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Vacuum Polarization and Proton-Proton Scattering*

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An attempt is made to verify the presence of effects due to vacuum polarization in available experimental data on proton-proton scattering. In spite of the smallness of these effects and relatively large errors in the data, it appears that the data substantiates the predicted effects of vacuum polarization on the electrostatic interaction of two protons, particularly on the assumption of a Yukawa shape for the nuclear potential. By correcting the available data for these effects, new values are obtained for the zero-energy scattering length and effective range of the nuclear interaction between two protons. The results in the notation of Blatt and Jackson are: $-R/a=3.704$, $r_0=2.76\times10^{-13}$ cm.

INTRODUCTION

THE scattering cross section for protons on protons is usually calculated on the basis that the potential energy is given by a simple Coulomb potential e^2/r plus a short range specifically nuclear potential $V(r)$. It was recently pointed out by the authors[1] that as a result of the phenomenon of vacuum polarization, there exists a further contribution to the potential energy of the form

$$V_{vp}(r)=\frac{2\alpha e^2}{3\pi r}\int_1^\infty e^{-2\kappa\xi r}\left(1+\frac{1}{2\xi^2}\right)\frac{(\xi^2-1)^{\frac{1}{2}}}{\xi^2}d\xi, \qquad (1)$$

where $\alpha=1/137.04$ is the fine structure constant and $\kappa=mc/\hbar$ is the reciprocal Compton wavelength of the electron, which, since it is of much longer range than the specifically nuclear potential, can lead to an appreciable effect on the scattering at low energies. It is the purpose of the present paper to examine available experimental data on proton-proton scattering to see whether the specific effects of vacuum polarization are discernible and also to investigate the changes in the usual analysis of proton-proton scattering data to obtain information about nuclear forces brought about by recognition of the contribution of vacuum polarization. Our work will follow closely the analysis and notation of Jackson and Blatt[2] in their recent review article. We shall also make extensive use of the recent work of Breit[3] and his collaborators.

We shall restrict our attention to the effect of vacuum polarization on the S-wave phase shift. In the absence of vacuum polarization, the radial wave function $u(r)$ satisfied the equation:

$$d^2u/dr^2+k^2u-u/Rr+MVu/\hbar^2=0, \qquad (2)$$

where $k^2=ME/2\hbar^2$, $R=\hbar^2/Me^2$. (E is the energy in the laboratory system; M is the mass of the proton.) The solution of (2) which is regular at the origin can be normalized so that it is asymptotic to

$$u(r)\sim\sin(kr-\eta\ln 2kr+\sigma+\delta)/\sin\delta, \qquad (3)$$

where $\eta=1/2kR$, $\sigma=\arg\Gamma(1+i\eta)$ is the Coulomb phase shift, and δ is the nuclear phase shift. For a deep short range nuclear potential V, it has been shown that the function K defined by

$$K=C^2kR\cot\delta+h(\eta), \qquad (4)$$

with

$$C^2=2\pi\eta/(e^{2\pi\eta}-1),$$
$$h(\eta)=\mathrm{Re}\{\Gamma'(-i\eta)/\Gamma(-i\eta)\}-\ln\eta,$$

is nearly a linear function of k^2 (or E) over a considerable range of energies. It is therefore now common to employ this function (or one similar) in the analysis of proton-proton scattering data.

Now one may readily show that the addition of the small potential $V_{vp}(r)$ to $V(r)$ in Eq. (1) leads to a change in the function K which to first order (and this is sufficiently accurate for our purpose) is given by the formula:

$$\Delta K=\frac{MC^2R}{\hbar^2}\int_0^\infty V_{vp}(r)u^2(r)dr. \qquad (5)$$

Thus knowledge of the solution u of Eq. (1) enables us to calculate ΔK. Outside the range of the nuclear potential, where only the Coulomb potential acts, u will be a linear combination of the regular and irregular Coulomb functions $F(r)$ and $G(r)$, which, properly normalized, takes the form

$$u(r)=F(r)\cot\delta+G(r). \qquad (6)$$

Thus only a knowledge of the phase shift δ is required to evaluate that part of the integral (5) which lies

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[1] L. L. Foldy and E. Eriksen, Phys. Rev. **95**, 1048 (1954).
[2] J. D. Jackson and J. M. Blatt, Revs. Modern Phys. **22**, 77 (1950).
[3] Yovits, Smith, Hull, Bengston, and Breit, Phys. Rev. **85**, 540 (1952).

outside the range of the nuclear potential. Within this range, however, detailed knowledge of the shape of the nuclear potential is required in order to have knowledge of the precise form of $u(r)$ in this range. It is therefore convenient to break up the integral (5) into two parts

$$\Delta K = \Delta_1 K + \Delta_2 K, \tag{7}$$

$$\Delta_1 K = \frac{MC^2 R}{2}\int_0^{e^2/mc^2} V_{vp} u^2 dr, \tag{8}$$

$$\Delta_2 K = \frac{MC^2 R}{2}\int_{e^2/mc^2}^{\infty} V_{vp} u^2 dr, \tag{9}$$

where we have arbitrarily chosen to make the division at the classical radius of the electron $e^2/mc^2 = 2.818 \times 10^{-13}$ cm since this is of the order of the range of nuclear forces. Thus $\Delta_2 K$ can be evaluated accurately without any specific assumptions concerning the shape of the nuclear potential over the energy range which will be of interest to us (0–5 Mev). In this energy range we have determined the appropriate δ at each energy from the formula

$$K \simeq -R/a + \tfrac{1}{2} R r_0 k^2, \tag{10}$$

with

$$-R/a = 3.755, \quad r_0 = 2.65 \times 10^{-13}\ \text{cm}, \tag{11}$$

which is sufficiently accurate for our purposes. We have computed $\Delta_2 K$ by numerical integration of (5) using tabulated Coulomb wave functions over the energy range from 0 to 1 Mev. For energies from 1 to 5 Mev the numerical integration becomes tedious and hence some analytic approximations to the wave functions were used. The results are given in Table I.

Since the contribution $\Delta_1 K$ is the same as that arising from a small short range addition to the nuclear potential, it should be an approximately linear function of energy. To obtain a reasonably accurate estimate of its magnitude, we have taken as an approximate wave function $u(r)$ the form

$$u(r) = (1 - e^{-\beta r})[F(r)\cot\delta + G(r)], \tag{12}$$

where β was taken to be 1.20×10^{13} cm^{-1}. This value has been chosen so as to give the proper effective range for the nuclear potential if the Coulomb potential is neglected within the range of the nuclear force. This evaluation of $\Delta_1 K$ is probably sufficiently accurate for all present purposes. The resultant $\Delta K = \Delta_1 K + \Delta_2 K$ is tabulated in Table I and plotted in Fig. 1. It can be seen that $\Delta_1 K$ is approximately a linear function of energy as anticipated.

The most characteristic feature of the contribution ΔK is its strong deviation from a linear variation with energy especially at low energies, a direct consequence of the relatively long range of the vacuum polarization potential. Unfortunately, this curvature is most manifest at very low energies where there are no accurate experiments on proton-proton scattering. Nevertheless, we shall show that the available data still appears to substantiate the reality of this effect though the limited accuracy of present experiments precludes an unambiguous conclusion. It will be noted further that in the energy range from 0.2 to 5 Mev, ΔK is of the order of one percent of observed K values in this range. Thus we may expect corrections of the order of one percent to the properties of the nuclear potential (zero-energy scattering length and effective range) derived from proton-proton scattering data when proper account is taken of the vacuum polarization contribution.

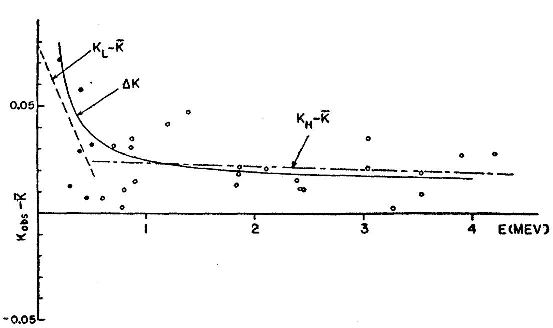

Fig. 1. Evidence for vacuum polarization contribution to proton-proton scattering. The line $K_L - \bar{K}$ represents the best least-squares fit to the low-energy data (full circles); the line $K_H - \bar{K}$ represents the best least-squares fit to the high-energy data (open circles). If vacuum polarization effects are present, then the curve ΔK should be a better fit to the experimental points than any single straight line.

EXPERIMENTAL EVIDENCE FOR THE VACUUM POLARIZATION CONTRIBUTION

We consider first the analysis of available experimental data on proton-proton scattering in the energy range from 0.2 to 4.2 Mev to find evidence for the vacuum polarization contribution. This would be a simple task if we had *a priori* knowledge of the exact form of the specifically nuclear interaction between two protons, for in this case we could calculate quite precisely the function K in the absence of vacuum polarization and the difference between the calculated and the observed values could be compared directly with our calculations of ΔK. However, since our knowledge of the specifically nuclear interaction is obtained from the observed values of K themselves, it is necessary to find some other approach.

An alternative procedure can be based on the fact that the shape of the curve of K as a function of energy for a strong short range nuclear interaction is quite different from the shape of the curve of ΔK as a function of energy at low energies. Thus for a short range nuclear interaction (in the absence of vacuum polarization) the function K can be accurately represented by a quadratic in E:

$$K = A + BE + CE^2, \tag{13}$$

over the energy range from 0 to 4 Mev. The constants A and B, for any given "shape" of the nuclear interaction potential between two protons, determine the depth and range of the potential, and conversely. For fixed A and B, however, the coefficient C depends on the shape of the potential and in this sense is said to be "shape-dependent." Thus leaving aside the question of the vacuum polarization contribution, if there were available sufficiently accurate proton-proton scattering data, it would be possible to determine all three coefficients A, B, and C and thus to obtain information about the depth, range, and shape of the potential. Unfortunately, data of this accuracy are not available and what are available are sufficient only to determine that C is very small and to determine A and B with an accuracy of the order of 1 percent.

However, if a value is *assumed* for the coefficient C, then it is possible to determine A and B to somewhat higher accuracy. Theoretical values for C have been computed by Jackson and Blatt for several potential shapes (employing, of course, the available approximate values of A and B) who find:

Square well: $C=0.0026$ (Mev)$^{-2}$,

Gaussian well: $C=0.0015$ (Mev)$^{-2}$,

Exponential well: $C=-0.0007$ (Mev)$^{-2}$,

Yukawa well: $C=-0.00433$ (Mev)$^{-2}$.

The present available data cannot distinguish between these, though the higher-energy data appear to favor a small negative value for C. We have assumed in what follows that the shape-dependent coefficient C is that appropriate to a Yukawa well. This assumption is favored by three considerations: (1) the higher-energy data mentioned above, (2) the charge-independence hypothesis,[4] and (3) the limited theory available for nuclear forces. Thus we assume that in the absence of vacuum polarization effects K can be represented by a quadratic expression:

$$K=A+BE-0.00433E^2 \tag{14}$$

to an accuracy of the order of $\pm 0.01E^2$, from 0–4 Mev.

Now, the vacuum polarization contribution to K cannot be represented by such a quadratic expression over this energy range because of its strong upward curvature at low energies. Hence we consider the following test for the presence of vacuum polarization effects in the proton-proton scattering data. We divide the available experimental data for K into two groups, those corresponding to energies in the range from 0.2 to 0.5 Mev and those in the range from 0.5 Mev to 4.2 Mev. The first group of data we fit by least squares to a quadratic of the form (14):

$$K_L=A_L+B_LE-0.00433E^2, \tag{15}$$

[4] J. Schwinger, Phys. Rev. **78**, 135 (1950).

Table I. Vacuum polarization contribution to proton-proton scattering.

E(Mev)	Δ_2K	ΔK
0.01	19.9	19.9
0.02	3.26	3.27
0.03	1.30	1.31
0.04	0.758	0.766
0.05	0.504	0.512
0.06	0.369	0.377
0.07	0.288	0.296
0.08	0.234	0.242
0.09	0.196	0.204
0.10	0.169	0.177
0.15	0.0984	0.1066
0.20	0.0699	0.0781
0.25	0.0547	0.0629
0.30	0.0454	0.0536
0.35	0.0391	0.0473
0.40	0.0345	0.0428
0.45	0.0310	0.0393
0.50	0.0283	0.0366
0.60	0.0244	0.0327
0.70	0.0216	0.0299
0.80	0.0196	0.0279
0.90	0.0180	0.0263
1.00	0.0167	0.0250
1.20	0.0148	0.0232
1.40	0.0135	0.0219
1.60	0.0124	0.0208
1.80	0.0116	0.0201
2.00	0.0110	0.0195
2.50	0.0098	0.0183
3.00	0.0090	0.0176
3.50	0.0084	0.0171
4.00	0.0079	0.0167

and the second group of data we fit to another expression of the same form:

$$K_H=A_H+B_HE-0.00433E^2, \tag{16}$$

If vacuum polarization effects are present, these two quadratics should not have the same coefficients; in fact we should find $A_L>A_H$ and $B_L<B_H$. But if we take the experimental values of K and correct them by subtracting the theoretical vacuum polarization contribution at each energy and carry out the same procedure on the corrected values K':

$$K_L'=A_L'+B_L'E-0.00433E^2, \tag{17}$$

$$K_H'=A_H'+B_H'E-0.00433E^2, \tag{18}$$

then we should find $A_L'=A_H'$ and $B_L'=B_H'$.

This is the procedure we have adopted. The experimental data which have been employed are the same as those used by Breit *et al.* in their recent analysis, with the following exceptions: we have dropped the experimental results of Ragan, Kanne, and Taschek as being of insufficient accuracy for our purpose, and we have included the recent data of Worthington, McGruer, and Findley[5] and of Cooper, Frisch, and Zimmerman.[6] These data are summarized in Table II.

[5] Worthington, McGruer, and Findley, Phys. Rev. **90**, 899 (1953). See also H. H. Hall and J. L. Powell, Phys. Rev. **90**, 912 (1953).

[6] Cooper, Frisch, and Zimmerman, Phys. Rev. **94**, 1209 (1954).

TABLE II. Experimental results employed in the least squares analyses.

E(Mev)	Weight	K[a]	ΔK	K'	Observers
0.2		3.8721	0.0781	3.7940	
0.3		3.8611	0.0536	3.8075	
0.4		3.9536	0.0428	3.9108	
0.45		3.9271	0.0393	3.8878	
0.5		3.9761	0.0366	3.9395	
0.6	0.0233	3.9976	0.0327	3.9649	HL[b]
0.7		4.0706	0.0299	4.0407	
0.8		4.0976	0.0279	4.0697	
0.9		4.1491	0.0263	4.1228	
0.3828	0.0300	3.917	0.0478	3.8692	CFZ[c]
0.670		3.9851	0.0307	3.9544	
0.776	0.00843	4.0781	0.0283	4.0498	HHT[d]
0.867		4.1536	0.0268	4.1268	
0.860		4.1461	0.0269	4.1192	
1.200		4.3176	0.0232	4.2944	
1.390		4.4126	0.0220	4.3906	
1.830		4.5846	0.0200	4.5646	
2.105	0.08338	4.7201	0.0192	4.7009	HKPP[e]
2.392		4.8476	0.0185	4.8291	
2.42		4.8566	0.0185	4.8381	
3.04		5.1641	0.0176	5.1465	
3.27	0.06438	5.2361	0.0173	5.2188	BFLSW[f]
3.53		5.3601	0.0171	5.3430	
1.855		4.6013	0.0199	4.5814	
1.858		4.6048	0.0199	4.5849	
2.425		4.8538	0.0185	4.8398	
3.037		5.1488	0.0176	5.1312	
3.527	0.1000	5.3687	0.0171	5.3516	WMF[g]
3.899		5.5442	0.0168	5.5274	
4.203		5.6811	0.0166	5.6645	

[a] Use has been made of the calculations of Yovits, Smith, Hull, Bengston, and Breit (see reference 3) and of H. H. Hall and J. L. Powell Phys. Rev. **90**, 912 (1953).
[b] N. P. Heydenburg and J. L. Little (see reference 3).
[c] Cooper, Frisch, and Zimmerman (see reference 6).
[d] Heydenburg, Hafstad, and Tuve, Phys. Rev. **56**, 1078 (1939).
[e] Herb, Kerst, Parkinson, and Plain, Phys. Rev. **55**, 998 (1939).
[f] Blair, Freier, Lampi, Sleator, and Williams, Phys. Rev. **74**, 553 (1948).
[g] Worthington, McGruer, and Findley, Phys. Rev. **90**, 899 (1953).

The weights assigned to the various data are those employed by Breit *et al.* and we have arbitrarily assigned a weight of 0.030 to the data of Cooper, Frisch, and Zimmerman, and a weight of 0.100 to the data of Worthington, McGruer, and Findley.

Carrying out the least-squares fits described above to the uncorrected data, we find

$$A_L=3.782,\quad B_L=0.366,\qquad A_H=3.729,\quad B_H=0.480. \tag{19}$$

When we make the least-squares fits to the data corrected for vacuum polarization, we obtain

$$A_L'=3.685,\quad B_L'=0.494,\qquad A_H'=3.705,\quad B_H'=0.482. \tag{20}$$

One sees that there is a decided improvement in the matching of the low- and high-energy fits for the corrected data, particularly in the more sensitive coefficient B, thus verifying, to the limited accuracy available, the presence of a vacuum polarization contribution to proton-proton scattering. A visual presentation of the results is given in Fig. 1.

It should be noted that because of the relatively large errors in the data we cannot rule out completely the possibility that the better agreement in the case of the corrected data is fortuitous and results from a fortunate combination of random errors in the experimental values or from some energy dependent systematic error. Furthermore, this agreement depends to some extent on our choice of the quadratic coefficient in our quadratic expression for K. Had we taken this coefficient to be zero, for example, we would not have found better agreement in the A coefficients after the vacuum polarization correction though the improvement in agreement in the B coefficients would not have been impaired. This uncertainty in our result also cannot be removed until sufficiently more accurate experimental data are available to determine the C coefficient from the data itself. Thus our claim that the experimental data substantiate the vacuum polarization contribution must be taken with some reserve.

We may add further that our conclusion is based also on the assumption that the only long-range interactions between two protons are the Coulomb potential and the vacuum polarization potential. However, part of the magnetic interaction between two protons may also behave like a long-range potential, and while it would appear that such an effect should be small, further investigation of this point would be important.[7]

PARAMETERS OF THE NUCLEAR INTERACTION

If we accept the above evidence for the existence of the vacuum polarization effect (and there is every reason from the theoretical viewpoint that it should exist) then we may now take the experimental K values corrected for the vacuum polarization effect to determine those parameters of the specific short-range nuclear interaction which are of interest. To this end we have fitted all of the corrected data up to 4.203 Mev with a single quadratic of the form (14) by least squares and find

$$\bar{K}=3.7043+0.4818E-0.00433E^2. \tag{21}$$

We have not bothered to include the meson tail effect, emphasized by Breit, as it should be small over this limited energy range. We then regard (21) as the best Yukawa fit available at present (in view of both the correction for vacuum polarization, which has about a 1 percent effect on the coefficients, and the inclusion of the more recent data). These coefficients correspond to a zero-energy scattering length and effective range, which in the notation of Jackson and Blatt can be written,

$$-R/a=3.704,\quad r_0=2.76\times10^{-13}\ \text{cm}. \tag{22}$$

[7] One of the present authors (E.E.) is planning to investigate this point.

The changes from previously derived values are appreciable; on the basis of data available at the time of their publication, Jackson and Blatt[2] derived the values:

$$(-R/a)_{JB}=3.755\pm0.024,$$
$$(r_0)_{JB}=(2.65\pm0.07)\times10^{-13}\text{ cm}$$

while the results of Breit *et al.*[3] correspond to

$$(-R/a)_B=3.744,\quad (r_0)_B=2.73\times10^{-13}\text{ cm}.$$

The values derived by Hall and Powell[5] on the basis of the Wisconsin data alone are:

$$(-R/a)_{HP}=3.72,\quad (r_0)_{HP}=2.79\times10^{-13}\text{ cm}.$$

One remark of interest in connection with the results expressed in (22) is their bearing on the charge independence hypothesis. Here one is interested in the comparison between the ^1S nuclear interaction between two protons and between a neutron and a proton. As has been pointed out by Schwinger,[4] a direct comparison of the scattering lengths is not possible since both of these include the effects of short-range magnetic interactions between the particles which are different in the two cases. However, Schwinger has made appropriate corrections for the latter effect on the assumption that the magnetic interaction is short range (and this may not be justified for part of the magnetic interaction between two protons, as has been pointed out above). If we trace the consequences of our change in the zero-energy scattering length for two protons through Schwinger's analysis, we find that the corrected neutron-proton and proton-proton scattering lengths (on the assumption of a Yukawa potential) differ by about 12 percent. This is not unsatisfactory since the magnetic corrections are made on the assumption that the magnetic moments associated with the particles can be considered point dipoles. Since it is expected that the magnetic moments are essentially associated with a spatial distribution of magnetization, of uncertain amount, and the resulting correction would be in a direction to reduce the above discrepancy, the above agreement is probably satisfactory. The considerable present uncertainty in the effective range of the singlet neutron-proton interaction makes the change in the effective range of the proton-proton interaction derived above of little importance in evaluating charge independence of nuclear forces from this feature of the potential. All in all one can say that our present knowledge of nuclear forces is entirely consistent with the charge independence hypothesis but does not unequivocally establish its validity.

In our analysis above, the effect of vacuum polarization on the S phase shift only has been considered. Since the vacuum polarization potential is of long range and the specifically nuclear ^3P state potential is relatively weak, we would expect relatively much larger effects in the P phase shift. It would be of interest to examine this point, since data on the P-wave phase shift are now available.[5]

In conclusion the authors would like to remark that the considerations described above add additional importance to the necessity for improved proton-proton scattering measurements at low energies. In particular accurate measurements at energies below 0.2 Mev would be very desirable. These would not only aid in establishing beyond doubt the existence of the vacuum polarization contribution, but would help to make the determinations of the zero-energy scattering length and effective range less dependent on assumptions about the shape of the nuclear potential.

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