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Use of the Flat-Space Metric in Einstein's Curved Universe, and the "Swiss-Cheese" Model of Space*

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Using a second metric tensor $\gamma_{\mu\nu}$ as proposed by Rosen, Gupta's supplementary condition for the gravitational field (which has the form of De Donder's coordinate condition) is written in general-covariant form. This supplementary condition appears to be of physical importance because of the use made of it by Gupta in the quantization of Einstein's gravitational field. This physical significance of the supplementary condition singles out a manifold of coordinate systems, which contains as sub-manifolds infinitely many metric spaces each allowing only "Lorentz transformations" leaving the $\gamma_{\mu\nu}$ constant. A particle is called "at rest" if it is not accelerated with respect to some of these "Lorentz" frames. Although the $\gamma_{\mu\nu}$-metric may be important in the formulation of the quantum theory of gravitons, it does not enter in the line element describing the results of physical measurements of time or distance, which are described by a line element containing as metric the gravitational tensor $g_{\mu\nu}$, so that space, flat with respect to hypothetical measurements by unrealistic rods keeping their γ-metric length on displacement, is actually found to be curved by physical measurements by realistic rods keeping their g-metric length on parallel displacement.

The static spherically symmetric gravitational field $g_{\mu\nu}$ in empty space around a singularity "at rest" is obtained in terms of conventional polar coordinates in its "Lorentz" rest system, in a form satisfying the supplementary condition. A simple relation is established between this new solution and the Schwarzschild solution for this static central field. The radial coordinate ρ used in the Schwarzschild solution, which is a convenient variable in the discussion of planetary motion, differs by a constant from the polar coordinate r in the "Lorentz" frame in which the point source of the field is at rest. Neither r nor ρ is equal to the radial distance R measured from the point source. If a picture of space is made on the γ scale, then space has holes where masses are located. ("Swiss-cheese" model of space.) This fact may be helpful in eliminating divergencies of field theory.

1. COVARIANT FORM OF THE SUPPLEMENTARY CONDITION

In his papers on the quantization of the gravitational field, Gupta[1–3] has stressed the importance of imposing the supplementary condition

$$\partial_\nu \mathfrak{g}^{\mu\nu} = 0 \tag{1}$$

on the contravariant gravitational tensor density.[4] Here, $\partial_\nu \equiv \partial/\partial x^\nu$, and

$$\mathfrak{g}^{\mu\nu} = g^{\mu\nu}(-g)^{\frac{1}{2}}; \quad g = \mathrm{Det}(g_{\alpha\nu}) = \mathrm{Det}(\mathfrak{g}^{\mu\nu}). \tag{2}$$

Since the Schwarzschild solution for the static field around a point mass at rest does not satisfy the condition (1), it is of interest to investigate in what way the conventional expression for the gravitational field around such a point source is to be modified in order to satisfy the supplementary condition imposed by Gupta.

This condition (1) in form is identical with the coordinate condition suggested by De Donder,[5] and used by Papapetrou[6] in deriving a new form for the gravitational equation. Its importance for creating a similarity between the formulas of the theories of gravitation and of electromagnetism was first suggested by Gupta.[2–3]

As the condition (1) apparently is not generally covariant, it singles out a category of coordinate systems in which it is valid. In one of these coordinate systems,[7] we introduce a symmetric tensor $\gamma_{\mu\nu}$, which in this chosen coordinate system shall have the components

$$\gamma_{00} = 1, \quad \gamma_{\mu k} = \delta_\mu{}^k, \quad (\mu = 0,1,2,3; \quad k = 1,2,3). \tag{3}$$

We call this tensor the γ metric or "flat-space metric." We also follow Rosen[8] in defining in every (arbitrary) coordinate system the "flat-space Christoffel symbols"[9]

$$\left.\begin{aligned} \boldsymbol{\Gamma}^\lambda{}_{\mu\nu} &= \boldsymbol{\gamma}^{\lambda\alpha}\Gamma_{\alpha,\mu\nu}, \\ \Gamma_{\alpha,\mu\nu} &= \tfrac{1}{2}(\partial_\mu\gamma_{\nu\alpha} + \partial_\nu\gamma_{\mu\alpha} - \partial_\alpha\gamma_{\mu\nu}), \end{aligned}\right\} \tag{4}$$

where $\boldsymbol{\gamma}^{\lambda\alpha}$ is defined by

$$\boldsymbol{\gamma}^{\lambda\alpha}\gamma_{\alpha\mu} = \delta_\mu{}^\lambda. \tag{5}$$

Then, the quantities

$$\mathfrak{g}^{\lambda\mu}{}_{|\nu} = \partial_\nu \mathfrak{g}^{\lambda\mu} + \boldsymbol{\Gamma}^\lambda{}_{\alpha\nu}\mathfrak{g}^{\alpha\mu} + \boldsymbol{\Gamma}^\mu{}_{\alpha\nu}\mathfrak{g}^{\lambda\alpha} - \boldsymbol{\Gamma}^\alpha{}_{\alpha\nu}\mathfrak{g}^{\lambda\mu} \tag{6}$$

will form a tensor under general coordinate transformations. This "covariant γ differentiation" of the tensor density $\mathfrak{g}^{\lambda\mu}$ has the same form as the usual covariant derivative of such a tensor density, except for the use

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[1] S. N. Gupta, Proc. Phys. Soc. (London) **A65**, 161 (1952).
[2] S. N. Gupta, Proc. Phys. Soc. (London) **A65**, 608 (1952).
[3] S. N. Gupta, Phys. Rev. **96**, 1683 (1954). I thank Professor Gupta for letting me read this paper before its publication.
[4] Different methods of quantizing the gravitational field, in which no such condition need to be imposed, were developed by Bergmann and collaborators, and by Pirani and Schild. However, since in those papers an interpretation of the commutation relations or an expression for the quantized fields in terms of annihilation and creation operators is not given, an application of those quantum theories of gravitation as yet is not directly possible. See P. G. Bergmann and J. H. M. Brunings, Revs. Modern Phys. **21**, 480 (1949); F. A. E. Pirani and A. Schild, Phys. Rev. **79**, 968 (1950); **87**, 452 (1952); P. G. Bergmann *et al.*, Phys. Rev. **80**, 81 (1950).
[5] T. De Donder, *La Gravifique Einsteinienne* (Gauthier-Villars, Paris, 1921).
[6] A. Papapetrou, Proc. Roy. Irish Acad. **A52**, 11 (1948).
[7] The arbitrariness introduced by this choice of coordinate system is discussed in the next section.
[8] N. Rosen, Phys. Rev. **57**, 147 (1940).
[9] We write $\boldsymbol{\gamma}^{\lambda\alpha}$, $\boldsymbol{\Gamma}^\lambda{}_{\mu\nu}$, etc., in boldface because we use $g^{\mu\nu}$ for raising indices without change in print.

of the flat-space Christoffel symbols $\mathbf{\Gamma}^\lambda{}_{\mu\nu}$ instead of the gravitational (curved-space) Christoffel symbols $\left\{\begin{matrix}\lambda\\ \mu\nu\end{matrix}\right\}$ defined by

$$\left\{\begin{matrix}\lambda\\ \mu\nu\end{matrix}\right\} = g^{\lambda\alpha}[\mu\nu,\alpha],$$
$$[\mu\nu,\alpha] = \tfrac{1}{2}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}), \quad (7)$$

where

$$g^{\lambda\alpha}g_{\alpha\mu} = \delta_\mu{}^\lambda. \quad (8)$$

We form the covariant γ-divergence

$$\mathfrak{g}^{\lambda\nu}{}_{|\nu} = \partial_\nu \mathfrak{g}^{\lambda\nu} + \mathfrak{g}^{\alpha\beta}\mathbf{\Gamma}^\lambda{}_{\alpha\beta}. \quad (9)$$

In the special frame of reference in which $\gamma_{\mu\nu}$ was first introduced, the $\mathbf{\Gamma}^\lambda{}_{\alpha\beta}$ by (4) with (3) obviously vanish. Because in this same coordinate system also (1) was assumed to be valid, we obtain for Gupta's supplementary condition Papapetrou's covariant expression[10]

$$\mathfrak{g}^{\lambda\nu}{}_{|\nu} = 0. \quad (10)$$

The category of coordinate systems in which (1) is valid therefore are *those for which the last term in* (9) *vanishes*. Apparently this term does not vanish in polar coordinate systems. Therefore, we shall have to use (10), and not (1), when we want to apply the supplementary condition in a polar coordinate system as usually is used in expressing the gravitational field around a point mass.

2. GAUGE MANIFOLD OF THE THEORY OF GRAVITATION

In the preceding section we have defined the tensor $\gamma_{\mu\nu}$ by giving its components the special-relativistic values (3) in one selected frame of reference in which the gravitational tensor density $\mathfrak{g}^{\mu\nu}$ satisfied De Donder's coordinate condition (1). By writing the latter equation in the generally covariant form (9), all restrictions on the coordinate system could then be dropped, and Eq. (9) could be interpreted as a covariant supplementary condition imposed on the tensor density $\mathfrak{g}^{\mu\nu}$ in combination with the tensor field $\gamma_{\mu\nu}$ and quantities derived from the latter.

The special coordinate systems for which De Donder's coordinate condition is valid can then be characterized by the property that in these frames of reference

$$\mathfrak{g}^{\alpha\beta}\mathbf{\Gamma}^\lambda{}_{\alpha\beta} = 0. \quad (11)$$

Since the $\mathbf{\Gamma}^\lambda{}_{\alpha\beta}$ form no tensor, this equation is not covariant: while it is satisfied in the special frame of reference selected for defining $\gamma_{\mu\nu}$ by (3), it will not hold in general after an arbitrary coordinate transformation. It will, however, remain valid after application of any coordinate transformation from the group of transformations that leaves (11) invariant. We shall call this particular group of coordinate transformations the "coordinate gauge group." The collection of coordinate systems obtainable from our original especially selected one by means of coordinate gauge transformations we shall call the *gauge manifold*.[11]

There is a subgroup of the coordinate gauge group of considerable interest to us.[12] This is the "Lorentz" group which leaves $\gamma_{\mu\nu}$ in its special-relativistic form (3). Obviously this leaves the $\mathbf{\Gamma}^\lambda{}_{\alpha\beta} = 0$ and therefore is part of the group leaving (11) valid.

The coordinate gauge group is much wider than this Lorentz group. It allows transformation to a new (primed) frame of reference, in which the $\mathbf{\Gamma}^{\lambda'}{}_{\alpha'\beta'}$ need no longer vanish identically anymore, while yet Eq. (11), with primes, is valid at every point:

$$\mathfrak{g}^{\alpha'\beta'}\mathbf{\Gamma}^{\lambda'}{}_{\alpha'\beta'} = 0. \quad (11')$$

In such a coordinate system the components $\gamma_{\mu'\nu'}$ will no longer have the simple values (3). On the other hand, Eq. (1) with primes will still be valid, on account of the primed equations (10), (9), and (11). This shows that the flat-space metric tensor γ (with components $\gamma_{\mu\nu}$), and the "Lorentz manifold" obtained by our Lorentz group from our originally selected coordinate system, both were not unambiguously determined. We might have selected the primed frame of reference as the special one, in which a tensor γ' might have been defined in such a way that its components $\gamma'_{\mu'\nu'}$ would take the special-relativistic values (3). While this would make $\gamma'_{\mu'\nu'} = \gamma_{\mu\nu}$, it would also make $\gamma'_{\mu'\nu'} \neq \gamma_{\mu'\nu'}$, (thence, $\gamma'_{\mu\nu} \neq \gamma_{\mu\nu}$), so that the tensors γ and γ', looked upon as "geometric objects," are different from each other. The new tensor γ' will then determine a new Lorentz manifold, consisting of the primed coordinate system and of all other "Lorentz frames" obtainable from it by the Lorentz group that leaves the components of γ' invariant.

As the tensors γ' and γ were different, there must also be a difference between the new Lorentz manifold (for which the components of γ' take the special-relativistic values) and the original Lorentz manifold (for which the components of γ took the special-relativistic values). Yet, in all regards, the new Lorentz manifold is as good as the old one, and the new γ' is as good a selection of a flat-space metric as the old γ was.

This freedom in choice corresponds to what may be called an arbitrariness of "gauge" of the gravitational

[10] Reference 6, Eq. (29b).

[11] We use here the word "manifold" in the meaning of "collection of coordinate systems obtainable from each other by a certain transformation group." As the word "manifold" is often used with a different meaning, maybe somebody can invent a more appropriate name for such a collection of coordinate systems.

[12] Another subgroup of interest is the "affine" group, of linear transformations with arbitrary constant coefficients. It is the fact that the gravitational Lagrangian density [Eq. (2) of reference 3] transforms as a scalar density under arbitrary infinitesimal affine transformations, that leads to the important Tolman relation [R. C. Tolman, Phys. Rev. **35**, 875 (1930), Eq. (7), and footnote 10 of reference 6], on which the derivation of Papapetrou's gravitational equation [Eq. (9) of reference 3] is based. In the "affine manifold" obtained by the affine group from our initial frame of reference, $\mathbf{\Gamma}^\lambda{}_{\mu\nu}=0$. A gauge transformation of γ as discussed in the text following Eq. (11') will be accompanied by a change-over to a different affine submanifold of the same gauge manifold.

field. There are two different ways of looking at the transformation from one gauge to another.

The first point of view deals with the purely geometric aspect of this transformation, and has the added advantage of showing a certain degree of analogy to the gauge transformations in electromagnetic theory. We first note that the gravitational metric tensor g (with components $g_{\mu\nu}$) is a geometric object with a well-defined physical meaning, as in any given coordinate system its components can be determined uniquely by physical measurements. Similarly, the electromagnetic field tensor F (with components $F_{\mu\nu} = -F_{\nu\mu}$) is a geometric object with a well-defined physical meaning. Therefore, these geometric objects g and F cannot be tampered with in a gauge transformation. On the other hand, in gravitational as well as in electromagnetic theory we also introduce auxiliary geometric objects which are subject to a certain arbitrariness and therefore may be submitted to gauge transformations. They are the potential fourvector A in the electromagnetic case, and the flat-space metric γ in the gravitational case.

In the gravitational as in the electromagnetic case, therefore, we may look upon a gauge transformation as an alteration in an auxiliary geometric object. Whether one wants at the same time to perform a coordinate transformation is, from this geometric point of view, completely irrelevant: Just as one can change the gauge of the electromagnetic potential fourvector without altering the coordinate system, we can alter $\gamma_{\mu\nu}$ into $\gamma'_{\mu\nu}$ without introducing at the same time a primed coordinate system. The $\gamma'_{\mu\nu}$ must, of course, satisfy the same covariant differential equation (10) with (9) and (4) as the original $\gamma_{\mu\nu}$ did. And, before all, they must satisfy the second-order differential equations which state that the curvature tensor derived from the metric γ vanishes identically. These conditions, to some extent, are comparable to the condition, imposed in electromagnetic theory, that the four-dimensional divergence of the potential fourvector shall vanish also after a gauge transformation.

However, in the gravitational case there is no relation between the g and the γ, comparable to the relation between the field strengths and the potentials in electromagnetism; except for the relation (10), which is considerably different in form.

In the second way of looking at the gauge transformation of the theory of gravitation, one confines oneself, before and after the gauge transformation, to coordinate systems taken from the Lorentz manifold in which the flat-space metric γ takes the special-relativistic value (3). Then, contrary to the electromagnetic case, a change of gauge of the gravitational field necessitates a corresponding (non-Lorentz) coordinate transformation, from a frame of reference belonging to the Lorentz manifold with the metric γ in special-relativistic form, to some coordinate system belonging to the Lorentz manifold with γ' in special-relativistic form.[13] This coordinate transformation, of course, causes a corresponding transformation of the tensor components $g_{\mu\nu}$ into $g_{\mu'\nu'}$. On the other hand, it leaves the new components $\gamma'_{\mu'\nu'}$ of the new tensor γ' equal to the old components $\gamma_{\mu\nu}$ of the old tensor γ. From this point of view, where one is not much interested in "geometric objects" and more in components with respect to chosen coordinate systems, one therefore says that the $\gamma_{\mu\nu}$ matrix is left invariant, and that the gauge transformation essentially consists of a non-Lorentz coordinate transformation within the coordinate gauge group admitted by De Donder's coordinate condition, together with the corresponding tensor transformation of $g_{\mu\nu}$. This non-Lorentz coordinate gauge transformation, from the old Lorentz manifold to the new one, is for a given gauge transformation determined but for arbitrary Lorentz transformations which may precede or follow the coordinate gauge transformation.

We have been very explicit on this point in order to prevent a confusion of the tensor transformation of $g_{\mu\nu}$ with corresponding coordinate transformation within the coordinate gauge group, on the one hand, with the gauge transformation of the geometric object the flat-space metric γ on the other hand. The latter was defined in such a way that the combined effect, of the coordinate and tensor transformation stressed in the second point of view, and the gauge transformation of the geometric object γ stressed in the first point of view, is just to keep $\gamma'_{\mu'\nu'} = \gamma_{\mu\nu}$.

3. ASYMPTOTIC BEHAVIOR OF THE GRAVITATIONAL FIELD, AND THE RESTRICTED GAUGE MANIFOLD

There is one point which still needs clarification, and this concerns the asymptotic behavior of the metric tensor $g_{\mu\nu}$. In proofs of the conservation of (matter plus gravitational) energy, it usually is assumed that asymptotically (= for increasing spacelike distances) the curvature of space tends to zero, and then it is postulated that the coordinate system asymptotically shall become a Lorentz system with special-relativistic value (3) for the gravitational metric $g_{\mu\nu}$.[14] The total energy and momentum then become components of a "free" fourvector with respect to Lorentz transformations of that Lorentz frame. The purpose of postulating a Lorentz frame in the outer regions is for avoiding the existence, in those regions, of gravitational energy or momentum with respect to the frame of reference used, as the existence of such energy or momentum in the

[13] As the Lorentz manifold is determined entirely and merely by the geometric object γ, and not by the potential fourvector A, a similar restriction to a Lorentz manifold imposed in electromagnetic theory does not necessitate a modification of this Lorentz manifold after an electromagnetic gauge transformation, and therefore electromagnetic gauge transformations even from our second point of view need not be accompanied by a coordinate transformation.

[14] See, for instance, C. Møller, *The Theory of Relativity* (Clarendon Press, Oxford, 1952), pp. 339–340.

outer region would upset the proof of the conservation of total energy and momentum.

As we selected, for our original definition of $\gamma_{\mu\nu}$ by (3), a frame of reference in which the laws of nature supposedly took their simplest form, we may then assume that the components of $g_{\mu\nu}$ just in that frame of reference asymptotically will take the values (3) wanted for the validity of the conservation laws for total energy and momentum. That is, in this particular frame of reference, $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ asymptotically become equal. As they both are tensors, this means that the geometric objects the tensors g and γ asymptotically become identical.

This desirability of keeping the metric $g_{\mu\nu}$ at spacelike infinity in the special-relativistic form (3) imposes an additional constraint on the allowable coordinate transformations, in addition to the postulate that the transformation shall belong to the coordinate gauge group leaving (11) invariant. The more restricted group of transformations, which throughout space leaves (11) invariant, but which asymptotically become Lorentz transformations leaving the values of the components of the metrics $\gamma_{\mu\nu}$ and $g_{\mu\nu}$ invariant, in the region at spacelike infinity where these two metrics become identical, we shall call the "restricted coordinate gauge group." The allowable coordinate systems obtainable by this group of transformations make up the "restricted gauge manifold."

Now, if in a new (primed) coordinate system obtained by such a restricted coordinate gauge transformation we replace the geometric object γ (components $\gamma_{\mu'\nu'}$) by a new geometric object γ' (components $\gamma'_{\mu'\nu'}$) in such a way that $\gamma'_{\mu'\nu'} = \gamma_{\mu\nu}$ is given by (3), then it is true that the tensors γ' and γ differ at finite distances. Asymptotically, however, the primed coordinate system differs from the unprimed one by a Lorentz transformation only, so that asymptotically the $\gamma_{\mu'\nu'}$ components of the original tensor γ were still equal to the old special-relativistic values $\gamma_{\mu\nu}$, and therefore the change to the new $\gamma'_{\mu'\nu'}$ was no change at all, or, asymptotically, $\gamma' = \gamma$. In other words, on account of this restriction imposed on the coordinate gauge group, all gauge transformations of the geometric object γ will leave this object unaltered at spacelike infinity, and not only γ but automatically also γ' will asymptotically be equal to the tensor g. Thus, gauge transformations of the gravitational field are restricted to those which vanish at spacelike infinity. If our second point of view is taken, this manifests itself in the fact that the components of the tensor $g_{\mu\nu}$, after as well as before the tensor transformation accompanying the restricted coordinate gauge transformation, asymptotically keep the special-relativistic values without change.

4. POINT PARTICLES AT REST

Although general relativity suggests complete equivalence of all coordinate systems, the importance of using the supplementary condition (1) in attempts to keep the quantum theory of Einstein's gravitational field reasonably simple suggests that the category of frames of reference in which the condition (1) is satisfied is singled out by the simplicity of form of certain laws of nature in such coordinate systems.[15] We have seen that this category of more or less fundamental or "absolute" frames of reference constitutes what we have called the gauge manifold, and that there were also reasons to restrict it further to what we have called the restricted gauge manifold, in which a special-relativistic flat-space metric given by (3) was determined by the asymptotic behavior of the gravitational metric $g_{\mu\nu}$. Further we found it expedient to restrict our choice of coordinate system even more, to one out of an infinite number of Lorentz manifolds, which are submanifolds of the restricted gauge manifold, and which can be obtained from each other by restricted coordinate gauge transformations, with a simultaneous gauge transformation of the geometric object the flat-space metric γ, which determines each Lorentz manifold by its special-relativistic form (3). The main advantage of confining oneself to one of such Lorentz manifolds is that it enables one to use the choice (3) of the γ tensor throughout space.

From a general-relativistic point of view, there is no such a thing as a "particle at rest." From a special-relativistic point of view, a particle can be called "at rest," at least with respect to a properly chosen Lorentz frame, if, and only if it is not accelerated with respect to any Lorentz frame.

The above considerations suggest an intermediate point of view. In gravitational theory, we shall call a particle "at rest" (with respect to a properly chosen frame of reference, of course) if within the restricted gauge manifold there is a Lorentz submanifold with

[15] N. Rosen, Phys. Rev. **57**, 150 and 154 (1940), suggests that the γ metric could perhaps be used in measuring the velocity of light. Such a use of $\gamma_{\mu\nu}$ is not clear to me, and seems rather doubtful, as Rosen himself assumes the $g_{\mu\nu}$ to govern physical motion of objects, and therefore also of clocks and measuring rods. As measurement means a comparison of an object and a measuring rod, and a measuring rod compared to itself always has a length 1, a measuring rod *a fortiori* keeps its length in physical parallel displacement, and so does $ds \equiv (g_{\mu\nu}dx^\mu dx^\nu)^{\frac{1}{2}}$, but not $d\sigma \equiv (\gamma_{\mu\nu}dx^\mu dx_\nu)^{\frac{1}{2}}$. (See small letter on page 152 of Rosen's paper.) I cannot understand Rosen's alternative "point of view" that the length of the rod could change, unless he means by "length" something different from the result of a measurement by a rod of physical reality.—For a more accurate description of the relation between $g_{\mu\nu}$ and the results of physical measurements [see Eqs. (62)–(64) on page 238 of reference 14], and for an explanation of the necessity of identifying the metric $g_{\mu\nu}$ governing in this way the results of measurements, with the metric $g_{\mu\nu}$ governing [by Eq. (85) on page 244 of reference 14] the geodesic describing the paths of planets and light rays, the reader is urged to read chapter VIII of reference 14 in full. The basic hypothesis underlying these reasonings there is italicized on page 223.—The mere fact that the conditions (1) [or (11)] and $g_{\mu\nu} \rightarrow \gamma_{\mu\nu}$ (for $r \rightarrow \infty$), used for selecting the frame of reference in which $\gamma_{\mu\nu}$ takes the special-relativistic form (3), allow for gauge transformations, is a strong indication that the $\gamma_{\mu\nu}$ cannot have as much physical meaning as the metric determining the results of physical measurements.

respect to which the particle is not accelerated. It is then possible to think of the particle as fixed in the origin of some frame of reference, of which the flat-space metric has the special-relativistic value (3), while the gravitational metric in this frame of reference satisfies the supplementary condition (1).

5. FLAT-SPACE POLAR COORDINATE SYSTEMS

From some frame of reference with coordinates T, x, y, z, in which the $\gamma_{\mu\nu}$ components are still taking the special-relativistic values (3), we transform to polar coördinates T, r, θ, φ by

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta. \quad (12)$$

In the following, let us denote by x^0, x^1, x^2, x^3 these polar coordinates:

$$x^0 = T(=ct), \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \varphi. \quad (13)$$

One easily finds

$$\gamma_{00} = -1, \quad \gamma_{11} = 1, \quad \gamma_{22} = r^2, \quad \gamma_{33} = r^2\sin^2\theta; \quad (14)$$

$$\gamma^{\mu\mu} = 1/\gamma_{\mu\mu} \quad \text{(no sum!)}; \quad (15)$$

$$\left.\begin{aligned} &\partial_1\gamma_{22} = 2r, \quad \partial_1\gamma_{33} = 2r\sin^2\theta, \\ &\partial_2\gamma_{33} = 2r^2\sin\theta\cos\theta; \end{aligned}\right\}(16)$$

$$\left.\begin{aligned} -\Gamma_{1,22} = \Gamma_{2,12} = \Gamma_{2,21} = r, \\ -\Gamma_{1,33} = \Gamma_{3,13} = \Gamma_{3,31} = r\sin^2\theta, \\ -\Gamma_{2,33} = \Gamma_{3,23} = \Gamma_{3,32} = r^2\sin\theta\cos\theta; \end{aligned}\right\}(17)$$

$$\left.\begin{aligned} &\mathbf{\Gamma}^1{}_{22} = -r, \quad \mathbf{\Gamma}^1{}_{33} = -r\sin^2\theta, \\ &\qquad\qquad\qquad \mathbf{\Gamma}^2{}_{33} = -\sin\theta\cos\theta, \\ &\mathbf{\Gamma}^2{}_{12} = \mathbf{\Gamma}^2{}_{21} = \mathbf{\Gamma}^3{}_{13} = \mathbf{\Gamma}^3{}_{31} = r^{-1}, \\ &\mathbf{\Gamma}^3{}_{23} = \mathbf{\Gamma}^3{}_{32} = \cot\theta. \end{aligned}\right\}(18)$$

(Components not listed vanish.)

By the above transformation (12) to polar coordinates, we have introduced a curvilinear coordinate system not belonging to the gauge manifold. Therefore, in this frame of reference we must use the supplementary condition in Papapetrou's general-covariant form (10). With the above values of the flat-space Christoffel symbols $\mathbf{\Gamma}^\lambda{}_{\mu\nu}$, the four conditions (10) become

$$0 = \mathfrak{g}^{0\nu}{}_{|\nu} = \partial_\nu\mathfrak{g}^{0\nu}, \quad (19\text{-}0)$$

$$0 = \mathfrak{g}^{1\nu}{}_{|\nu} = \partial_\nu\mathfrak{g}^{1\nu} - r(\mathfrak{g}^{22} + \mathfrak{g}^{33}\sin^2\theta), \quad (19\text{-}1)$$

$$0 = \mathfrak{g}^{2\nu}{}_{|\nu} = \partial_\nu\mathfrak{g}^{2\nu} + 2r^{-1}\mathfrak{g}^{12} - \mathfrak{g}^{33}\sin\theta\cos\theta, \quad (19\text{-}2)$$

$$0 = \mathfrak{g}^{3\nu}{}_{|\nu} = \partial_\nu\mathfrak{g}^{3\nu} + 2r^{-1}\mathfrak{g}^{13} + 2\mathfrak{g}^{23}\cot\theta. \quad (19\text{-}3)$$

6. STATIC CENTRAL FIELD SATISFYING THE SUPPLEMENTARY CONDITION AROUND A SOURCE AT REST AT THE ORIGIN

We start by using the Lorentz frame, belonging to the restricted gauge manifold, in which the source of a gravitational field is at rest "at the origin." We then transform by (12) to polar coordinates. We ask for the spherically symmetric static gravitational field $g_{\mu\nu}$ surrounding the source at O.

For reasons of symmetry the expression for the square of the line element ds or time element $d\tau$ in this field will be given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = u^2dr^2 + v^2(d\theta^2 + \sin^2\theta d\varphi^2) - w^2dT^2 = -d\tau^2, \quad (20)$$

where u, v, and w are functions of r only. From (20), (13), and (2) we obtain

$$\left.\begin{aligned} &(-g)^{\frac{1}{2}} = uv^2w\sin\theta; \quad \mathfrak{g}^{00} = -uv^2w^{-1}\sin\theta, \\ &\mathfrak{g}^{11} = u^{-1}v^2w\sin\theta, \quad \mathfrak{g}^{22} = uw\sin\theta, \\ &\mathfrak{g}^{33} = uw(\sin\theta)^{-1}. \end{aligned}\right\}(21)$$

Inserting this in the Eqs. (19), we find that Eqs. (19-0), (19-2), and (19-3) are satisfied automatically, while the condition (19-1) takes the form

$$d(u^{-1}v^2w)/dr = 2ruw. \quad (22)$$

This condition is not satisfied by the Schwarzschild solution

$$ds^2 = -d\tau^2 = (1 - 2m/\rho)^{-1}d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2) - (1 - 2m/\rho)dT^2, \quad (23)$$

if we would interpret ρ here as the coordinate r occurring in (12) and (22), and then would find $\rho(\rho - 2m)$ for $(u^{-1}v^2w)$ and 1 for (uw) in Eq. (22). Therefore, the coordinate system for which the Schwarzschild solution is valid is not the coordinate system obtained by conventional transformation to polar coordinates from the Lorentz frame in which the source of the field is at rest, and in which the supplementary condition takes the simple form (1).

7. NEW SOLUTION FOR THE STATIC FIELD AROUND A PARTICLE AT REST

We want to find the static spherically symmetric field $g_{\mu\nu}$ corresponding to the line element given by (20), and satisfying the supplementary condition (10), that is, (22). This field, around a possible singularity in the center, must satisfy the gravitational equations for empty space, that is,

$$0 = R_{\mu\nu} = \tfrac{1}{2}\partial_\mu\partial_\nu\ln(-g) - \partial_\alpha\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix} + \begin{Bmatrix}\alpha\\ \mu\beta\end{Bmatrix}\begin{Bmatrix}\beta\\ \nu\alpha\end{Bmatrix} - \tfrac{1}{2}\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}\partial_\alpha\ln(-g). \quad (24)$$

From (20) with (13) and from (7)–(8) we find, if primes indicate differentiations with respect to r:

$$g_{00}=-w^2,\quad g_{11}=u^2,\quad g_{22}=v^2,\quad g_{33}=v^2\sin^2\theta;\quad (25)$$

$$\left.\begin{aligned}
&\begin{Bmatrix}0\\10\end{Bmatrix}=\begin{Bmatrix}0\\01\end{Bmatrix}=\frac{w'}{w},\quad \begin{Bmatrix}1\\00\end{Bmatrix}=\frac{ww'}{u^2},\quad \begin{Bmatrix}1\\11\end{Bmatrix}=\frac{u'}{u},\\
&\begin{Bmatrix}1\\22\end{Bmatrix}=-\frac{vv'}{u^2},\quad \begin{Bmatrix}1\\33\end{Bmatrix}=-\frac{vv'\sin^2\theta}{u^2},\\
&\begin{Bmatrix}2\\33\end{Bmatrix}=-\sin\theta\cos\theta,\\
&\begin{Bmatrix}2\\12\end{Bmatrix}=\begin{Bmatrix}2\\21\end{Bmatrix}=\begin{Bmatrix}3\\13\end{Bmatrix}=\begin{Bmatrix}3\\31\end{Bmatrix}=\frac{v'}{v},\\
&\begin{Bmatrix}3\\23\end{Bmatrix}=\begin{Bmatrix}3\\32\end{Bmatrix}=\cot\theta.
\end{aligned}\right\}\quad (26)$$

Thence, the Eqs. (24) take the form

$$0=\frac{u^2}{w^2}R_{00}=-\frac{w''}{w}+\frac{u'w'}{uw}-2\frac{v'w'}{vw},\qquad (27\text{-}0)$$

$$0=\quad R_{11}=\quad \frac{w''}{w}-\frac{u'w'}{uw}-2\frac{u'v'}{uv}+2\frac{v''}{v},\qquad (27\text{-}1)$$

$$0=\frac{u^2}{v^2}R_{22}=\frac{u^2R_{33}}{v^2\sin^2\theta}=-\frac{u^2}{v^2}+\frac{v'w'}{vw}-\frac{u'v'}{uv}+\frac{v''}{v}+\frac{v'^2}{v^2}.\qquad (27\text{-}2)$$

As shown in Appendix A, these equations are solved by

$$u^2=v'^2v/(v-2m),\quad w^2=K^2(v-2m)/v.\qquad (28\text{a-b})$$

Here, v may still be an arbitrary function of r; $v'=dv/dr$; m and K are undetermined constants.

We now insert (28) in our condition (22). This yields

$$d[v(v-2m)/v']=2rdv.\qquad (29)$$

As shown in appendix B, the general solution of this equation is

$$r=C_1\left[1+\frac{v-m}{2m}\ln\left(\frac{v-2m}{v}\right)\right]+C_2(v-m).\qquad (30)$$

If we assume that measuring rods measure ds,[15] then radial distances from the position $v=2m$ are given, according to (20) with (28a), by

$$R=\int udr=\int_{2m}^{v}[v/(v-2m)]^{\frac{1}{2}}dv$$
$$=[v(v-2m)]^{\frac{1}{2}}+m\ln[v^{\frac{1}{2}}+(v-2m)^{\frac{1}{2}}]-m\ln[v^{\frac{1}{2}}-(v-2m)^{\frac{1}{2}}].\qquad (31)$$

We postulated that $g_{\mu\nu}\rightarrow\gamma_{\mu\nu}$ for $r\rightarrow\infty$, so, asymptotically, we should find $v^2\rightarrow r^2$, thence $r\rightarrow v$. Expanding (30) for large r in powers of (m/v), we obtain

$$r\rightarrow C_1[-\tfrac{1}{2}(m/v)^2+\cdots]+C_2(v-m),\qquad (32)$$

so that the best we can do is putting

$$C_2=1.\qquad (33)$$

While the term with C_1 is unimportant for $r\rightarrow\infty$, it becomes predominant near the singularity at $v=2m$, where u^2, by (28a), would go through infinite and invert sign, so that R would become imaginary for $r<2m$. If ρ is the "effective" radius of a circle around the origin, defined as $(2\pi)^{-1}\times$ the circumference of such a circle, then, by (20),

$$\rho=v.\qquad (34)$$

Apparently, among the real circles around the origin, the one with smallest circumference is the one for which $v=2m$, for which

$$R=0,\quad \rho=2m,\quad (\text{"zero circle" around } O).\qquad (35)$$

If we want to keep $r=$ finite for this zero circle, we see from (30) that we must choose

$$C_1=0.\qquad (36)$$

Since for large values of r we want to find $g_{00}\rightarrow\gamma_{00}$, we should have $w^2\rightarrow 1$; therefore, by (28b),

$$K=1.\qquad (37)$$

Combining (28), (30), (33), (36), (37), and (20), we find

$$\left.\begin{aligned}v&=r+m,\\ u^2&=v/(v-2m)=(r+m)/(r-m),\\ w^2&=(v-2m)/v=(r-m)/(r+m);\end{aligned}\right\}(38)$$

$$d\tau^2=\left(\frac{r-m}{r+m}\right)dT^2-\left(\frac{r+m}{r-m}\right)dr^2-(r+m)^2(d\theta^2+\sin^2\theta d\varphi^2).\qquad (39)$$

8. "SWISS-CHEESE" MODEL OF SPACE

Now, imagine a picture of our space on the γ scale, that is, at $T=0$, drawing points P_1 and P_2 a distance $\int_{P_1}^{P_2}d\sigma$ apart, although their actual distance is $\int_{P_1}^{P_2}ds$. (See footnote 15.) On the γ scale, our space looks flat and is an ordinary Euclidean xyz-space, with polar coordinates $r\theta\varphi$. (This flatness is, of course, due to the distortion introduced by not drawing on the g-scale with metric $g_{\mu\nu}$.) Now, we have seen that, in our actual space, points with $v<2m$ (so by (38) with $r<m$) should be excluded as they would lie at imaginary distance from the zero circle. In our flat picture of space, this means that there is a spherical hole in space, or a cylindrical hole in four-dimensional space-time ("Swiss-cheese" model of space). On the γ scale, this hole has a radius m and a circumference $2\pi m$. Actually

(on the g-scale), it has a radius $R=0$, but a circumference $2\pi v = 4\pi m = 4\pi GM_0/c^2$. [Compare Eq. (49) of Sec. 10.] For instance, for a neutron, this circumference would measure 1.55×10^{-51} cm. Because the circumference consists of many points, it is not completely correct to call the source of our static field a point singularity, and it is better to call it a singularity *at* the origin, than to call it a singularity *in* the origin.

Papapetrou[6] has found that, on the γ scale, the gravitational equation on account of (1) may be written in the form

$$\square \mathfrak{g}^{\mu\nu} = (16\pi G/c^4)\mathfrak{T}^{\mu\nu}, \tag{40}$$

where

$$\square = \gamma^{\mu\nu}\partial_\mu\partial_\nu, \tag{41}$$

while the symmetric total energy density $\mathfrak{T}^{\mu\nu}$ includes the spin as well as orbital energy density, of the gravitational field as well as of matter. Gupta has stressed the importance of this fact in the quantum theory of gravitation.[1-3] The result (40)–(41) indeed suggests use of the γ scale in defining the rules of quantization of the field. This, however, seems to imply that in momentum representation of fields we should expand fields in terms of "plane waves" $\exp(ik_\mu x^\mu)$, where the coordinates $x^\mu(=\int d\sigma)$ form a fourvector with respect to the Lorentz group determined by the tensor $\gamma_{\mu\nu}$. We now have found that this Lorentz space has holes where matter is located. Consequently, there is a natural cut-off radius for the fields around such a hole, which will help to suppress high-momentum states in the expansions of such fields. To what extent this will provide a means of eliminating the divergencies encountered in the quantum theory of fields is a matter which deserves careful investigation.

9. RELATION OF OUR SOLUTION TO SCHWARZSCHILD'S SOLUTION

If, instead of the polar coordinate

$$r = (x^2+y^2+z^2)^{\frac{1}{2}}$$

introduced in the transformation (12), we introduce the effective radius ρ of Eq. (34) as our radial parameter, then (38) gives

$$\rho = v = r + m, \quad dr = d\rho, \tag{42a-b}$$

$$\left.\begin{aligned} u^2 &= \rho/(\rho-2m) = (1-2m/\rho)^{-1}, \\ w^2 &= 1-2m/\rho. \end{aligned}\right\} \tag{43}$$

Inserting this in (20), we obtain the Schwarzschild solution (23). On account of (42b), the transformation (42a) has not changed at all the values of $g_{\mu\nu}$ (of u, v, and w) in transforming from our to Schwarzschild's coordinates. Schwarzschild's radial coordinate (the effective radius ρ) is simply larger by the constant m than our polar coordinate r obtained by (12) starting from our Lorentz frame of reference.

By (42), the supplementary condition (22) takes the form

$$d(u^{-1}v^2w)/d\rho = 2(\rho-m)uw, \tag{44}$$

which evidently is satisfied by (42)–(43).

10. MOTION OF A PLANET IN OUR STATIC CENTRAL FIELD

The variable ρ is not only the effective radius used in calculating the circumference of a circle around the origin. To its importance adds its usefulness in the discussion of the motion of a "planet" around this origin. This motion is described in the well-known way by the geodesic determined by

$$d^2x^\mu/d\tau^2 + \begin{Bmatrix} \mu \\ \alpha\beta \end{Bmatrix}(dx^\alpha/d\tau)(dx^\beta/d\tau) = 0. \tag{45}$$

The reader will remember[16] that one integral of motion is given by the angular momentum of this planet,

$$L = Mc\rho^2 d\varphi/d\tau, \tag{46}$$

where $M=$ mass of planet, and where we have assumed that the plane of the motion is given by $\theta = \pi/2$. Another integral of motion is the energy, which is given by

$$E = -Mc^2 g_{00}dx^0/d\tau = Mc^2(1-2m/\rho)(dT/d\tau). \tag{47}$$

Notice that here we have to use ρ, and not r or R. Also, Binet's method for obtaining the orbit of the planet can be used if one introduces ρ^{-1} (not r^{-1}) as a new variable. In first approximation one thus finds Kepler's elliptic orbits corresponding to a Newtonian central field of force with potential energy

$$V = -GMM_o/\rho, \tag{48}$$

where the "mass of the sun" M_o is related to our constant m by

$$m = GM_o/c^2. \tag{49}$$

In second approximation one obtains

$$\rho^{-1} = A(1-e\cos\Gamma\varphi), \tag{50}$$

where $A = a^{-1}(1-e^2)^{-1}$, if $a=$ half the major axis and $e=$ eccentricity of the approximately elliptic orbit, of which the advance of the perihelion per period is approximately given by Einstein's

$$\eta \approx 2\pi(1-\Gamma) \approx 6\pi GM_o[ac^2(1-e^2)]^{-1}. \tag{51}$$

11. DISCUSSION

We have found that the coordinate ρ appearing in the Schwarzschild line element (23) remains a most convenient radial parameter in discussing such problems as planetary motion, and that by the simple relation (42) it is related to the polar coordinate r obtained by (12) from the "flat" Lorentz frame $xyzT$

[16] See, for instance, P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, Inc., New York, 1947), pp. 212–217.

in which the source singularity of the field is at rest. By "flat" we mean, of course, "γ flat," that is, the vanishing of the curvature which would follow from a metric $\gamma_{\mu\nu}$. Judged after the identification of the planetary motion (45) of particles in the $g_{\mu\nu}$-field with geodesics, as well as from the results of physical measurements made in the $g_{\mu\nu}$-field (see footnote 15), spacetime notwithstanding the flat metric $\gamma_{\mu\nu}$ has the curved aspect described by the gravitational metric tensor $g_{\mu\nu}$. In this regard, the flat "picture" of space may be called distorted. However, this does not take away the usefulness of this flat picture in visualizing space, as a help in calculations, and as a basis for a simple quantum theory of fields. In particular, we have noted that in the presence of electrically neutral[17] point particles (or, rather, "bubble particles") there may be "holes" in this flat picture of space, that is, there may be real values of the orthogonal coordinates xyz in our "Lorentz frame," to which no "real" points correspond. (The "reality" of a point is here judged after the reality of its $g_{\mu\nu}$-measured spacelike distance from other points.)

In discussing the effect of a gravitational field $g_{\mu\nu}$ on the motion of a planetary particle, we have not considered the effect of this particle itself on the gravitational field; that is, we have omitted gravitational self-interaction. This may seem particulary serious as the gravitational fields from various matter sources are not really additive, due to the nonlinearity of Einstein's gravitational equations. Gupta circumvents the latter difficulty by considering the nonlinearity in the field equations as the action of the gravitational energy (including spin energy) as another source of the gravitational field $\mathfrak{g}^{\mu\nu}$ satisfying Papapetrou's otherwise linear wave equation (40). This makes it possible in principle to solve in successive approximations for the gravitational field from given matter sources.[2–3]

APPENDIX A

Adding Eqs. (27-0) and (27-1), and multiplying by $(-v/2v')$, we find

$$d[\ln(uw/v')]/dr = 0; \quad \text{thence,} \quad uw = Kv', \tag{A.1}$$

where K is some constant. Multiplying Eq. (27-2) by v/v' we find

$$d[\ln(wvv'/u)]/dr = u^2/vv'. \tag{A.2}$$

Multiplying this by wvv'/u, and using (A.1) in the right member, we obtain

$$\frac{d}{dr}\left(\frac{wvv'}{u}\right) = \left(\frac{wvv'}{u}\right)\frac{d}{dr}\ln\left(\frac{wvv'}{u}\right) = wu = K\frac{dv}{dr},$$

thence,

$$wvv'/u = K(v-2m), \tag{A.3}$$

where m is some other constant. Again using Eq. (A.1), we find from (A.3)

$$u^2(v-2m) = K^{-1}uwvv' = vv'^2. \tag{A.4}$$

This yields Eq. (28a). Squaring (A.1) and using (A.4) we obtain

$$u^2w^2 = K^2v'^2 = K^2u^2(v-2m)/v, \tag{A.5}$$

which gives Eq. (28b).

Finally, Eq. (27-0), multiplied by $(-w/w')$, yields

$$d[\ln(u^{-1}v^2w')]/dr = 0, \quad \text{or} \quad w' \propto u/v^2. \tag{A.6}$$

This equation does not impose any restriction on v. In fact, the derivative of the square root of Eq. (28b) yields already

$$w' = K\frac{d}{dr}\left[\frac{v-2m}{v}\right]^{\frac{1}{2}} = \frac{mKv'}{v^2}\left(\frac{v}{v-2m}\right)^{\frac{1}{2}}, \tag{A.7}$$

or, by Eq. (28a),

$$w' = mKu/v^2, \tag{A.8}$$

so that (A.6) is automatically fulfilled as a consequence of (28a–b).

APPENDIX B

Introduce new variables x and y by

$$v = m(x+1), \quad r = xy, \tag{B.1}$$

and take x as the independent variable, so that

$$\frac{d(x+1)}{dr} = \left(\frac{dr}{dx}\right)^{-1}\frac{d(x+1)}{dx} = \left[y + x\frac{dy}{dx}\right]^{-1}. \tag{B.2}$$

This changes Eq. (29) into

$$d[(x^2-1)(y+xdy/dx)] = 2xydx, \tag{B.3}$$

or, if we put $dy/dx = p$,

$$(4x^2-2)p + x(x^2-1)dp/dx = 0. \tag{B.4}$$

Integration yields

$$p = C_1x^{-2}(x^2-1)^{-1} = dy/dx; \tag{B.5}$$

thence,

$$y = C_1\left[\frac{1}{x} + \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)\right] + C_2'. \tag{B.6}$$

By (B.1) this gives Eq. (30).

[17] Fields surrounding interacting particles may prevent the occurrence of holes in our flat picture of space.