A dense electron beam can be taken to be $I_0 \sim 10^5 A$, where I_0 is in amperes and A, the cross-sectional area of the beam, is in square meters.[14] If, in accordance with Eq. (51), we take $\epsilon \sim 10^{-1}\lambda v_0/c$, and if we consider $A \sim \lambda^2$, then $n \sim 2\times 10^{14}\lambda^3$, where λ is in meters. It is thus evident that n becomes small with the wavelength.

[14] This is the order of magnitude of the current in the densest beams used in microwave tubes.

If we want to know at what wavelength n is just large enough to produce an increment in field equal to the initial uncertainty in field, we set n equal to 400 and find that the wavelength is of the order of a tenth of a millimeter.

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Freundlich Red-Shift Formula

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Freundlich has attempted to explain various anomalous observations by suggesting that the red shift be reinterpreted as an effect proportional to radiation density and length along the path of a photon. For a star of radius R the contribution of the outside radiation field exceeds that within the atmosphere (thickness l_0) by a factor of the order of $R/l_0=1000$ for average stars; this makes Freundlich's original formula untenable. The question is here examined whether the objections to Freundlich's conception can be removed if one reduces his constant of proportionality by 10^{-3}. To give the conception a full trial the calculations are made realistically, taking into account the extension of the star, and the deviations from Lambert's law of emission which result from the radiative transfer conditions in the photosphere. The resultant formula for the radiation shift gives an approximate fit with solar observations out to near the limb without retaining, as Freundlich does, an additional relativity shift equal to one fifth of the expected value. However, the rise of the observed shift at the limb to the gravitational value (and even higher) is unaccounted for. The serious objections to Freundlich's interpretation from other astronomical evidence still stand, though the order of magnitude of the discrepancies is in some instances considerably reduced.

THERE are systematic anomalies, as compared with the expected gravitational values, in the red shifts of the sun,[1] of many of the hotter stars,[2] and of the cool supergiant M stars.[3] The situation with the white dwarfs is not altogether clear: there is some doubt about the consistency of the observations on Sirius B with proposed models; on the other hand Popper[4] has recently found a fairly good verification of the gravitational shift in 40 Eridani B.

In an attempt to correlate the various anomalous red shifts Freundlich[5] has suggested that there is a red shift $\Delta\lambda/\lambda$ proportional to the radiation density U in the path of a photon and to the path length. Presumably such a shift, if it exists, is due to some as yet unformalized action of a radiation field upon a photon; none of the accepted processes for light scattering have anywhere near the size of cross section required to produce such an effect.[6] Thus, there is a natural reserve about accepting Freundlich's interpretation. A preliminary question to be settled before one even considers interpretation is whether the suggested proportionality of shift to U and path length really is valid.

For paths of length l along which U is constant, the

[1] C. E. St. John, Astrophys. J. **67**, 195 (1928); Freundlich, Brunn, and Bruck, Z. Astrophys. **1**, 43 (1930); M. G. Adam, Monthly Notices Roy. Astron. Soc. **108**, 446 (1948); L. Spitzer, Monthly Notices Roy. Astron. Soc. **110**, 216 (1950). Spitzer finds reasons why the collisional shifts are likely to be to the violet rather than the red, helping to explain the observed defect from the gravitational values. The articles by L. Goldberg and C. E. Moore, in *The Sun*, edited by G. P. Kuiper (University of Chicago Press, Chicago, 1953), contain useful bibliographies on the line shift problem.

[2] This is shown by the dependence of the K-term on spectral class of stars: R. J. Trumpler and H. F. Weaver, *Statistical Astronomy* (University of California, Berkeley, 1953), pp. 291, 354, 566; H. F. Weaver, *Vistas in Astronomy* (Pergamon Press, London, 1954). Even after Weaver's recent reanalysis of the data, and correcting for the gravitational effect, there seems to be a residual shift of 1–2 km/sec for O and B stars. Also in some Wolf-Rayet stars, which are much hotter yet, extraordinary red shifts have been demonstrated: O. C. Wilson, Astrophys. J. **91**, 394 (1940); **109**, 76 (1949).

[3] W. S. Adams and E. MacCormack, Astrophys. J. **81**, 119 (1935). These, like the solar discrepancies, have been ascribed to large systematic motions in the atmospheres; however various investigators do not agree about the likelihood of such large motions.

[4] D. M. Popper, Astrophys. J. **120**, 316 (1954).

[5] E. F. Freundlich, Proc. Phys. Soc. (London) **A67**, 192 (1954); Phil. Mag. **45**, 303 (1954).

[6] D. Ter Haar, Phil. Mag. **45**, 320 (1954). It should be remarked however that, in a recent analysis, T. Neugebauer, Acta Phys. Acad. Sci. Hung. **IV**, 31 (1954) has pointed out the enormous preponderance, under certain conditions, of small-angle forward scattering (the "Mie effect") such as would be requisite to produce red shift phenomena; he has suggested that neutrino-photon scattering may occur with sufficient probability to serve as an alternative to the relativistic explanation of the Hubble (cosmic) shift.

suggested shift may be written

$$\Delta\lambda/\lambda = BU\int dl = BUl = AT^4 l, \quad (1)$$

where A and B are constants, and T is the cavity-radiation temperature along the path. Freundlich evaluates A empirically, from data on B-type stars, to be 2×10^{-29} deg^{-4} cm^{-1}.

In analyzing the red shift across the solar disk, Freundlich assumes that the length of path l through the atmosphere is equal to $l_0 \sec\Theta$, l_0 being the thickness of the atmosphere and Θ the angle between the outward solar radius and the line to the observer (Fig. 1). The formula (1) then gives

$$(\Delta\lambda/\lambda)_{\text{inside}} = BUl_0 \sec\Theta. \quad (1')$$

This expression, corrected by an added adjusted constant, is represented by the dashed curve in Fig. 2. (Freundlich interprets the added constant, *ad hoc*, as a relativity shift equal to $\frac{1}{5}$ the expected value.)

Freundlich's basic conception was not applied consistently by him, since, except in one case, he considered only the effect of the radiation field in the atmosphere of the star. Several people have since made calculations including the effect of the radiation field outside a star both on one of its own photons and on that of another star passing near it. The conclusions seem incompatible with observational data both from individual[6] and binary stars,[7] because the outside contribution (see below) turns out to be a thousand times greater than the contribution (1).

In view of the peristent anomalies, however, it is worth while to ask whether it is possible to salvage Freundlich's attempted correlation by analyzing the data on the basis of an outside shift instead, and revaluating his constant A accordingly. That this might possibly yield the center-limb increase in shift across the solar disk is suggested by the fact that a photon coming from near the limb has a longer path in the vicinity of the sun than a photon coming straight away from the center. The M-supergaint and cosmic shift evidence cited by Freundlich is of course not compatible with attributing the bulk of the shift to an outside effect, and their reinterpretation would have to be dropped.

For generality, we consider the shift in λ for a photon originating at any point P in the neighborhood of a principal star ☉, which we shall refer to conveniently as "solar" (Fig. 1). With suitable reinterpretation this may be applied to a stellar photon nearly grazing the sun at time of total eclipse, or to a photon originating in the companion of the principal star in a binary system. Let the angle between the outward solar radius vector and the line of sight at P be Θ. Let ρ be the variable radial distance given as a fraction of the solar radius; α is the value of ρ where the photon originates and κ is the component of α normal to the photon path. If P' is any point on the line of sight, and θ the corresponding angle, the following relations hold:

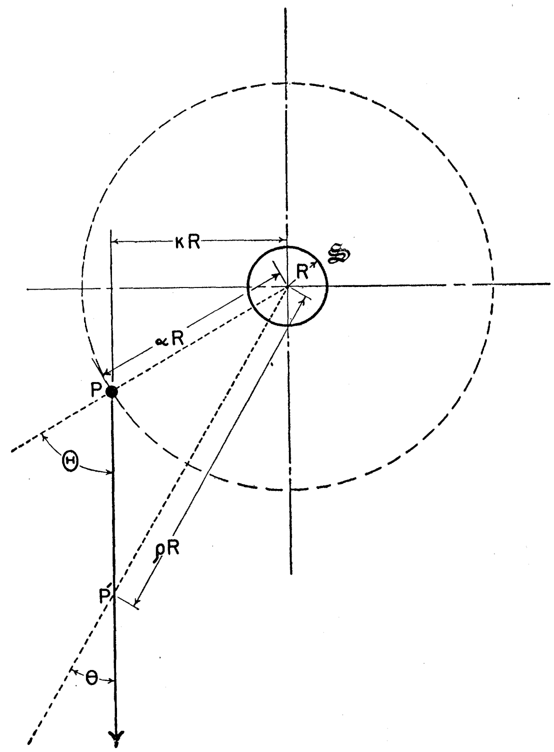

FIG. 1. Variables involved in the calculation of the effect of the radiation density around a primary star ☉ upon a photon originating at P and moving along the path PP'.

$$\kappa/\rho = \sin\theta, \quad d\rho/d(l/R) = \cos\theta. \quad (2)$$

The fractional red shift in the light on the path from P to E (the observer) is then

$$(\Delta\lambda/\lambda)_{\text{outside}} = B\int_P^E U(P')dl$$

$$= BR\int_\alpha^\infty U(\rho)(1/\cos\theta)d\rho, \quad (3)$$

where the distance to the observer is set equal to infinity. On the point-source model used by Ter Haar[6] and Helfer,[7]

$$U = u_1/\rho^2, \quad (4)$$

one finds

$$(\Delta\lambda/\lambda)_{\text{outside}} = Bu_1 R\Theta/\kappa = Bu_1(R/\alpha)\Theta/\sin\Theta. \quad (4')$$

The proper value to be given to the constant u_1 will be indicated below. At the surface of the sun $\alpha = 1$ and the contribution (4′) would certainly far outweigh (1′) since $R/l \gtrsim 10^3$. The dependence on Θ in (4′) is too

[7] H. L. Helfer, Phys. Rev. **96**, 224 (1954).

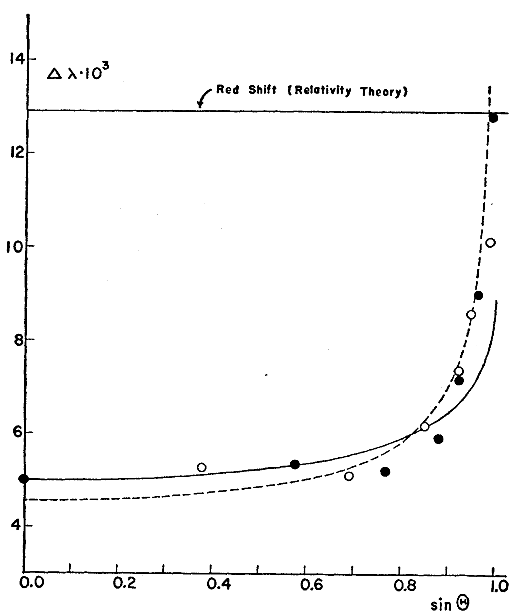

FIG. 2. The red shift $\Delta\lambda$ of solar lines as a function of the angle Θ between the solar radius to a point on its surface and the line of sight from that point. The full circles represent Adam's data (Oxford, 1948) (reference 1), the open circles Freundlich's data (Potsdam, 1930) (reference 1). The dashed curve represents Freundlich's secant formula based upon considering only the atmospheric contribution to the radiation shift. The full curve represents the outside contribution according to Eq. (5″) with Freundlich's constant reduced to 10^{-3} of his value.

gradual to represent the observed variation across the sun (Fig. 2) since (with a suitably adjusted B) it rises from the initial value of 5×10^{-3} to only $7.8\times10^{-3}A$ at $\Theta=\pi/2$, whereas the observed rise is quite steep near the limb. One may not reject (3) outright on this basis, however, since (4) has been derived assuming the point source model of the radiation field, which can lead to appreciable error near a star where the shift effect is greatest.

A more careful calculation,[8] taking into account the extension of the (spherical) source and the proper angular distribution of intensity emitted from the photosphere, shows that the radiation field near a star can be fairly well represented by

$$U(\rho)=U_1[1-(1-1/\rho^2)^{\frac{1}{2}}],\qquad [U_1=(7/4)\text{Emittance}/c]. \quad (5)$$

The difference in general form between $U(\rho)$ in (5), and the point-source expression (4) takes account of the extension of the source. On the other hand the numerical factor (7/4 rather than 2, as it would be with Lambert's law) in the expression for U_1 represents approximately the effect of the law of radiance from the outer boundary of the photosphere resulting from radiative transfer conditions within. [We note that for (4) to be an approximation to (5) u_1 must be taken *one-half* as large as U_1.]

Using (2), setting $1/\rho\equiv\sin\varphi$ and $1/\alpha\equiv\sin\Phi$, and substituting (5) in (3) we find

$$\left(\frac{\Delta\lambda}{\lambda}\right)_{\text{outside}}=BU_1R\int_\theta^\Phi\frac{1-\cos\varphi}{\sin^2\varphi\Delta(\varphi)}\cos\varphi d\varphi,\qquad [\Delta(\varphi)\equiv(1-\kappa^2\sin^2\varphi)^{\frac{1}{2}}].$$

The integral may be expressed in terms of elliptic integrals of the first and second kind, and gives

$$\left(\frac{\Delta\lambda}{\lambda}\right)_{\text{outside}}=BU_1R\left[\mathcal{E}_\alpha(\Theta)-\frac{1-\cos\Phi}{\sin\Phi}\cos\Theta\right], \quad (5')$$

where we have introduced the function

$$\mathcal{E}_\alpha(\Theta)=E(\Phi,\kappa), \qquad (\kappa\equiv\alpha\sin\Theta\le1)$$

$$\mathcal{E}_\alpha(\Theta)=\kappa[E(\Theta,1/\kappa)-(1-1/\kappa^2)F(\Theta,1/\kappa)]. \qquad (\kappa\equiv\alpha\sin\Theta\ge1)$$

(The second argument in E and F is the modulus.) At the surface of the sun $\alpha=1$ and $\Phi=\pi/2$. We then have

$$(\Delta\lambda/\lambda)_{\text{outside}}=BU_1R[\mathcal{E}_1(\Theta)-\cos\Theta]. \quad (5'')$$

This formula, adjusted to coincide with the observational point at $\Theta=0$, is shown by the full curve in Fig. 2. As we see, it is an improvement on the point-source representation (4′). However, it still falls appreciably short of the full limb value, rising only to about 8.8×10^{-3} A. Thus (5′) cannot be regarded as very satisfactory.

From the respective calculated and observed quantities at $\Theta=0$:

$$\mathcal{E}_1(0)-\cos0=\pi/2-1,\quad (\Delta\lambda/\lambda)_{\text{obs}}=5\times10^{-3}/6100,$$

and the known values of the radius and emittance for the sun, we find

$$U_1=(7/4c)\times6.25\times10^{10}\text{ erg cm}^{-3},\qquad B=5.7\times10^{-18}\text{ cm}^2\text{ erg}^{-1},$$

which corresponds to an approximate value

$$A=(7\sigma/4c)B=2\times10^{-32}\text{ deg}^{-4}\text{ cm}^{-1}, \quad (6)$$

one thousand times smaller than Freundlich's value.

The counter-evidence to Freundlich's hypothesis, assembled by Helfer from binary star data, is still decisive. In Table I we have listed again the double-star systems selected by Helfer, where both components are of the same type. Assuming rough equality in the radii, we have first calculated on the basis of (5′), using A as given in (6), the fixed shift in the light of each component due to its own radiation field (column 6). In the last column of Table I we have calculated the minimum, middle, and maximum observable radiation shifts over a half-period of revolution [at $\Theta=\sin^{-1}(1/\alpha)$,

[8] A calculation of the radiation flux field outside a radiatively transferring spherical source has been made [M. A. Melvin (to be published)].

TABLE I. A reanalysis of Helfer's data from five binary star systems by which Freundlich's red-shift interpretation was tested. The results in column 7 are obtained upon the basis of formula (5'). For the minimum and middle shifts the values obtained by means of the point-source model do not differ appreciably from the more exact values given in column 7; for the maximum shift the difference is appreciable and is indicated in parentheses.

1	2	3	4	5	6	7
Star name	Eff. temp. °K	Mean radius in units of the solar radius	Radial distance between stars in units of their radius	Observed semi-amplitudes expressed in km/sec	Calculated fixed shift of each star due to own radiation field, km/sec	Calculated min middle and max shift in a half-period, due to radiation field of companion km/sec
TT Aur.	18 000	4.3	2.7	197; 246	188	36 ; 55 ; 102 (259)
WW Aur.	8000	2.1	6.0	116; 135	3.6	0.3; 0.5; 0.9 (5.3)
TX Her.	10 000	1.6	6.7	121; 140	6.7	0.5; 0.8; 1.3 (10)
Z Vul.	18 000	4.3	3.5	96; 214	188	28 ; 42 ; 81 (268)
Y Cyg.	25 000	5.9	4.8	245; 241	960	101 ; 156 ; 304 (1405)

$\pi/2$, and $\pi-\sin^{-1}(1/\alpha)]$, in the light of one component due to the radiation field of its companion. It is seen that, while the variations in shift are now within the semiamplitudes of the radial velocity curves, they are still, except for two of the systems, considerably outside of observational error.

In view of the reduced magnitude of the stellar-statistical K-term which emerges from Weaver's recent analysis,[1] there is no longer any possibility of correlating the solar shift values with those found in O and B stars—that is if one adheres to the temperature fourth power (radiation density) dependence, or any reasonable modification thereof. This is illustrated forcibly again by the large values in column 6 of Table I.

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Entropy of Radiation

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In view of the recent publication of a derivation of the expression for the entropy flux of radiation, in terms of its spectral specific intensity, a brief comparative study of various known methods for obtaining the entropy formula is presented.

IN a recent article P. Rosen presents an interesting discussion of the question of how to obtain an expression for the entropy flux of electromagnetic radiation of specified intensity.[1] The very problem itself implies, in its general formulation, a well-defined entropy concept for nonequilibrium radiation. Rosen's derivation of the above-mentioned expression starts with the ordinary statistical definition of entropy, in this case for a system of photons (bosons) in terms of their density in phase space, followed by the substitution of this density by the equivalent expression in terms of the specific intensity of the corresponding radiation.

The explicit solution to the problem may be expressed in "differential" form as a relation connecting K_ν, the specific radiation (energy) intensity of linearly polarized light of frequency ν, and L_ν, the corresponding specific entropy intensity. K_ν is defined in such a manner that $K_\nu d\nu d\sigma \cos\theta d\Omega$ represents radiation energy pr. sec (power) in the frequency band $d\nu$ which passes through an element of area, $d\sigma$, and inside an element of solid angle, $d\Omega$, in a direction which makes an angle θ with the normal to the area. L_ν is defined correspondingly. The desired relation turns out to be the following:

$$L_\nu=(k\nu^2/c^2)[(1+x)\ln(1+x)-x\ln x],\quad x\equiv c^2K_\nu/h\nu^3, \quad (1)$$

where k = Boltzmann's constant, h = Planck's constant, and c = velocity of light.

In passing to present his elegant derivation of what amounts to this formula, Rosen makes a statement to the effect that while Planck has previously derived a functional relation, $L_\nu=\nu^2 f(K_\nu/\nu^3)$, no explicit expression was given by the latter[2] for $f(K_\nu/\nu^3)$.

In the opinion of the present author, however, the explicit relationship for L_ν in terms of K_ν was indeed known to Planck, even prior to the advent of the Bose-Einstein statistics proper.[3] It seems worth while, therefore, to present a brief comparison of various methods of proving the entropy formula (1).

As previously mentioned, Rosen bases his method on

[1] P. Rosen, Phys. Rev. 96, 555 (1954).

[2] M. Planck, *The Theory of Heat* (The MacMillan Company, New York, 1949).

[3] Bose, Z. Physik 26, 178 (1924).