PHYSICAL REVIEW VOLUME 98, NUMBER 3 MAY 1, 1955

Structure of the Scattering Matrix*

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(Received December 23, 1954)

It is shown that in field theory the structure of the scattering matrix for a certain variety of processes, such as those between a meson and a nucleon, a photon and a charged particle, etc., can be analyzed making use of relativistic invariance and microscopic causality. General formulas have been obtained which enable one to express scattering matrices for real as well as virtual processes explicitly by means of a parametric representation. The ideas are closely related to those which have been used by several authors in the analysis of the Green's functions for interacting fields.

I. INTRODUCTION

FORMAL aspects of scattering matrices, apart from the dynamical characteristics of the individual cases, have been studied extensively. Generally speaking, a scattering process may be described either by means of the S matrix of Heisenberg, or by means of the R matrix of Wigner, and each has its own advantages as well as disadvantages. In nonrelativistic quantum theory of nuclei, the latter has proved of great use, while in quantum field theory the former has mainly been exploited so far. It remains to be seen whether an R matrix formalism can be developed successfully in the relativistic case.[1]

The structure of these matrices is restricted *a priori* by various formal requirements which reflect the structure of the underlying theory. In field theory, relativistic invariance and invariances under other transformation groups must be imposed. This has been widely utilized in analyzing the S matrix. Another remarkable and important requirement is causality, which also seems to be a fundamental property of our physical world. Recently, Gell-Mann, Goldberger, and Thirring[2] have applied it to the scattering of γ rays by charged particles in field theory with interesting consequences of practical use.

Like relativistic invariance, causality in the present field theory may be called microscopic causality, in the sense that two simultaneous measurements at different points (on a space like surface) should not interfere with each other, however close they may be. This enables one to draw some conclusions on the analytic behavior of the scattering matrix as a function of energy, which expresses itself by a relation between absorptive and dispersive parts of the matrix elements of different energies, a relation obtained by Kramers and Heisenberg[3] a long time ago.

The purpose of the present paper is to exploit both relativity and causality more thoroughly than has been done so far, and try to get more detailed information on the nature of some scattering matrices in field theory, such as that between a nucleon and a meson. This is made possible by virtue of a convenient formula which expresses the scattering matrix as a product of two operators in the Heisenberg representation at different space-time points. Relativistic invariance and the causality condition, which demands commutativity of the two operators outside of each other's light cone, then enable one to write down the general form of the matrix element as a parametric integral representation involving certain unknown functions.

This phase of considerations is closely related to that concerning the so-called Δ_F' or S_F' functions which, originally introduced by Dyson[4] and Schwinger,[5] and studied along an interesting line by Källén and Lehman,[5] are fundamental in the current quantum field theory. In fact, the present results may be instructive not only as a theory of the scattering matrix, which has direct bearing on observation, but also for throwing some additional light on the fundamental character of quantum field theory.

Although we shall not attempt here any application of the results to particular problems, a brief discussion of their physical significance will be made in the last section.

II. DERIVATION OF THE SCATTERING MATRIX

Formal but exact expressions for certain types of the scattering matrices, such as that for the scattering of a gamma ray by a particle, have been obtained recently by several authors.[6] Indeed this can be done in various different ways, either by using perturbation theory explicitly or without it, but leading essentially to the same result. Since our purpose is to study the structure

* Supported by a grant from the U. S. Atomic Energy Commission.
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[1] The only attempt so far along this line seems to be that of Schwinger: J. Schwinger, Phys. Rev. **74**, 1439 (1948).
[2] Gell-Mann, Goldberger, and Thirring, Phys. Rev. **95**, 1612 (1954). Earlier literature on causality is cited in this paper.
[3] H. A. Kramers and W. Heisenberg, Z. Physik. **31**, 681 (1925); also R. Kronig, Physica **12**, 543 (1926).
[4] F. J. Dyson, Phys. Rev. **75**, 1736 (1949); J. Schwinger, Proc. Nat. Acad. Sci. U. S. **37**, 452 and 455 (1951).
[5] G. Källén, Helv. Phys. Acta **25**, 417 (1952), Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 12 (1953); H. Lehman, Nuovo cimento **11**, 342 (1954). See also M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).
[6] F. Low, Phys. Rev. **97**, 1392 (1955); M. L. Goldberger, Phys. Rev. **97**, 508 (1955). For the particular case of the low-energy limit, see also N. Kroll and M. Ruderman, Phys. Rev. **93**, 233 (1954); Deser, Thirring, and Goldberger, Phys. Rev. **94**, 711 (1954).

of such scattering matrices, it may be appropriate to show the derivation of the starting formulas more or less in our own way.

As an example, let us consider the scattering of a pseudo-scalar meson by a nucleon. We assume that the latter can be described by the Dirac spinor field which may interact with any other fields in addition to the meson field under consideration. In order that the theory yield meaningful results in the current field theoretical sense, however, it is further necessary that the fields and interactions be of the renormalized type. Understanding this, we shall assume that the meson-nucleon interaction is given by the interaction Hamiltonian:

$$H_{\text{int}} = -\sum_{\alpha=1}^{3} gi\bar{\psi}\gamma_5\tau_\alpha\psi\varphi_\alpha, \tag{1}$$

where ψ and $\bar{\psi}$ are the nucleon field operators, φ_α the meson field operator, and the τ_α are the conventional isotopic spin matrices.

In order to pay particular attention to this interaction, let us take an interaction representation in which the entire Hamiltonian has been eliminated except for the above part, Eq. (1). Then we construct the S-matrix which describes the meson-nucleon interation only:

$$S \equiv S(\infty, -\infty) = \sum_{n=0}^{\infty}(-i)^n \int_{-\infty}^{\infty}(dx_1)\int_{-\infty}^{x_1}(dx_2)\cdots \times\int_{-\infty}^{x_{n-1}}(dx_n)H_{\text{int}}(x_1)\cdots H_{\text{int}}(x_n) = P\exp\left[-i\int_{-\infty}^{\infty}H_{\text{int}}(x)(dx)\right], \tag{2}$$

where $(dx)\equiv dxdydzdt$ and $\hbar=c=1$. P stands for the time-ordered product of the operators. The matrix elements for the meson-nucleon scattering can be obtained from the expression

$$\lim F_{\alpha\beta}(x,x') \equiv \lim_{\substack{t\to+\infty \\ t'\to-\infty}} (\Psi_f, P(S, \varphi_\alpha(x), \varphi_\beta(x'))\Psi_a). \tag{3}$$

Ψ_a and Ψ_f are the initial and final states of the nucleon (without the meson interaction) which we assume to be realized at $t=-\infty$ and $t=+\infty$, respectively. Taking the limits indicated in (3), we get the initial and final states of the combined system of a free meson and a nucleon, since $\varphi(x)$ operating directly on Ψ creates one meson states.

The expression (3), being of the typical time-ordered form, may be converted into the so-called normal form: we divide $\varphi(x)$ into creation (negative frequency) part $\varphi^-(x)$ and annihilation (positive frequency) part $\varphi^+(x)$, displace them all the way to the left and right of S, respectively, and avail ourselves of the relations $\varphi^+\Psi_a = \Psi_f\varphi^- = 0$.[7] Knowing the structure of S, this can be carried out without difficulty, and one arrives at the following result:

$$\begin{aligned} F_{\alpha\beta}(x,x') &= \delta_{\alpha\beta}\Delta_F(x-x') + \int\int \Delta_F(x-y)\Delta_F(x'-y') \\ &\quad\times\left(\Psi_f, P\left(S, -i\frac{\delta H_{\text{int}}}{\delta\varphi_\alpha(y)}, -i\frac{\delta H_{\text{int}}}{\delta\varphi_\beta(y')}\right)\Psi_a\right)(dy)(dy') \\ &= \delta_{\alpha\beta}\Delta_F(x-x') + \int\int \Delta_F(x-y)\Delta_F(x'-y') \\ &\quad\times(\Psi_f, P(S, -g\bar{\psi}\gamma_5\tau_\alpha\psi(y), -g\bar{\psi}\gamma_5\tau_\beta\psi(y'))\Psi_a)(dy)(dy'). \end{aligned} \tag{4}$$

Δ_F is the Feynman propagation function for the meson. In the limit $t\to+\infty$, $t\to-\infty$, the Δ_F's describe the propagation of real meson waves in the wave zone. Omitting the first term corresponding to the direct propagation without interaction, the scattering matrix for a meson of four-momentum k_μ, isotopic spin β to that of four-momentum l_μ, isotopic spin α is obtained as

$$M_{\alpha\beta}(l,k) = \int\int e^{ikx'}e^{-ilx}(dx)(dx')M_{\alpha\beta}(x,x'),$$

$$M_{\alpha\beta}(x,x') = (\Psi_f, P(S, g\bar{\psi}\gamma_5\tau_\alpha\psi(x), g\bar{\psi}\gamma_5\tau_\beta\psi(x'))\Psi_a), \tag{5}$$

apart from some trivial factors.

We can convert $M_{\alpha\beta}(x,x')$ into a form which looks more useful for our purposes. The ordered operator appearing in Eq. (5) is in the mixed representation, one side being referred to $t=-\infty$, and the other to $t=+\infty$. A Heisenberg representation (referred to $t=-\infty$) will be obtained by writing

$$M_{\alpha\beta}(x,x') = (S^{-1}\Psi_f, S^{-1}P(S, g\bar{\psi}\gamma_5\tau_\alpha\psi(x), g\bar{\psi}\gamma_5\tau_\beta\psi(x'))\Psi_a). \tag{6}$$

Since Ψ_f is a stationary state of the nucleon, however, $S^{-1}\Psi_f = \Psi_f$ apart from a phase factor which may be neglected.[8] On the other hand, it is easily seen that

$$\begin{aligned} &S^{-1}P(S, g\bar{\psi}\gamma_5\tau_\alpha\psi(x), g\bar{\psi}\gamma_5\tau_\beta\psi(x')) \\ &= \begin{cases} S(-\infty,\infty)S(\infty,x)g\bar{\psi}\gamma_5\tau_\alpha\psi(x)S(x,x') \times g\bar{\psi}\gamma_5\tau_\beta\psi(x'), S(x',-\infty), & t\geq t' \\ S(-\infty,\infty)S(\infty,x')g\bar{\psi}\gamma_5\tau_\beta\psi(x')S(x'x) \times g\bar{\psi}\gamma_5\tau_\alpha\psi(x)S(x,-\infty), & t\leq t' \end{cases} \\ &= \begin{cases} S(-\infty,x)g\bar{\psi}\gamma_5\tau_\alpha\psi(x)S(x,-\infty)S(-\infty,x') \times g\bar{\psi}\gamma_5\tau_\beta\psi(x'), S(x',-\infty), & t\geq t' \\ S(-\infty,x')g\bar{\psi}\gamma_5\tau_\beta\psi(x')S(x',-\infty)S(-\infty,x) \times g\bar{\psi}\gamma_5\tau_\alpha\psi(x)S(x,-\infty), & t\leq t' \end{cases} \\ &= P(g\bar{\boldsymbol{\psi}}\gamma_5\tau_\alpha\boldsymbol{\psi}(\mathbf{x}), g\bar{\boldsymbol{\psi}}\gamma_5\tau_\beta\boldsymbol{\psi}(\mathbf{x}')), \end{aligned}$$

[7] A convenient formula for this procedure is given in T. Kinoshita and Y. Nambu, Phys. Rev. 94, 598 (1954), Appendix.

[8] Actually the phase factor is to be removed by renormalization as stated below.

with

$$\bar{\boldsymbol{\psi}}\boldsymbol{\gamma}_5\boldsymbol{\tau}_\alpha\boldsymbol{\psi}(\mathbf{x})\equiv S(-\infty,x)\bar{\psi}\gamma_5\tau_\alpha\psi(x)S(x,-\infty),\ \text{etc.} \quad (7)$$

The boldface operators are in the Heisenberg representation which coincides with the interaction representation at $t=-\infty$. The last condition, however, is irrelevant, and we can write in general:

$$M_{\alpha\beta}(x,x')=(\Phi_f,\ P(g\bar{\boldsymbol{\psi}}\boldsymbol{\gamma}_5\boldsymbol{\tau}_\alpha\boldsymbol{\psi}(\mathbf{x}),\ g\bar{\boldsymbol{\psi}}\boldsymbol{\gamma}_5\boldsymbol{\tau}_\beta(\mathbf{x}'))\Phi_a), \quad (8)$$

with the Φ's being the rigorous Heisenberg state vectors for the nucleon including the meson interaction.

In the above derivation the problem of renormalization has been omitted. In order to take account of it, we would have to start from a renormalized Lagrangian[9]:

$$L_{\text{int}}=gZ_1i\bar{\psi}\gamma_5\tau_\alpha\psi\varphi_\alpha+\tfrac{1}{2}Z_3\delta\mu^2\varphi_\alpha\varphi_\alpha +Z_2\delta m\bar{\psi}\psi+\lambda Z_4(\varphi_\alpha\varphi_\alpha)^2. \quad (9)$$

The effect is that we can eliminate all the infinities and interpret the masses and the coupling constant as the observed ones in the actual calculation of Eq. (8) by means of perturbation theory. Since, however, we are not primarily concerned with such explicit calculations, but rather interested in finding out the general structure of M, we shall simply use Eq. (8) supposing that the renormalization has been carried out.

The results obtained in this section can be extended to other scattering problems such as the scattering of light by charged particles.

III. STRUCTURE OF THE SCATTERING MATRIX

According to the last results, the scattering matrix (of a meson-nucleon system) can be obtained from the quantity M defined by Eqs. (5) and (8). We want to study in this section the general properties of expressions of the form

$$M=\langle q,t|A_\alpha(x)A_\beta(x')|p,s\rangle. \quad (10)$$

Here $|p,s\rangle$ (or $|q,t\rangle$) is the eigenstate of a real particle (fermion or boson) with the four-momentum p_μ(or q_μ), spin and other internal coordinates s (or t). $A_\alpha(x)$ is a local operator (defined below) at point x with spin and other indices α in the Heisenberg representation.

From the requirements of the quantum field theory we know offhand that M should have the following properties:

(1) Under the translation

$$x_\mu\to y_\mu=x_\mu+a_\mu,\quad x_\mu'\to y_\mu'=x_\mu'+a_\mu, \quad (11)$$

the operators $A_\alpha(x)$, $A_\beta(x')$ are transformed according to

$$\begin{aligned}A_\alpha(x)&\to A_\alpha(y)=\exp(iP_\mu a_\mu)A_\alpha(x)\exp(-iP_\mu a_\mu),\\ A_\beta(x')&\to A_\beta(y')=\exp(iP_\mu a_\mu)A_\beta(x')\exp(-iP_\mu a_\mu),\end{aligned} \quad (12)$$

[9] See for example, P. T. Matthews, Phil. Mag. **62**, 221 (1951); J. C. Ward, Phys. Rev. **84**, 897 (1951).

where P_μ is the energy-momentum operator of the total system. M transforms accordingly as

$$\begin{aligned}M\to M'&=\langle q,t|\exp(iP_\mu a_\mu)A_\alpha(x)A_\beta(x')\exp(-iP_\mu,a_\mu)|p,s\rangle\\ &=\exp[-i(p-q)_\mu a_\mu]M,\end{aligned} \quad (13)$$

since

$$P_\mu|p,s\rangle=p_\mu|p,s\rangle,\quad P_\mu|q,t\rangle=q_\mu|q,t\rangle.$$

(2) Under a homogeneous Lorentz transformation R: $x_\mu\to y_\mu=\sum_\nu c_{\mu\nu}x_\nu$ without time reversal, the state vectors transform contragrediently to the operators, so that

$$\begin{aligned}|p,s\rangle\to R|p,s\rangle&=\textstyle\sum_t \epsilon C_{st}|c_{\mu\nu}p_\nu,t\rangle\\ &\equiv\epsilon|Rp,Rt\rangle,\\ A_\alpha(x)\to RA_\alpha(x)R^{-1}&=\textstyle\sum_\beta\epsilon' C_{\alpha\beta}A_\beta(c_{\mu\nu}x_\nu)\\ &\equiv\epsilon' A_{R\alpha}(Rx),\end{aligned} \quad (14)$$

where C_{st} and $C_{\alpha\beta}$ are the spinor or tensor transformation coefficients corresponding to R; ϵ and ϵ' are the intrinsic parities which take the values ±1 or $\pm i$ (the latter is possible in case of spinor quantities). It follows then that M transforms under R as

$$\begin{aligned}M(z)\to M'(z)&=\langle qt|R^{-1}\cdot RA_\alpha(x)A_\beta(x')R^{-1}\cdot R|ps\rangle\\ &=\pm M(Rz)=M(z),\end{aligned} \quad (15)$$

with z standing for all the arguments appearing in M. In other words, if we denote by u_s, v_t, n_α, m_β the unit spinors or tensors representing the direction of polarization of the state vectors and operators, M is a scalar or pseudo-scalar quantity made up of p, q, x, x', u, v, n, and m. Especially it must be linear in each of the polarization tensors u, v, n, and m in view of Eq. (14).

(3) Other invariances. If the theory also is invariant under other types of transformations such as gauge transformation, charge conjugation, or isotopic spin transformation, they must be reflected in the nature of M in a similar way. It will not be necessary to enter into details here.

(4) Causality. Apart from the invariance under various transformation groups, the conventional field theory requires another property which may be called (microscopic) causality. We use the term here in a well-specified sense that two measurements at spatially separated points do not interfere; or in other words, a disturbance in the space-time does not propagate faster than the light velocity. This entails the property that two Heisenberg operators at space-like points x and x' commute (or anticommute) each other except possibly when $x=x'$.[10] We shall call such operators local operators.

[10] Rigorously speaking, not all Hermitian operators are physically observable. A spinor operator like $\psi(x)+\psi^*(x)$ cannot be an observable quantity. [E. P. Wigner, Z. Physik **133**, 101 (1952); Wick, Wightman, and Wigner, Phys. Rev. **88**, 101 (1952).] However, we may modify the causality statement by saying that two spinor operators should anticommute for space-like points. In the following we assume that the A's contain even numbers of spinor operators. However, extension to the general case is easy.

We now define the following four quantities:

$$M_1=\langle q,t|\{A_\alpha(x),A_\beta(x')\}|p,s\rangle,$$
$$M_2=\langle q,t|[A_\alpha(x),A_\beta(x')]|p,s\rangle,$$
$$M_3=\epsilon(x-x')M_2,$$
$$\epsilon(x-x')=\begin{cases}1 & t>t'\\ 0 & t=t'\\ -1 & t<t',\end{cases}\qquad(16)$$
$$M_4=\langle q,t|P(A_\alpha(x),A_\beta(x'))|p,s\rangle=\tfrac{1}{2}M_1+\tfrac{1}{2}M_3.$$

It is the consequence of causality that (*a*) M_2 and M_3 are zero if $(x-x')^2>0$; (*b*) M_3 and M_4 do not depend on the choice of the time axis (except possibly for $x=x'$, which case needs special consideration). Thus all these M's must share the invariance properties described in 1. In addition, we note that (*c*) M_1, M_3, and M_4 are symmetric, while M_2 is antisymmetric, under the simultaneous exchange $x\leftrightarrow x'$ and $\alpha\leftrightarrow\beta$. Moreover, if A_α and A_β are Hermitian, M_1 and M_3 are Hermitian matrix elements, while M_2 is anti-Hermitian.

We now go over to the momentum representation of the M's and try to study their nature. For this purpose, it must further be assumed that such Fourier transforms really exist and are meaningful. It is not always the case, however, for an arbitrary operator $A_\alpha(x)$ in the present field theory. So we explicitly assume here that the M's are such quantities that, at least after renormalization, they are finite and have Fourier transforms.

The Fourier transforms are as follows:

$$M_i(k,l)\delta^4(k-l+p-q)\equiv\frac{1}{(2\pi)^4}\int\int e^{-ilx}e^{ikx'}M_i(x,x')(dx)(dx').\qquad(17)$$

The delta function on the left-hand side, expressing energy-momentum conservation, is a consequence of Eq. (13). We can define, therefore, three independent vectors:

$$\begin{aligned}&P\equiv\tfrac{1}{2}(p+q),\ Q\equiv\tfrac{1}{2}(p-q)=\tfrac{1}{2}(l-k),\ K\equiv\tfrac{1}{2}(k+l);\\ &p=P+Q,\ q=P-Q,\ k=K-Q,\ l=K+Q,\end{aligned}\qquad(18)$$

of which only K will be regarded as variable. Further, we get

$$\begin{aligned}(2\pi)^4M_{1,2}(K)\delta^4(k-l+p-q)&=\langle q,t|A_\alpha(-l)A_\beta(k)\pm A_\beta(k)A_\alpha(-l)|p,s\rangle\\ &=\textstyle\sum_z(\langle q,t|A_\alpha(-l)|z\rangle\langle z|A_\beta(k)|p,s\rangle\\ &\qquad\pm\langle q,t|A_\beta(k)|z\rangle\langle z|A_\alpha(-l)|p,s\rangle),\end{aligned}\qquad(19)$$
$$A_\alpha(k)=\int e^{ikx}A_\alpha(x)(dx),\quad\text{etc.},$$

the summation being over a complete set of orthogonal states; and with the aid of the formula

$$\epsilon(x)=\frac{1}{\pi i}\int_{-\infty}^{\infty}P\frac{e^{isx_0}}{s}ds,\quad(P\equiv\text{principal value}),$$

we have

$$M_3(K)=\frac{1}{\pi i}\int\frac{1}{K_0-K_0'}\delta^3(\mathbf{K}-\mathbf{K}')M_2(K')(dK').\qquad(20)$$

Corresponding to (*c*), we observe that M_1, M_3, and M_4 are symmetric, while M_2 is antisymmetric against the change $K\rightarrow-K$, $\alpha\leftrightarrow\beta$.

Our main task is now to see the implications of (*a*) and (*b*) on the nature of the Fourier transforms. Equation (20) already suggests the dependence of M_3 on the time component of K. As a function of K_0, M_3 should go to zero at most as $1/K_0$ as $|K_0|\rightarrow\infty$ if

$$\int_{-\infty}^{\infty}M_2(K_0)dK_0=\frac{1}{(2\pi)^3}\int M_2(\mathbf{r},t;\,\mathbf{r}',t)\times\exp(i\mathbf{l}\mathbf{r}+i\mathbf{k}\mathbf{r}')d\mathbf{y},\quad\mathbf{y}=\mathbf{r}-\mathbf{r}',$$

the integrand of which is the commutator of two operators with equal times, is assumed to be finite. Naturally $M_2(K_0)$ goes to zero more strongly than $1/K_0$.

On the other hand, we have requirements of relativistic invariance for the M's, whereas Eq. (20) is not an invariant expression since there enters a special time axis which has nothing to do with the intrinsic nature of the system. Thus we are led to expect that the fraction in Eq. (20), which gives the K_0 dependence of M_3, could effectively be transformed into inverse polynomials of the scalars K^2, (PK), and (QK), and that such inverse polynomials could be decomposed into partial fractions of real values if considered as functions of K_0.

To see this, it is convenient to construct the quantities

$$M_\pm=M_2\pm M_3.\qquad(21)$$

They have the property that M_+ (M_-) is different from zero only in the future (past) light cone: $t\geq t'$ ($t'\geq t$). The corresponding Fourier transform $M_\pm(K_0)$ can be continued analytically into the upper (lower) complex plane, and has no singularities there. Let us then take the function

$$G_\pm(x)=\exp(ia_\mu x_\mu)\left(1+\frac{1}{i}b_\mu\frac{\partial}{\partial x_\mu}\right)\Delta_\pm^{(m)}(x),\qquad(22)$$

where $\Delta_\pm(x)$ is the retarded (advanced) propagation function for a field with mass m:

$$\begin{aligned}&(\square^2-m^2)\Delta_\pm(x)=-\delta^4(x),\\ &\Delta_\pm(0)=0,\quad\partial\Delta_\pm(0)/\partial t=\pm\delta^3(\mathbf{r}),\\ &\Delta_\pm(x)=0\quad\text{for}\quad\begin{cases}t<0\\ t>0,\end{cases}\end{aligned}\qquad(23)$$

and a_μ, b_μ are arbitrary vectors. The G's therefore share the same property of being zero except in one of the light cones. The Fourier transforms of the G's are

$$G_\pm(k)=[1+(b,k)]\left[P\frac{1}{(k+a)^2+m^2}\mp i\pi\epsilon(k+a)\delta((k+a)^2+m^2)\right] \quad (24)$$
$$=\frac{1+(b,k)}{(k+a\pm i\eta)^2+m^2},$$

with an infinitesimal time-like vector η pointing to future ($\eta_0>0$).

Let $\mathfrak{M}_\pm$ be a set of functions $G(x)$ which are nonzero only in the future ($t\geq 0$) or past ($t\leq 0$), and such that $G(k_0)=O(1/k_0)$ for $|k_0|\rightarrow\infty$. Then such functions have the following properties:

(1) If $G_1(x)$ and $G_2(x)$ belong to $\mathfrak{M}_+$, then $G_3(x)\equiv aG_1(x)+bG_2(x)$ also belongs to $\mathfrak{M}_+$.
(2) If $G_1(x)$ and $G_2(x)$ belong to $\mathfrak{M}_+$, then $G_3(x)\equiv\int G_1(x-y)G_2(y)(dy)$ also belongs to $\mathfrak{M}_+$, and $G_3(k)=G_1(k)G_2(k)$.

The same is true for the set $\mathfrak{M}_-$. Since the function (22) obviously belongs to $\mathfrak{M}$ and cannot be factored further without impairing the relativistic invariance, we may conclude that the set $\mathfrak{M}_\pm$ can be generated by elementary functions of the form (22) according to the above rules (1) and (2). We further observe that

$$\frac{1}{(k+a\pm i\eta)^2+m^2}\cdot\frac{1}{(k+a'\pm i\eta)^2+m'^2}$$
$$=\int_0^1 dx[\{(k+a\pm i\eta)^2+m^2\}x+\{(k+a'\pm i\eta)^2+m'^2\}(1-x)]^{-2}$$
$$=-\left(\frac{\partial}{\partial m^2}+\frac{\partial}{\partial m'^2}\right)\int_0^1 dx[(k+xa+(1-x)a'\pm i\eta)^2+x(1-x)(a-a')^2+xm^2+(1-x)m'^2]^{-1},$$
$$\frac{1}{(k+a\pm i\eta)^2+m^2}\cdot\frac{1}{(k+a'\pm i\eta)^2+m'^2}k_\mu$$
$$=-\frac{1}{2}\left(\frac{\partial}{\partial a_\mu}+\frac{\partial}{\partial a_\mu'}\right)\int_0^1 dx[(k+xa+(1-x)a'\pm i\eta)^2+x(1-x)(a-a')^2+xm^2+(1-x)m'^2]^{-1}$$
$$+\frac{1}{2}\left(\frac{\partial}{\partial m^2}+\frac{\partial}{\partial m'^2}\right)\int_0^1 dx(xa_\mu+(1-x)a_\mu')\times[(k+xa+(1-x)a'\pm i\eta)^2+x(1-x)(a-a')^2+xm^2+(1-x)m'^2]^{-1}, \quad (25)$$

which enables one to reduce a product of two $G_\pm$'s into a linear combination of single $G_\pm$'s. Thus an arbitrary function belonging to $\mathfrak{M}_\pm$ may always be expressed as

$$G(k)=\int\frac{1}{(k+a\pm i\eta)^2+m^2}\rho^{(1)}(m^2,a_\mu)dm^2(da)+\int\frac{k_\mu}{(k+a\pm i\eta)^2+m^2}\rho_\mu^{(2)}(m^2,a_\nu)dm^2(da), \quad (26)$$

provided that we formally understand by the ρ's functions of the variables m and a_μ with sufficiently general character, including perhaps derivatives of delta functions.

The integrand of Eq. (25) is not exactly of the form Eq. (22) since the corresponding mass term $x(1-x)\times(a-a')^2+xm^2+(1-x)m'^2$ is not necessarily positive definite. If it is negative, the corresponding $G_\pm(x)$ will be superficially nonzero also outside of the light cone. In the physical consideration below, however, it is natural to assume that such a contribution does not arise actually. This situation is related to the fact that the representation (26) is not unique, a point which will be discussed later.

In order to apply the above results to the case under consideration, let us for the moment assume that there are no spin or tensor variables involved. We have then three vectors K, P, and Q available, on which the M's should depend. Thus the vectors a_μ and ρ_μ must be linear combinations of P and Q. It is then easy to write down formulas for the M's explicitly:

$$M_\pm(K)=\int\int\int\frac{\rho_1(m^2,\alpha,\beta)+\rho_2(m^2,\alpha,\beta)(K,P)+\rho_3(m^2,\alpha,\beta)(K,Q)}{(K+\alpha P+\beta Q\pm i\eta)^2+m^2}dm^2d\alpha d\beta. \quad (27)$$

The ρ's are taken to be equal for both M_+ and M_-, since otherwise it leads to inconsistencies as we shall see. From Eq. (27) it follows that

$$\begin{aligned}
M_2(K)&=\pi\int\int\int\epsilon(K+\alpha P+\beta Q)\delta((K+\alpha P+\beta Q)^2+m^2)[\rho_1+\rho_2(K,P)+\rho_3(K,Q)]dm^2d\alpha d\beta,\\
M_1(K)&=\pi\int\int\int\delta((K+\alpha P+\beta Q)^2+m^2)[\rho_1+\rho_2(K,P)+\rho_3(K,Q)]dm^2d\alpha d\beta,\\
M_3(K)&=P\int\int\int\frac{1}{(K+\alpha P+\beta Q)^2+m^2}[\rho_1+\rho_2(K,P)+\rho_3(K,Q)]dm^2d\alpha d\beta,\\
M_4(K)&=-i\int\int\int\frac{1}{(K+\alpha P+\beta Q)^2+m^2-i\epsilon}[\rho_1+\rho_2(K,P)+\rho_3(K,Q)]dm^2d\alpha d\beta.
\end{aligned} \quad (28)$$

In the last formula, ϵ represents a small positive constant.

In Eq. (28), M_2 and M_3 are the immediate consequences of Eq. (27). Before proving the formulas for M_1 and M_4, let us first note restrictions on the ρ's.[11]

(1) In the first place, the symmetry properties (c) require that

$$
\begin{aligned}
&\rho_1(m^2,\alpha,\beta)=\rho_1(m^2,-\alpha,-\beta)=\rho_1{}^*(m^2,\alpha,-\beta),\\
&\rho_2(m^2,\alpha,\beta)=-\rho_2(m^2,-\alpha,-\beta)=\rho_2{}^*(m^2,\alpha,-\beta),\\
&\rho_3(m^2,\alpha,\beta)=-\rho_3(m^2,-\alpha,-\beta)=-\rho_3{}^*(m^2,\alpha,-\beta).
\end{aligned}
\tag{29}
$$

(2) According to Eq. (19), M_1 and M_2 are zero for a given value of K unless a real process is possible in the intermediate state z, the energy-momentum of which is either $p+k=P+K$ or $p-l=P-K$ with positive time components, and each factor $\delta((K+\alpha P+\beta Q)^2+m^2)$ in M_2 of Eq. (28) should correspond to some such process. Since, on the other hand, the particle considered is assumed to be stable, the mass of the intermediate state cannot be less than that of the initial state μ. We have thus two relations:

$$
\begin{aligned}
&(K+\alpha P+\beta Q)^2+m^2=0,\\
&(P\pm K)^2+\mu^2\leq 0,\quad P_0\pm K_0>0,
\end{aligned}
\tag{30}
$$

which must be compatible with each other. From Eq. (30) we get

$$\mu^2-m^2+P^2-(\alpha P+\beta Q)^2\pm 2(K,\ (1\mp\alpha)P\mp\beta Q)\leq 0,$$

or

$$
\begin{aligned}
\mu^2-m^2+&[(1\mp\alpha)P\mp\beta Q]^2\\
&\pm 2(K+\alpha P+\beta Q,\ (1\mp\alpha)P\mp\beta Q)\leq 0.
\end{aligned}
\tag{31}
$$

Equation (31) is satisfied for any possible K only if $(1\mp\alpha)P\mp\beta Q=0$, or if K, $K+\alpha P+\beta Q$, and $(1\mp)P\mp\beta Q$ are all time-like vectors with

$$\pm\operatorname{sgn}K_0=\pm\operatorname{sgn}(K+\alpha P+\beta Q)=\operatorname{sgn}[(1\mp\alpha)P\mp\beta Q],$$

and

$$m+\{-[(1\mp\alpha)P\mp\beta Q]^2\}^{\frac{1}{2}}\geq\mu. \tag{32}$$

Considering Eq. (29) and the properties of P and Q:

$$-P^2=\mu^2+Q^2\geq\mu^2,$$

we conclude that

$$|\alpha|+|\beta|(Q^2/\mu^2+Q^2)^{\frac{1}{2}}\leq 1. \tag{33}$$

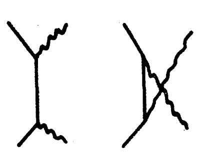

FIG. 1. The lowest (second-) order Feynman diagams for the scattering process between a φ meson (full line) and a χ meson (wavy line).

[11] As was mentioned above, the ρ's may contain derivatives of the delta function up to infinite order. But in order that the Fourier transform M_2 in Eq. (28) have no extraordinary behavior, arbitrarily high derivatives clearly should not occur. We assume here that this is actually the case.

We see from this also that the second (first) term of Eq. (19) does not contribute if $(K+\alpha P+\beta Q)_0$ is positive (negative), and so $M_1=M_2$ $(-M_2)$ accordingly. This proves the formulas for M_1 and M_4 in Eq. (28). If we took in Eq. (27) different ρ's for $M_\pm$, we would get for M_2 an additional term similar to M_3. The above arguments rule this out.

(3) If we take a K such that $(K,Q)=l^2-k^2=0$, then

$$(K+\alpha P+\beta Q)^2+m^2=K^2+2\alpha(K,P)+\alpha^2P^2+\beta^2Q^2+m^2,$$

so that ρ_1 and ρ_2 are functions of β^2. When this equation is combined with Eq. (29), we then see that

$$\rho_1(m^2,\alpha,\beta),\quad \rho_2(m^2,\alpha,\beta)\ \text{are real.} \tag{34}$$

For the particular case $Q=p-q=0$, which corresponds to the forward scattering of real or virtual mesons, M_1 must be positive definite:

$$
\begin{aligned}
(2\pi)^4M_1\delta^4(Q)=\textstyle\sum_z[&\langle p|A(-k)|z\rangle\langle z|A(k)|p\rangle\\
&+\langle p|A(k)|z\rangle\langle z|A(-k)|p\rangle]\\
=\textstyle\sum_z[&|\langle p|A(k)|z\rangle|^2+|\langle p|A(-k)|z\rangle|^2]\geq 0.
\end{aligned}
\tag{35}
$$

This should hold for an arbitrary $K=k$, so that

$$\int\int\int[\rho_1+y\rho_2]\delta((1-\alpha^2)m^2+x+2\alpha y)dm^2d\alpha d\beta\geq 0 \tag{36}$$

for $Q=0$ and any x and y.

The formulas Eq. (28) have been obtained on the assumption that there are no spin or tensor variables involved. In case there are such variables, we must construct the ρ's with the correct transformation properties. For example, in the case of Sec. 2,

$$A_\alpha=gi\bar{\psi}\gamma_5\tau_\alpha\psi,$$

so that $M_{\alpha\beta}$ is a scalar, but we have four vectors K, P, Q, and γ_μ available; of course it must also depend on the τ spin. Thus we have

$$
\begin{aligned}
M_{1\alpha\beta}=\int\int\int&\delta((K+\alpha P+\beta Q)^2+m^2)dm^2d\alpha d\beta\\
&\times\{\delta_{\alpha\beta}[\rho_1+(K,P)\rho_2+(K,Q)\rho_3+(K,\gamma)\rho_4]\\
&+\tau_{\alpha\beta}[\rho_5+(K,P)\rho_6+(K,Q)\rho_7+(K,\gamma)\rho_8]\},\ \text{etc.,}
\end{aligned}
\tag{37}
$$

$$\tau_{\alpha\beta}=\frac{1}{2i}(\tau_\alpha\tau_\beta-\tau_\beta\tau_\alpha).$$

Scalars like (P,γ) and (Q,γ) have been eliminated by means of the equation of motion for the free nucleon.

A little care is needed, for example, in the case of the current operator for a scalar particle:

$$J_\mu=-ie[\varphi^*(\partial_\mu-ieA_\mu)\varphi-(\partial_\mu+ieA_\mu)\varphi^*\varphi],$$

where A_μ is the electromagnetic field. In this case the commutators of the J_μ's at the same point do depend on the coordinate system, so that the previous invariance arguments cannot be applied directly, although the gauge invariance enables one to overcome the difficulty. We will not go into details here.

In concluding this section, it seems appropriate to point out that the representation Eq. (26) is not a unique one. We first observe that the quantities

$$K\equiv 1/[(k+a)^2+b],\quad K_\mu\equiv k_\mu K \tag{38}$$

satisfy the relations

$$\begin{gathered}\frac{\partial K_\mu}{\partial b}+a_\mu\frac{\partial K}{\partial b}-\frac{1}{2}\frac{\partial K}{\partial a_\mu}=0,\\ \frac{\partial}{\partial b}[(b^2+a^2)K]+a_\mu\frac{\partial K_\mu}{\partial m^2}+\frac{1}{2}\frac{\partial K_\mu}{\partial a_\mu}=0.\end{gathered}\tag{39}$$

From this follows that if the ρ's are defined as

$$\begin{gathered}\rho=(b^2+a^2)\frac{\partial\lambda}{\partial b}+\frac{\partial}{\partial b}(\lambda_\mu a_\mu)-\frac{1}{2}\frac{\partial\lambda_\mu}{\partial a_\mu},\\ \rho_\mu=a_\mu\frac{\partial\lambda}{\partial m^2}+\frac{1}{2}\frac{\partial\lambda}{\partial a_\mu}+\frac{\partial\lambda_\mu}{\partial b},\end{gathered}\tag{40}$$

where λ and λ_μ are functions of a_μ and b, and if

$$\lambda,\lambda_\mu\rightarrow 0 \tag{41}$$

uniformly on the boundary of the domain of a_μ and b, then Eq. (26) is identically zero after partial integration. In other words, the ρ's are indeterminate to within arbitrary additive functions (40) with the condition (41). Thus the restrictions found before for the ρ's should not be interpreted as necessary consequences of the representation (26), but rather as a specification to a certain extent of the "gauge" by means of physical considerations. The precise meaning of such a "gauge" transformation is not clear. It may also be suggested that this freedom could be used to eliminate one of the ρ's explicitly, but it is not clear whether we can in general satisfy the boundary condition (41). It should further be noted that Eq. (39) implies

$$\mathcal{G}K=\mathcal{G}K_\mu=0,\quad \mathcal{G}\equiv\sum_\mu\frac{\partial^2}{\partial a_\mu{}^2}+4b\frac{\partial^2}{\partial b^2},$$

so that each of the ρ's by itself is not unique either.

On the other hand, the degree of freedom of a scattering matrix can be checked easily. A scattering matrix, which is a complex quantity, depends in the spinless case on three parameters: total energy, relative momentum, and scattered angle if we let the mass of the scattered particle be arbitrary; if we let both the incident and outgoing particles be separately arbitrary, we need four parameters. In the first case we have $(K,Q)=0$, so that M_4 in Eq. (28) contains two real functions ρ_1, ρ_2 of three parameters m^2, α, and β; this is reasonable. In the latter case, we have a greater degree of freedom, and it is natural that a new function ρ_3 should appear. Since, however, we have been treating M as a function of K only, and not of P or Q, such arguments as these cannot be complete.

IV. CORRESPONDENCE TO PERTURBATION THEORY

It is not difficult to verify the preceding formulas with the perturbation theory, at least in the lowest few orders. To avoid unnecessary complications, let us take a neutral scalar field φ of mass μ interacting with another neutral scalar field χ with mass κ according to the interaction

$$H_{\text{int}}=-g\varphi^2\chi. \tag{42}$$

The quantity to be calculated is

$$M_4=\langle q|P(g\varphi^2,g\varphi'^2)|p\rangle,$$

which describes, according to Sec. 2, the scattering of a χ meson by a φ meson. The lowest order perturbation, of order g^2, corresponding to the two Feynman diagrams in Fig. 1, gives immediately

$$\begin{aligned}M_4{}^{(2)}&=g^2[\Delta_F{}^{(\mu)}(p+k)+\Delta_F{}^{(\mu)}(p-l)]\\ &=-ig^2\left(\frac{1}{(P+K)^2+\mu^2-i\epsilon}+\frac{1}{(P-K)^2+\mu^2-i\epsilon}\right),\end{aligned}\tag{43}$$

so that

$$\rho_1{}^{(2)}=g^2\delta(m^2-\mu^2)\delta(1-|\alpha|)\delta(\beta),\quad \rho_2{}^{(2)}=\rho_3{}^{(2)}=0.$$

In the fourth-order approximation, there are eight corrections to each of the second-order diagrams, which are classified into five kinds in Fig. 2.

(a) Represents the self-energy correction to the initial or final meson line, which amounts only to the renormalization of the amplitude.

(b) Is the self-energy correction to the intermediate meson-line. This can be obtained by replacing the Δ_F-functions in Eq. (43) by the modified ones Δ_F'. As has been shown by several authors,[5] Δ_F' has the structure

$$\Delta_F{}'^{(\mu)}(P\pm K)=-i\int_{\mu^2}^{\infty}\frac{1}{(P\pm K)^2+m^2-i\epsilon}\rho(m^2)dm^2,$$

which is also clear from our present argument. Thus, for this part,

$$\rho_{1b}{}^{(4)}=f(m^2)\delta(1-|\alpha|)\delta(\beta),\quad \rho_{2b}{}^{(4)}=\rho_{3b}{}^{(4)}=0, \tag{44}$$

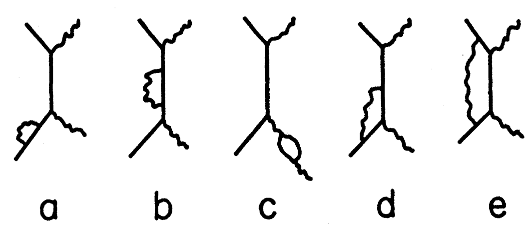

FIG. 2. Typical fourth-order corrections to one of the lowest order diagrams.

where $f(m^2)$ may be calculated in the g^4-approximation.

(c) Is the self-energy correction to the incident χ-meson line. This can again be obtained by multiplying Eq. (43) by

$$(k^2+\kappa^2)\Delta'^{(\kappa)}(k)=\int_{\kappa^2}^{\infty}\frac{k^2+\kappa^2}{k^2+m^2}\rho_\kappa(m^2)dm^2.$$

Now

$$\frac{1}{(P\pm K)^2+\mu^2}\frac{1}{k^2+m^2}=\frac{1}{(P\pm K)^2+\mu^2}\cdot\frac{1}{(K-Q)^2+m^2}$$

$$=\int_0^1 dx\frac{1}{\{x[(P\pm K)^2+\mu^2]+(1-x)[(K-Q)^2+m^2]\}^2}$$

$$=\int_0^1 dx\frac{1}{\{[K\pm xP-(1-x)Q]^2+x^2\mu^2+(1-x)m^2\}^2}.$$

The correction to the outgoing χ line is obtained by replacing Q by $-Q$. Hence

$$\rho_{1c}{}^{(4)}=\Bigg[\{k^2-\alpha^2\mu^2-m^2+((1+|\beta|)^2-\alpha^2)Q^2\}\frac{\partial}{\partial m^2}-\left(2+|\alpha|\frac{\partial}{\partial|\alpha|}+(1+|\beta|)\frac{\partial}{\partial|\beta|}\right)\Bigg]f_c(m^2,|\alpha|,|\beta|),$$

$$f_c=\delta(1-|\alpha|-|\beta|)\frac{1}{1-|\alpha|}\rho_\kappa\left(\frac{m^2-\alpha^2\mu^2}{1-|\alpha|}\right),\quad (45)$$

$$\rho_{2c}{}^{(4)}=\rho_{3c}{}^{(4)}=0,$$

which satisfies the inequalities (32) and (33).

The corrections (d) and (e), although more complicated, can be calculated in a similar way. We give below only the results.

$$M_d{}^{(4)}=i\frac{g^4}{(2\pi)^4}\int\left\{\frac{1}{((p-t)^2+\mu^2)[(p-t+k)^2+\mu^2](t^2+\kappa^2)[(p+k)^2+\mu^2]}+\text{similar terms}\right\}(dt),$$

$$M_e{}^{(4)}=i\frac{g^4}{(2\pi)^4}\int\left\{\frac{1}{((p-t)^2+\mu^2)[(p-t+k)^2+\mu^2][(q-t)^2+\mu^2](t^2+\kappa^2)}+\text{similar terms}\right\}(dt),$$

$$\rho_{1d}{}^{(4)}=\left(\frac{g^2}{4\pi}\right)^2\delta(1-|\alpha|-|\beta|)\int_0^1\int_0^1 dxdy\,\frac{\delta'(m^2-f_d(\alpha,x,y))}{(1-x)(1-y)}dxdy,$$

$$f_d(\alpha,x,y)=2(1-|\alpha|)Q^2+\frac{(1-|\alpha|)x}{(1-x)(1-y)}\kappa^2+\left\{1+(1-|\alpha|)\left(1-\frac{|\alpha|}{1-x}+\frac{1-x}{1-y}+\frac{1}{y(1-x)}\right)\right\}\mu^2,\quad (46)$$

$$\rho_{1e}{}^{(4)}=\frac{1}{2}\left(\frac{g^2}{4\pi}\right)^2\theta(1-|\alpha|-|\beta|)\int_0^1\frac{dz}{(1-z)^2}\delta'(m^2-f_e(\alpha,x,y)),$$

$$f_e(\alpha,x,y)=\frac{1}{z(1-z)}\{z[(1-|\alpha|)^2-\beta^2]Q^2+[1-z|\alpha|+z(1-|\alpha|)^2]\mu^2+z|\alpha|\kappa^2\},$$

$$\rho_2{}^{(4)}=\rho_3{}^{(4)}=0,$$

$$\theta(x)=\begin{cases}1 & x\geq 0\\ 0 & x=0.\end{cases}$$

ρ_2 and ρ_3 have been found to be zero in the present example. This is due to the special choice of the interaction, and may not be a general rule. Indeed, if the φ and χ fields interact with another spinor field ψ through a Hamiltonian

$$f_1\bar{\psi}\psi\varphi+f_2\bar{\psi}\psi\chi,$$

there will be a contribution to the scattering from diagrams represented in Figs. 3, and ρ_2 and ρ_3 will not be zero since the nucleon propagation functions contain momenta also in the numerator.

The effect of the renormalization has not been discussed so far. If we do not carry out renormalization, what happens is that the ρ's get multiplied by the renormalization constants Z introduced in Eq. (9), and also in general a (divergent) constant term must be added to M_3 and M_4 of Eq. (28), which makes our causality considerations invalid. In the present example, this occurs in the case of the diagrams in Fig. 3.

V. DISCUSSION

The present results are of interest for two reasons. First, we see the general structure of certain types of the scattering matrix in an explicit form. Second, it is an extension of the work developed by Källém, Lehman, and others[5] who studied the structure of quantities

$$\langle P(A_\alpha(x),A_\beta(x'))\rangle_0,\quad (47)$$

where the matrix element is taken with respect to the vacuum rather than one particle states.

As a theory of the scattering matrix, we can compare the results with the dispersion relation of Kramers and Heisenberg[3] between the real and imaginary components of a scattering matrix:

$$\mathrm{Re}[M(\omega)-M(0)]=\frac{2\omega^2}{\pi}\mathbf{P}\int_0^\infty \frac{d\omega'}{\omega'}\frac{\mathrm{Im}[M(\omega')]}{\omega'^2-\omega^2}, \quad (48)$$

where $M(\omega)$ is the forward scattering matrix of a real particle as a function of the energy ω. If we put, correspondingly, in Eq. (28):

$$Q=k-l=0, \quad K^2=k^2=-\kappa^2=\text{const}, \quad (49)$$

and regard it as a function of $(P,K)=-\mu\omega$ (in the rest system of the scatterer), then it is easy to see that each integrand,

$$\frac{1}{\alpha^2P^2+m^2-\kappa^2-2\alpha\mu\omega-i\epsilon}+\frac{1}{\alpha^2P^2+m^2-\kappa^2+2\alpha\mu\omega-i\epsilon},$$

$$\frac{-\mu\omega}{\alpha^2P^2+m^2-\kappa^2-2\alpha\mu\omega-i\epsilon}+\frac{\mu\omega}{\alpha^2P^2+m^2-\kappa^2+2\alpha\mu\omega-i\epsilon},$$

satisfies Eq. (48) separately.

Our formula thus gives more information about the scattering matrix in that it is not restricted to the forward scattering of a real particle, and is in an explicit form. On the other hand, it must be admitted that this has been achieved at the cost of stronger assumptions on the properties of M. For one thing, the nature of the functions ρ is actually not clear. The same seems to be true in the case of Eq. (47), although the derivation there looks simpler and more convincing.

It may be stressed, on the other hand, that only causality and the transformation properties of the system have been utilized in deriving the results. As long as these conditions are satisfied, the formulas should apply to any system.

As a formula for the scattering matrix, its use may perhaps lie in that it gives some insight into the relation between real and virtual processes. One may also be able to determine the scattering matrix or the ρ's by a method other than the perturbation method.

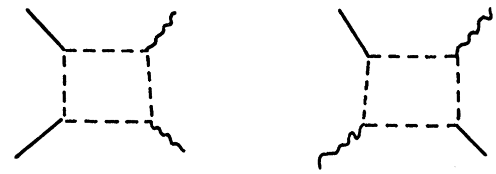

FIG. 3. Scattering of two mesons through an intermediate spinor field indicated by broken lines.

The implication in field theory is, among other things, that renormalization is necessary not only to secure convergence, but also to satisfy causality. For the imaginary part of the radiative corrections to a process, which simply corresponds to the occurrence of a new reaction, is finite at least in the lowest order, while the real part is divergent and indefinite. Since causality implies a definite relation between the real and imaginary parts, we would get inconsistencies without renormalization. It also suggests that the infinities could also be disposed of by the causality requirements, which really seems to be the meaning of Lehman's procedure of calculating renormalized quantities.

One of the limitations of our results is that the scattering matrix is considered as a function of one variable vector K only, so that the ρ's still depend on P and Q in an unknown way. But now that the general method of approach has been found, it does not seem too difficult to get the complete information about the scattering matrix insofar as it can be obtained without actually solving the equations of motion. Also this would enable one to discuss a wider variety of matrix elements involving products of more than two Heisenberg operators.

VI. ACKNOWLEDGMENT

This work was started when the author was staying at the University of Wisconsin under U. S. Atomic Energy Commission sponsorship, for which he is indebted to Professor R. G. Sachs. Thanks are also due to Professor M. L. Goldberger for his interest and stimulating discussions on the subject.