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Classical Field Theory of Nuclear Forces*

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An attempt to explain nuclear properties with the help of a classical nuclear potential whose quanta are π mesons leads to involved nonlinear interactions. We therefore used a potential whose quanta are neutral scalar mesons. Saturation properties, empirical binding energies, and observed nuclear densities are obtained if the potential depends on the velocity of the nucleons. This velocity dependence accounts for a number of additional nuclear properties.

1. NUCLEAR POTENTIAL

THE nuclear shell model[1] explains certain nuclear properties in terms of the independent motion of nucleons in an appropriately chosen potential. This theory differs significantly from the Hartree model in atomic physics. In the latter, a central body establishes a first crude approximation of the atomic potential, whereas, in the former, the nucleons themselves completely determine the nuclear potential. Bohr and Mottelson[2] have worked out the consequences of this difference; the agreement between their theory and experimental data provides further support for the idea of an average potential within the nucleus.

The energy difference between the last nucleon in a closed nuclear shell and the first nucleon in the next nuclear shell is approximately 3 Mev. Therefore, collisions between the independently moving nucleons should not disturb energy states by as much as 3 Mev; otherwise, the ordering of states predicted by the shell model would be altered. Gamma-ray absorption[3] indicates that the energy corresponding to the frequency of nucleons in their oribital motion is approximately 20 Mev. Thus, the average distance traversed by nucleons between collisions appears to exceed 20/3 nuclear radii, a conclusion that cannot be easily reconciled with the cross section for collisions between free nucleons.[4]

The study of the two-body problem in atomic physics led to an understanding of complicated atoms in terms of simple two-body potentials describing electrostatic forces. The corresponding study in nuclear physics has not been so successful. Nuclear interactions, in contrast to atomic interactions, are strong, which has the consequence that at high energies, the multiple production of nuclear quanta (mesons) is the rule, whereas the multiple production of electromagnetic quanta is a rare event. Consequently, in nucleon-nucleon collisions, several mesons may be expected in virtual states and the description of a nucleus in terms of additive two-body interactions may well be impossible.

On the other hand, the simple regularities exhibited by heavy nuclei encourage the belief that the many-body problem of nuclear physics might be treated more easily than the two-body problem. For a heavy nucleus in which the expectation value for the number of mesons present is considerably greater than one, the mesons obeying Bose's statistics will tend to occupy the same quantum state. The wave function of this quantum state will correspond to a classical potential of nuclear forces. The greater the number of mesons present, the closer the approach to the behavior of a classical field. Thus, strong interactions may validate the classical limit and thereby simplify the theory in the many-body case.

The empirical facts of the shell model show that the potential—and therefore the meson wave function—has the following properties. It is constant inside the nucleus. It does not depend on the mass number A except for the lightest nuclei. It extends roughly over a sphere of $1.2A^{\frac{1}{3}}\times 10^{-13}$ cm radius.[5] It falls to zero within a distance of 1 or 2 times 10^{-13} cm.

This potential forms the basis for the remainder of our discussion. We shall disregard its connection with two-body interactions between free nucleons or between nucleons within the nucleus. We see at present no way in which a smooth potential can be derived from interactions between elementary particles. Having postulated the existence of a classical potential, the exploration of its properties leads to some surprising conclusions.

2. TYPE OF THE MESON FIELD

We shall assume an interaction of the nucleons with the meson field in the form

$$\sum_l \psi^* O_l \psi \Phi_l(\phi). \tag{1}$$

Here ψ is the nucleon wave function, O_l a linear operator, ϕ the amplitude of the meson field, and Φ_l an arbitrary function of ϕ.

Expression (1) is bilinear in ψ and ψ^*. Higher powers

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[1] M. G. Mayer, Phys. Rev. **75**, 1969 (1949); Haxel, Jensen, and Suess, Phys. Rev. **75**, 1766 (1949).
[2] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab. Mat.-fys. Medd. **27**, No. 16 (1953).
[3] R. Nathans and J. Halpern, Phys. Rev. **93**, 437 (1954).
[4] See, however, Brueckner, Levinson, and Mahmoud, Phys. Rev. **95**, 217 (1954); K. A. Brueckner, Phys. Rev. **96**, 508 (1954).
[5] V. L. Fitch and J. Rainwater, Phys. Rev. **92**, 789 (1953); A. N. Cooper and E. M. Henley, Phys. Rev. **92**, 801 (1953); Hofstadter, Fechter, and McIntyre, Phys. Rev. **92**, 978 (1953).

of the wave functions would imply that nucleons interact with the meson field only if they are in contact with each other. Such an interaction will not lead to independent motion of nucleons.

We shall show that Φ_l must be a simple scalar and an isotopic singlet; otherwise simple nuclear properties cannot be represented with a potential such as described above. The main purpose of the present section is to prove this proposition.

For the sake of simplicity, we shall test the interaction on closed-shell nuclei with equal numbers of protons and neutrons.

In the strict sense of the shell model a nuclear state is defined by filling definite orbits by nucleons which move in a classical potential. This means that the total wave function can be written as a product of two factors. One is the antisymmetrical orbital function of the nucleons and the other is the dependence of a probability amplitude on the meson occupation numbers. In the classical limit a single smooth spatial meson state is occupied and the probability amplitudes can be replaced by average occupation numbers.

A more general function is the sum of several such shell-model states. The terms will differ both in the orbits filled by nucleons and in the average occupation numbers of mesons. In this case the common spatial meson functions can still be called the classical nuclear potential.

The nuclear binding energy is the average value of (1) over the sum of products. It differs from the strict shell-model value by matrix elements of (1) connecting different shell-model states. Due to the smooth spatial dependence of the meson function and to the exclusion principle, only the orbits near the top of the momentum-distribution will contribute. Such terms cannot be proportional to A but rather to $A^{\frac{2}{3}}$. Therefore the average of (1) over shell-model states in the strict sense must account for the main part of nuclear binding.

The invariance of the Hamiltonian requires that (1) be a simple scalar. If Φ_l is a pseudoscalar the same is true of the factor multiplying it. A pseudoscalar expression $\psi^*O_l\psi$ gives a vanishing average over any pure shell-model state. Therefore a pseudoscalar Φ_l can be ruled out.

The same conclusion holds if Φ_l is a component of any vector or tensor. The factor $\psi^*O_l\psi$ must then be a similar component and will give a zero average over any closed-shell nucleus.

Therefore Φ_l must be a simple scalar.

The conservation of isotopic spin means that (1) is an isotopic singlet. Since ψ^* and ψ are isotopic doublets, their product is either a singlet or a triplet. If $\psi^*O_l\psi$ is a singlet Φ_l is also a singlet. Similarly, if $\psi^*O_l\psi$ is an isotopic triplet the same is true of Φ_l.

If $\psi^*O_l\psi$ is a triplet, it will have components in which the nucleon charges change. These do not contribute to average values over strict shell-model states. One component is not connected with a change of charge. It can be shown that the contributions of protons and neutrons to this term are equal and opposite and cancel when the same orbits are occupied by protons and neutrons.

Therefore Φ_l must be an isotopic singlet.

We have made calculations with a nonlinear interaction $\psi^*\psi\Phi(\phi)$, where Φ is an algebraic function of the isotopic singlet ϕ^2. We obtained saturation and proper binding by an appropriate choice of Φ. However, we found excessively larger nuclear surface energies which we might possibly avoid by introducing into the Hamiltonian a term depending on ϕ alone to a higher power than two. Such a term corresponds to a direct interaction between π-mesons. While we do not see that this series of assumptions is necessarily wrong, it seems preferable to keep simple linear interactions and to introduce Φ as the amplitude of some appropriate meson field.

3. LINEAR COUPLING

If Φ is considered as a meson-amplitude, we obtain the following simple picture. According to the above results, the meson is scalar and neutral. Furthermore, the interaction (1) is linear in the meson field.

The scalar neutral meson need not be an elementary particle in any sense of the word. It may be a virtual state composed of other mesons. It may be even a superposition of such virtual states. It may decay into π mesons so quickly that it cannot be observed. It may be related to mesons like a sound-quantum is related to electrons and nuclei. In any case, we assume that nuclear interactions follow in first approximation from a linear coupling with the meson field.

The simplest Hamiltonian valid for the interior of nuclei can be written in terms of a meson wave function and of a properly antisymmetrized product wave function constructed from single nucleon functions ψ_j:

$$H_1=\int\{\hbar^2(2m)^{-1}\sum_j |\nabla\psi_j|^2 +\mu^2c^4\phi^2-\hbar cg\phi\sum_j |\psi_j|^2\}d\tau. \quad (2a)$$

In the interaction term, the constant g has the dimensions of an electric charge. The nucleon mass and the meson mass are designated by m and μ. Since ϕ is an isotopic singlet, nucleons retain their charge and identical potentials act upon neutrons and protons.[6] The kinetic energy of the meson field, proportional to $(\nabla\phi)^2$, will be included later in a discussion of surface effects.

4. VELOCITY DEPENDENCE OF THE INTERACTION

The Hamiltonian H_1 has two shortcomings. It does not explain saturation and it predicts too large a neutron excess in heavy nuclei.

[6] This has the consequence that the protons occupy a smaller sphere than the neutrons [M. H. Johnson and E. Teller, Phys. Rev. 93, 357 (1954)].

Treating the nucleons as a degenerate Fermi gas, the first term in H_1 is proportional to $\rho^{5/3}$, where ρ is the number of nucleons per unit volume. When the integrand is minimized with respect to ϕ for fixed ρ, the last two terms give a negative contribution proportional to ρ^2. The nucleus therefore tends to high ρ values; in the absence of surface effects ρ becomes infinite.

The Coulomb energy which must be added to H_1 plays an essential part in fixing the proton to neutron ratio. If ρ is held fixed and the proton to neutron ratio is varied, the energy minimum occurs when the kinetic energy at the top of the proton Fermi distribution differs from the energy at the top of the neutron distribution by the Coulomb potential. This minimum occurs in heavy nuclei for too large a neutron excess.[7] Therefore "symmetry forces" were introduced[8] which stabilize nuclei with a small neutron excess. These forces are compatible with pairwise interactions between nucleons. However, a potential based on neutral mesons leaves no room for symmetry forces.

Both difficulties can be remedied by adding to H_1 a velocity-dependent term.[9] The Hamiltonian becomes

$$H_2=H_1+\int\{\hbar^3\mu^{-2}c^{-1}f\phi\sum_j|\nabla\psi_j|^2\}d\tau, \tag{2b}$$

where another coupling constant f has been introduced which again has the dimensions of an electric charge. The new term, proportional to the kinetic energy of the nucleons, is equivalent to an effective nucleon mass, m_{eff}, which depends on ϕ. Thus the Hamiltonian H_2 follows from H_1 if m is replaced by m_{eff},

$$m_{\text{eff}}/m=[1+2m\hbar f\phi\mu^{-2}c^{-1}]^{-1}. \tag{3}$$

The velocity-dependent term in H_2 is positive and increases more rapidly than ρ^2. The nuclear potential, therefore, has a minimum at a finite ρ value. In our present formulation, the minimum is not absolute; for very high ρ values arbitrarily low energies can be reached by making ϕ negative. The Hamiltonian H_2 can no longer be valid when ρ becomes high and the motion of the nucleons relativistic. At the actual densities prevailing in nuclei, we hope that H_2 is a good approximation.

The empirically known nuclear radii and the exclusion principle fix the nucleon momenta. For $m_{\text{eff}}<m$ nucleon velocities and kinetic energies will be increased. Therefore, a smaller neutron excess produces the difference in kinetic energies at the top of the Fermi distributions necessary to balance the Coulomb potential.

[7] W. G. McMillan, Phys. Rev. **92**, 210 (1953).

[8] E. Wigner, Phys. Rev. **51**, 947 (1937).

[9] A velocity dependence was introduced by K. A. Brueckner (unpublished) and he was led to similar values as those we find. Brueckner started from pairwise interactions with repulsive cores (see reference 4). His approach is therefore quite different but the consequences seem to be effectively the same.

5. THE COUPLING CONSTANTS

Various authors have summarized the binding energy of nuclei in semiempirical formulas[10] which separately itemize the Coulomb and surface energies. According to the same formulas, the binding energy of a nucleon in the nuclear fluid is 15.5 Mev. The last number, together with latest measurements[5] of nuclear density, 1.4×10^{38} cm^{-3}, suffice to determine the coupling constants f and g.

Treating the nucleons as degenerate Fermi gases with equal numbers of neutrons and protons, (2a) and (2b) become

$$H_2/A=1.805\hbar^2m^{-1}\rho^{\frac{2}{3}}+3.61\hbar^3\mu^{-2}c^{-1}\rho^{\frac{2}{3}}f\phi+\mu^2c^4\rho^{-1}\phi^2-\hbar cg\phi. \tag{4}$$

The explicit appearance of the meson mass may be removed by introducing the quantities

$$w=\mu c^2\phi, \tag{5a}$$

$$F=(\hbar\mu^{-1}c^{-1})^3f, \tag{5b}$$

$$G=\hbar\mu^{-1}c^{-1}g. \tag{5c}$$

$$H_2/A=1.805\hbar^2m^{-1}\rho^{\frac{2}{3}}+3.61\rho^{\frac{2}{3}}Fw+w^2\rho^{-1}-Gw. \tag{6}$$

The energy, (6), reaches a minimum when ρ and w satisfy the conditions

$$1.20\hbar^2m^{-1}\rho^{\frac{2}{3}}+2.41\rho^{\frac{2}{3}}Fw=w^2\rho^{-1}, \tag{7}$$

$$3.61\rho^{\frac{2}{3}}Fw=Gw-2w^2\rho^{-1}. \tag{8}$$

By eliminating w^2/ρ between (7) and (8), we find

$$2.41\hbar^2m^{-1}\rho^{\frac{2}{3}}+8.44\rho^{\frac{2}{3}}Fw=Gw, \tag{9}$$

$$H_2/A=0.60\hbar^2m^{-1}\rho^{\frac{2}{3}}-2.41\rho^{\frac{2}{3}}Fw. \tag{10}$$

We can now make use of the empirical results, $H_2/A=-15.5$ Mev and $\rho=1.4\times10^{38}$ cm^{-3}. We find from (10), (9), and (7) in succession

$$\rho^{\frac{2}{3}}Fw=9.22\text{ Mev}, \tag{11}$$

$$Gw=104.8\text{ Mev}, \tag{12}$$

$$w^2/\rho=35.7\text{ Mev}. \tag{13}$$

Finally from (13), (12), and (5c) and from (13), (11), and (5b), we have

$$g^2(\hbar c)^{-1}=2.97[\rho(\hbar\mu^{-1}c^{-1})^3]^{-\frac{2}{3}}, \tag{14}$$

$$f^2(\hbar c)^{-1}=0.0236[\rho(\hbar\mu^{-1}c^{-1})^3]^{-2}. \tag{15}$$

We shall later estimate from the surface energy that

$$\mu c^2\cong500\text{ Mev}. \tag{16}$$

Equations (14) and (15) then give

$$g^2(\hbar c)^{-1}\cong71, \tag{17}$$

$$f^2(\hbar c)^{-1}\cong311. \tag{18}$$

[10] N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939); E. Fermi, *Nuclear Physics* (University of Chicago Press, Chicago, 1950); A. E. S. Green, Phys. Rev. **95**, 1006 (1954).

We are, therefore, dealing with strong coupling.

6. COULOMB AND SURFACE ENERGIES

Simple electrostatics determines C, the Coulomb energy per nucleon.

$$C = 0.60e^2ZA^{-1}R^{-1}. \quad (19)$$

In (19), Z is the nuclear charge and R the nuclear radius. We have neglected the nonuniformity of the charge distribution and the correlation between proton positions produced by the exclusion principle. Eliminating R by means of the empirical density, 1.4×10^{38} cm^{-3}, (19) becomes

$$C = 0.73Z^2A^{-4/3} \text{ Mev}. \quad (20)$$

Green's semiempirical binding energy formula[10] gives for S, the surface energy per nucleon,

$$S = 18A^{-\frac{1}{3}} \text{ Mev}. \quad (21)$$

To minimize $H_2/A+S+C$, we must know how C and S depend on ρ and w for a nucleus with Z and A held fixed. It is then clear

$$C\sim\rho^{\frac{1}{3}}w^0. \quad (22)$$

We take

$$S\sim\rho^{-\frac{2}{3}}w^2. \quad (23)$$

In (23) we have assumed that the surface energy is proportional to the surface area, that the thickness of the surface layer does not change when ρ is varied, and that the main contribution to S is proportional to $(\nabla w)^2$, the kinetic energy of the meson field. None of these assumptions is accurately valid.

As $C+S$ is small compared to H_2/A, a first-order perturbation calculation suffices. We find the new value w^* of the meson field amplitude which now satisfies the minimum conditions

$$w^*/w = 1-(Gw)^{-1}[(5+0.83\xi)C+(5+6.63\xi)S] \times(1.66\xi-18\xi^{-1})^{-1}, \quad (24)$$

$$\xi = \hbar^{-2}m\rho^{-\frac{2}{3}}Gw. \quad (25)$$

We also find the new value ρ^* of the nucleon density produces insignificant changes. With numerical values of the previous section, we have

$$\xi = 9.34. \quad (26)$$

Taking as examples the light nucleus Ca^{40} and the heavy nucleus U^{238}, Eqs. (20), (21), (24), and (26) now give

$$Gw^*(Ca^{40}) = 77 \text{ Mev}, \quad (27)$$

$$Gw^*(U^{238}) = 83 \text{ Mev}. \quad (28)$$

The effective nuclear potential is nearly independent of A at a value of 80 Mev. This high value reflects the large kinetic energy which is a consequence of the small effective mass. In fact (3) may be written

$$m_{\text{eff}}/m = [1+2m\hbar^2Fw^*]^{-1} = 0.435. \quad (29)$$

A Hamiltonian capable of describing surface effects must contain the kinetic energy of the meson field. Such a Hamiltonian is

$$H_3 = H_2 + \int c^2\hbar^2|\nabla\phi|^2d\tau$$
$$= H_2 + \int (\hbar\mu^{-1}c^{-1})^2|\nabla w|^2d\tau. \quad (30)$$

The presence of $|\nabla w|^2$ in the integrand prevents w from dropping sharply to zero at the surface.

To estimate roughly the connection between surface energy and meson mass, we suppose that ϕ and ρ are constant in a sphere of radius R, that ρ drops suddenly to zero at the radius R, and that ϕ goes to zero with a constant slope in a layer of thickness $L(L\ll R)$. Then (30) gives

$$SA = (3L/R)[\tfrac{1}{3}+(\hbar\mu^{-1}c^{-1}L^{-1})^2]\int w^2d\tau. \quad (31)$$

For $L=\sqrt{3}\hbar\mu^{-1}c^{-1}$, the right side of (31) assumes its minimum value,

$$SA = (2L/R)Aw^2/\rho = 116\hbar\mu^{-1}c^{-1}\rho^{\frac{1}{3}}A^{\frac{2}{3}} \text{ Mev}. \quad (32)$$

In (32) the value of w^2/ρ from (13), reduced according to (24), has been used. Comparing (32) with (21) and using $\rho^{\frac{1}{3}} = 0.52\times10^{13}$ cm^{-1}, we obtain

$$\mu c^2 = (116/18)\hbar c\rho^{\frac{1}{3}} = 660 \text{ Mev}. \quad (33)$$

The termination of ρ on a sharply defined surface is not very realistic. The fact that ρ and ϕ will decrease together near the surface should reduce the surface energy for a given value of μ. To compensate for this, μ should be decreased. Therefore, 660 Mev is an upper limit which should be reduced to perhaps 500 Mev.

Equation (32) with μ set equal to the π meson mass (140 Mev), gives an excessively large surface energy. This conclusion has been verified by detailed numerical integrations.

According to (13), the total meson rest energy within a nucleus is $35.7A$ Mev. The actual value should be lower in the ratio $(w^*/w)^2$ which reduces $35.7A$ Mev to $21A$ Mev. The meson rest energy reaches the value 500 Mev at approximately $A=24$, that is magnesium. For $A<24$, the expectation number of mesons is less than one, for $A>24$, it is greater than one. Hence, for nuclei heavier than magnesium, classical ideas should begin to apply.

7. INFLUENCE OF THE EFFECTIVE MASS ON NUCLEAR PROPERTIES

The kinetic energy of a neutron at the top of the momentum distribution in a nucleus is given by

$$p_{\max}^2(2m)^{-1} = (9\pi/4)^{\frac{2}{3}}\hbar^2(2mR^2)^{-1}N^{\frac{2}{3}}, \quad (34)$$

where N is the number of neutrons in the nucleus. Equation (34) holds for protons with N replaced by Z. The difference between maximum kinetic energies is

10.7 Mev in U^{238} if the normal nucleon mass is used for m. The Coulomb potential, on the other hand, is 21 Mev. To reconcile this difference, m_{eff} must be 0.51 times the normal mass, which compares reasonably well with m_{eff} found in (29).

In addition to the proton-neutron ratio, a small effective mass can reveal itself in several other ways. The interaction terms containing g and f in the Hamiltonian cancel for a nucleon whose momentum is 1.86×10^{-14} g cm sec^{-1}. Such a nucleon with a kinetic energy of 65 Mev in a free state has the same momentum inside and outside the nucleus. Consequently, it will not be refracted by the nuclear field. Therefore, the potential scattering for nucleons of 65 Mev should vanish. Actually[11] the potential scattering decreases above 60 Mev and vanishes at considerably higher energies. Our theory predicts this effect at too low an energy. Furthermore, our theory indicates that at high nuclear energies the repulsive term proportional to f should cause a reappearance of the potential scattering which has not been observed.

It seems that the repulsive effects of the f term do not continue to increase proportionally to the momentum square as we have assumed. We have indeed remarked earlier that such a continued increase would lead to a collapse of the nucleus and we must assume that the quadratic term describes the momentum dependence only at low energies.

[11] J. deJuren and B. Moyer, Phys. Rev. **81**, 919 (1951); A. E. Taylor and E. Wood, Phil. Mag. **44**, 95 (1953).

The momentum distribution of nucleons in the nucleus is fixed by the nuclear radius and the exclusion principle. A small effective mass therefore means a high nucleon velocity and high frequencies for resonance processes. According to calculations,[12] the principal resonances for gamma-ray absorption should occur at $\hbar\omega$ values somewhat below 10 Mev. The observed frequencies are approximately twice the calculated ones. We expect that a value $m_{\text{eff}}/m=0.5$ will remove most of the disagreement between experiment and calculation.

It has been pointed out[13] that the Thomas precession will cause a spin-orbit coupling of the correct sign with nuclei. The precession, however, gave too small value for the spin-orbit coupling. The strength of the coupling is crudely $\hbar\omega v^2/c^2$, where ω is the orbital frequency of the nucleon. Both ω and v are proportional to m_{eff}^{-1} and therefore the Thomas precessions will suffice in our model to explain the spin-orbit coupling. In fact, $\hbar\omega$ seems to lie between 15 and 20 Mev and v^2/c^2 is approximately 0.2 so that the spin-orbit coupling should be almost 4 Mev.

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[12] E. D. Courant, Phys. Rev. **82**, 703 (1951).
[13] D. R. Inglis, Phys. Rev. **56**, 1175 (1939); Maria Goeppert Mayer, Phys. Rev. **78**, 16 (1950).