}}}\frac{1}{|\mathbf{k}-\mathbf{k}'-\mathbf{q}'-\mathbf{q}''|^2}\\ \times[\bar{u}_s(\mathbf{k}')\gamma_4u_r(\mathbf{k})][(\mathbf{k}\cdot\mathbf{e}')/(kq')-(\mathbf{k}'\cdot\mathbf{e}')/(k'q')]\\ \times[(\mathbf{k}\cdot\mathbf{e}'')/(kq'')-(\mathbf{k}'\cdot\mathbf{e}'')/(k'q'')].\end{aligned} \tag{12}$$

Let us now denote the angle between $\mathbf{k}$ and $\mathbf{k}'$ as γ. Further, let the angles made by $\mathbf{q}'$ with $\mathbf{k}$ and $\mathbf{k}'$ be denoted as α' and β', respectively, and the angles made by $\mathbf{q}''$ with $\mathbf{k}$ and $\mathbf{k}'$ be denoted as α'' and β'' respectively. Since it is evident from the denominators in (12) that the major contribution to the cross section arises from small values of the above angles, we can simplify these denominators as

$$|\mathbf{k}'+\mathbf{q}'+\mathbf{q}''-\mathbf{k}|^2\approx(|\mathbf{k}'|+|\mathbf{q}'|+|\mathbf{q}''|-|\mathbf{k}|)^2+k_0^2\gamma^2$$
$$\approx\kappa^4(q_0'+q_0'')^2/(4k_0^4)+k_0^2\gamma^2, \quad (13)$$

$$kq'=-\tfrac{1}{2}q_0'k_0(\kappa^2/k_0^2+\alpha'^2), \quad \text{etc.}, \quad (14)$$

where we have also made use of the conservation relations

$$\mathbf{k}+\mathbf{q}-\mathbf{k}'-\mathbf{q}'-\mathbf{q}''=0, \quad k_0-k_0'-q_0'-q_0''=0. \quad (15)$$

Using (13) and (14), we can express (12) as

$$K+K'=\frac{-2iZe^4}{(c\hbar)^2}\frac{1}{(q_0'q_0'')^{\frac{1}{2}}}$$
$$\times\frac{1}{[\kappa^4(q_0'+q_0'')^2/(4k_0^4)+k_0^2\gamma^2]}[\bar{u}_s(\mathbf{k}')\gamma_4u_r(\mathbf{k})]$$
$$\times\left[\frac{(\mathbf{k}\cdot\mathbf{e}')}{k_0(\kappa^2/k_0^2+\alpha'^2)}-\frac{(\mathbf{k}'\cdot\mathbf{e}')}{k_0(\kappa^2/k_0^2+\beta'^2)}\right]$$
$$\times\left[\frac{(\mathbf{k}\cdot\mathbf{e}'')}{k_0(\kappa^2/k_0^2+\alpha''^2)}-\frac{(\mathbf{k}'\cdot\mathbf{e}'')}{k_0(\kappa^2/k_0^2+\beta''^2)}\right]. \quad (16)$$

3. CROSS SECTION FOR DOUBLE BREMSSTRAHLUNG

In order to find the cross section for double bremsstrahlung, we have to square the element (16). Then, averaging over the spin states of the electron in the initial state, summing over the spin states of the electron in the final state, and also summing over the states of polarization of the photons $\mathbf{q}'$ and $\mathbf{q}''$, we get

$$\sum\langle|K+K'|^2\rangle_{\text{Av}}$$
$$=\frac{4Z^2e^8}{(c\hbar)^4}\frac{1}{(q_0'q_0'')^3}\frac{1}{[\kappa^4(q_0'+q_0'')^2/(4k_0^4)+k_0^2\gamma^2]^2}$$
$$\times\left\{\frac{\alpha'^2}{(\kappa^2/k_0^2+\alpha'^2)^2}+\frac{\beta'^2}{(\kappa^2/k_0^2+\beta'^2)^2}-\frac{\alpha'^2+\beta'^2-\gamma^2}{(\kappa^2/k_0^2+\alpha'^2)(\kappa^2/k_0^2+\beta'^2)}\right\}$$
$$\times\left\{\frac{\alpha''^2}{(\kappa^2/k_0^2+\alpha''^2)^2}+\frac{\beta''^2}{(\kappa^2/k_0^2+\beta''^2)^2}-\frac{\alpha''^2+\beta''^2-\gamma^2}{(\kappa^2/k_0^2+\alpha''^2)(\kappa^2/k_0^2+\beta''^2)}\right\}. \quad (17)$$

The cross section σ_2 for the process under consideration is related to the quantity (17) as

$$\sigma_2=(1/2!)V^{-2}\sum_{q'}\sum_{q''}(2\pi)^{-2}\int d\omega\mathbf{k}'^2$$
$$\times d|\mathbf{k}'|/dk_0'\sum\langle|K+K'|^2\rangle_{\text{Av}}, \quad (18)$$

where $d\omega$ is an element of solid angle along $\mathbf{k}'$. Putting

$$k_0'=k_0, \quad d|\mathbf{k}'|/dk_0'=1,$$
$$V^{-1}\sum_{q'}=(2\pi)^{-3}\int d\mathbf{q}', \quad \text{etc.}, \quad (19)$$

we have

$$\sigma_2=(1/2!)(2\pi)^{-8}\int d\mathbf{q}'\int d\mathbf{q}''\int d\omega k_0^2$$
$$\times\sum\langle|K+K'|^2\rangle_{\text{Av}}. \quad (20)$$

We now choose our coordinate axes in such a way that $\mathbf{k}$ is along the z axis, and $\mathbf{k}'$ lies in the xz plane. Then, we can write the components of $\mathbf{k}$ and $\mathbf{k}'$ as

$$\mathbf{k}=(0,0,|\mathbf{k}|), \quad \mathbf{k}'=(|\mathbf{k}'|\sin\gamma,0,|\mathbf{k}'|\cos\gamma). \quad (21)$$

We further assume that the azimuthal angles of $\mathbf{q}'$ and $\mathbf{q}''$ around the z axis are ϕ' and ϕ'' respectively, so that the components of $\mathbf{q}'$ and $\mathbf{q}''$ are

$$\mathbf{q}'=(q_0'\sin\alpha'\cos\phi', q_0'\sin\alpha'\sin\phi', q_0'\cos\alpha'),$$
$$\mathbf{q}''=(q_0''\sin\alpha''\cos\phi'', q_0''\sin\alpha''\sin\phi'', q_0''\cos\alpha''). \quad (22)$$

Since the angles γ, α', β', α'', and β'' have been assumed to be small, it follows that

$$\beta'^2=\alpha'^2+\gamma^2-2\alpha'\gamma\cos\phi',$$
$$\beta''^2=\alpha''^2+\gamma^2-2\alpha''\gamma\cos\phi'', \quad (23)$$

$$\int d\mathbf{q}'=\int dq_0'q_0'^2\int_0^\delta\alpha'd\alpha'\int_0^{2\pi}d\phi',$$
$$\int d\mathbf{q}''=\int dq_0''q_0''^2\int_0^\delta\alpha''d\alpha''\int_0^{2\pi}d\phi'', \quad (24)$$

$$\int d\omega=2\pi\int_0^\delta\gamma d\gamma, \quad (25)$$

where δ represents the upper limit of integration for the angles α', α'', and γ. We can choose any suitable value for this upper limit, provided that δ^2 is large compared with κ^2/k^2 but small compared with 1.

Using (17), (24), and (25), we can express (20) as

$$\sigma_2=\frac{16Z^2}{\pi}\left(\frac{e^2}{4\pi c\hbar}\right)^4\int\frac{dq_0'}{q_0'}\int\frac{dq_0''}{q_0''}\int_0^\delta\gamma d\gamma$$
$$\times\frac{k_0^2}{[\kappa^4(q_0'+q_0'')^2/(4k_0^4)+k_0^2\gamma^2]^2}I'I'', \quad (26)$$

where

$$I'=\frac{1}{2\pi}\int_0^{\delta}\alpha' d\alpha'\int_0^{2\pi}d\phi'\left\{\frac{\alpha'^2}{(\kappa^2/k_0^2+\alpha'^2)^2}+\frac{\beta'^2}{(\kappa^2/k_0^2+\beta'^2)^2}-\frac{\alpha'^2+\beta'^2-\gamma^2}{(\kappa^2/k_0^2+\alpha'^2)(\kappa^2/k_0^2+\beta'^2)}\right\}, \quad (27)$$

$$I''=\frac{1}{2\pi}\int_0^{\delta}\alpha'' d\alpha''\int_0^{2\pi}d\phi''\left\{\frac{\alpha''^2}{(\kappa^2/k_0^2+\alpha''^2)^2}+\frac{\beta''^2}{(\kappa^2/k_0^2+\beta''^2)^2}-\frac{\alpha''^2+\beta''^2-\gamma^2}{(\kappa^2/k_0^2+\alpha''^2)(\kappa^2/k_0^2+\beta''^2)}\right\}. \quad (28)$$

Substituting (23) in (27), and carrying out the integration over the azimuthal angle ϕ', we get

$$I'=\int_0^{\delta}\alpha' d\alpha'\left\{\frac{\gamma^2+2\kappa^2/k_0^2}{(\kappa^2/k_0^2+\alpha'^2)[(\kappa^2/k_0^2+\alpha'^2+\gamma^2)^2-4\alpha'^2\gamma^2]^{\frac{1}{2}}}-\frac{\kappa^2}{k_0^2}\frac{(\kappa^2/k_0^2+\alpha'^2+\gamma^2)}{[(\kappa^2/k_0^2+\alpha'^2+\gamma^2)^2-4\alpha'^2\gamma^2]^{\frac{3}{2}}}-\frac{\kappa^2}{k_0^2}\frac{1}{(\kappa^2/k_0^2+\alpha'^2)^2}\right\}. \quad (29)$$

The above integral can be easily evaluated, but the resulting expression is a rather complicated function of γ. However, we find approximately

$$\begin{aligned} I'&=\tfrac{1}{3}k_0^2\gamma^2/\kappa^2 && (\text{for } \gamma^2<4\kappa^2/k_0^2),\\ &=\log(k_0^2\gamma^2/\kappa^2)-1 && (\text{for } 4\kappa^2/k_0^2<\gamma^2<\delta^2), \end{aligned} \quad (30)$$

and we also note that $I''=I'$. Hence, substituting (30) in (26), and integrating over γ, we obtain

$$\sigma_2\approx\frac{7}{4\pi}\frac{16Z^2\alpha^4\hbar^2c^2}{3\mu^2}\int\frac{d\omega'}{\omega'}\int\frac{d\omega''}{\omega''}, \quad (31)$$

with

$$\alpha=e^2/4\pi c\hbar,\quad \mu=\kappa c\hbar,\quad \omega'=q_0'c\hbar,\quad \omega''=q_0''c\hbar, \quad (32)$$

where α is the fine structure constant, μ is the rest energy of the electron, and ω' and ω'' are the energies of the emitted photons.

In order to estimate the influence of screening on double bremsstrahlung, we note that the minimum value of the quantity $|\mathbf{q}|$ appearing in the integral (6) is

$$|\mathbf{q}|_{\min}=|\mathbf{k}|-|\mathbf{k}'|-|\mathbf{q}'|-|\mathbf{q}''| \approx\tfrac{1}{2}\kappa^2(q_0'+q_0'')/k_0^2, \quad (33)$$

and we also note that the square of the above quantity appears on the right-hand side of (13). It is well known[6] that roughly the effect of screening is to replace the above minimum value of $|\mathbf{q}|$ by

$$Z^{\frac{1}{3}}\alpha\kappa. \quad (34)$$

Hence, taking into account the effect of screening, we have instead of (26):

$$\sigma_2'\approx\frac{16Z^2}{\pi}\left(\frac{e^2}{4\pi c\hbar}\right)^4\int\frac{dq_0'}{q_0'}\int\frac{dq_0''}{q_0''}\int_0^{\delta}\gamma d\gamma \times\frac{k_0^2}{[Z^{\frac{2}{3}}\alpha^2\kappa^2+k_0^2\gamma^2]^2}I'I''. \quad (35)$$

According to the present approximations, the above integral gives the same result as (26), which shows that the effect of screening on double bremsstrahlung at high energies is negligible.

4. DISCUSSION

In the extreme relativistic case, the cross section for the ordinary bremsstrahlung due to an incident electron of energy E is approximately given by[6]

$$\sigma_1=\frac{16Z^2\alpha^3c^2\hbar^2}{3\mu^2}\int\frac{d\omega}{\omega}\log\frac{2E^2}{\mu\omega}, \quad (36)$$

$$\sigma_1'=\frac{16Z^2\alpha^3c^2\hbar^2}{3\mu^2}\int\frac{d\omega}{\omega}\log(183Z^{-\frac{1}{3}}), \quad (37)$$

where σ_1 denotes the cross section without screening, σ_1' denotes the cross section with screening, and ω is the energy of the emitted photon. For a rough estimate, we may take the upper and lower limits of integration for ω as E and μ, respectively, so that we have

$$\sigma_1=\tfrac{3}{2}(16Z^2\alpha^3c^2\hbar^2/3\mu^2)[\log(E/\mu)]^2, \quad (38)$$

$$\sigma_1'=(16Z^2\alpha^3c^2\hbar^2/3\mu^2)\log(E/\mu)\log(183Z^{-\frac{1}{3}}). \quad (39)$$

Choosing the upper and lower limits of integration for ω' and ω'' also in a similar way, we obtain from (31)

$$\sigma_2=(7\alpha/4\pi)(16Z^2\alpha^3c^2\hbar^2/3\mu^2)[\log(E/\mu)]^2. \quad (40)$$

Comparing (40) with (38) and (39), we find that even at $E\approx 10^{14}$ ev the probability for double bremsstrahlung is quite small compared with that for the ordinary bremsstrahlung.

Using the approximations, described in this paper and the earlier one,[4] we can also calculate the cross section for bremsstrahlung involving the production of n photons. The cross section for this process is found to be

$$\sigma_n=\frac{8\pi Z^2\alpha^2}{n!}\left(\frac{2\alpha}{\pi}\right)^n\int\frac{dq_0'}{q_0'}\cdots\int\frac{dq_0^{(n)}}{q_0^{(n)}}\int\gamma d\gamma \times\frac{k_0^2}{[\kappa^4(q_0'+\cdots+q_0^{(n)})^2/(4k_0^4)+k_0^2\gamma^2]^2}I'^n, \quad (41)$$

[6] W. Heitler, *Quantum Theory of Radiation* (Clarendon Press, Oxford, 1954).

where the integral I' is given by (27). Substituting (30) in (41), we can easily see that even at very high energies σ_n rapidly decreases as n increases.

The above situation is quite different from that in the case of the electron-positron annihilation. For, we have already shown[4] that in electron-positron annihilation at high energies multiple production of several photons can easily take place. This difference between the two cases is due to the following reasons: In high energy bremsstrahlung the average angle, through which the electron is deflected, is of the order of μ/E. Further, it follows from the general treatment in Sec. 5 of reference 4 that the photons emitted by an energetic electron mainly lie within a cone of angle μ/E around the direction of motion of the electron. Therefore, in bremsstrahlung the photons emitted by the initial and the final electrons interfere strongly, which greatly reduces the cross section. On the other hand, there is no appreciable interference between the photons emitted by the electron and the positron during their annihilation.

It is interesting to note that if $\gamma \gg \mu/E$, we obtain from (41) and (30)

$$\sigma_n = \frac{8\pi Z^2\alpha^2}{n!}\left(\frac{4\alpha}{\pi}\right)^n \int \frac{dq_0'}{q_0'} \cdots \int \frac{dq_0^{(n)}}{q_0^{(n)}} \int \gamma d\gamma \times \frac{1}{k_0^2\gamma^4}\left(\log\frac{k_0\gamma}{\kappa}\right)^n$$

or

$$\sigma_n = \frac{\sigma_0}{n!}\left(\frac{4\alpha}{\pi}\right)^n \left(\log\frac{E}{\epsilon}\right)^n \left(\log\frac{k_0\gamma}{\kappa}\right)^n, \tag{42}$$

where σ_0 is the cross section for the radiationless scattering of an electron through an angle γ, and ϵ is the lower limit to the energy of emitted photons. The above result is in agreement with that obtained by following the general treatment in Sec. 5 of reference 4, if we ignore the interference between the photons emitted by the initial and the final electrons. This confirms the fact that if an energetic electron is deflected through an angle, which is large compared with μ/E, there is very little interference between the photons emitted by the initial and the final electrons. In such a case, the probability for multiple photon production is appreciable, but a high-energy bremsstrahlung process with a large deflection of the incident electron is a very rare event.

5. GENERAL REMARKS ON MULTIPLE PHOTON PRODUCTION

The present investigation also throws some light on multiple photon production in nuclear collisions. For, according to the present view, a proton, too, is described by the Dirac equation, and therefore the treatment in Sec. 5 of reference 4 also holds for collisions involving high energy protons. Hence, in high-energy nuclear collisions sometimes multiple photon production is bound to take place along with the production of mesons and other particles. Moreover, a proton possesses an anomalous magnetic moment, which might also appreciably increase the cross section for multiple photon production by protons.

It further follows from the discussion in the preceding section that even at very high energies, the probability for multiple photon production is appreable only if there is no interference between the photons emitted in the initial and the final states. This condition is evidently satisfied in the following two cases: (1) an energetic charged particle is deflected through an angle, which is large compared with μ/E; (2) an energetic charged particle is annihilated, captured or converted into a neutral particle. Therefore, we should specially look for multiple photon production in nuclear collisions under the above conditions.