where we have used $p_E^2=ME$. This integral has been evaluated, using the Gaussian wave function of Eq. (2), to give a ratio of (taking $E=\mu$)

$$\Delta E_R^*=1.09R_{\text{abs}}. \quad (13)$$

This result completely specifies the real and imaginary parts of the energy shift associated with the absorption process, if the absorption rate is known. To determine the rate, we use directly the results of BSW or, equivalently, their results for the mean free path for absorption of the mesons in nuclear matter, which is simply related to the absorption rate. The result for the ratio of the energy shift to the K-shell binding energy $E_Z=\frac{1}{2}\mu(\alpha Z)^2$ is

$$\frac{\Delta E^*}{E_Z}=\left(\frac{Z^2}{985}+i\frac{Z^2}{2150}\right). \quad (14)$$

It is interesting to note that the equivalent potential felt by the meson in uniform nuclear matter at normal density is

$$V^*=(3.66+1.68i)\text{ Mev}. \quad (15)$$

To the energy shift associated with absorption is to be added the effect from scattering. The result given by Deser *et al.* for the effect from scattering is:

$$\frac{\Delta E^{sc}}{E_Z}=-\frac{4Z\alpha\mu}{3}[2Za_1+(3N+Z)a_3], \quad (16)$$

where a_1 and a_3 are the scattering amplitudes for the isotopic spin $\frac{1}{2}$ and $\frac{3}{2}$ states, with values of $0.16/\mu$ and $-0.11/\mu$ respectively. There can be no contribution to the level broadening from this term, contrary to the result of Deser *et al.* since the charge-exchange scattering is energetically forbidden in most light nuclei. Combining these results, we find (for $N=Z$)

$$\frac{\Delta E}{E_Z}=\frac{Z^2}{456}+\frac{Z^2 i}{2150}. \quad (17)$$

This final result for the level shift is about twice as large as the result obtained when absorption is neglected; in addition, as remarked above, almost all of the level width comes from the absorption effect leading to star formation.

The experiments[2] seem to show a somewhat smaller shift than calculated here; the discrepancy may arise from the assignment of the low-energy meson scattering phase shifts which have been used.

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Applications of Causality to Scattering

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The optical dispersion relations are extended to the scattering of massive particles and are applied to the nuclear interaction of the pion-nucleon system. The sign of the forward scattering amplitude is unambiguously inferred from measured total cross sections and found to agree with that determined from Coulomb interference.

INTRODUCTION

THERE is a considerable history of investigations on those relations among cross sections and scattering amplitudes which are independent of the underlying model. For quantum mechanical systems it has proved convenient to remove all reference to the details of the interactions by expressing these relations in terms of Heisenberg's[1] S-matrix. The requisites of special relativity and of conservation of probabilities then are succinctly stated as the Lorentz invariance and unitarity of the S-matrix. Unitarity leads to the familiar relation

$$4\pi\,\text{Im}\,f=k\sigma, \quad (1)$$

where f is the amplitude for elastic scattering in the forward direction, σ is the total cross section, and k the wave number of the incident particle.

In recent years there has been a renewed interest in the consequences for scattering of causality,[2,3] the fact that signals cannot propagate faster than with the speed of light.

Kramers[4] and Kronig[5] showed the consequence that,

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for electromagnetic waves in matter, the real and imaginary parts of the index of refraction are not independent. For any frequency ω they satisfy the equation

$$\mathrm{Re}\,[n(\omega)-n(0)]=P\int_0^\infty \frac{2\omega^2\,\mathrm{Im}\,n(\omega)}{\pi\omega'(\omega'^2-\omega^2)}. \quad (2)$$

Equation (2) implies that $n(\omega)$, defined for complex values of ω by analytic continuation from its measured value when ω is on the positive real axis, is an analytic function for $\mathrm{Im}\,\omega>0$ with at most a simple pole at $|\omega|=\infty$.

According to Gell-Mann, Goldberger, and Thirring[3] such an analyticity property is valid for field theoretic systems and for quanta with mass: the S-matrix for forward elastic scattering is analytic as a function of the energy E of the incident quantum if $\mathrm{Im}\,E>0$. The poles of this function on the real axis are simply related to the energy and asymptotic wave function of any bound states that may exist. In directions other than forward the S-matrix will in general have the same analytic behavior except for an essential singularity at infinity.[6]

The analytic behavior for forward scattering enables one to express the real part of the forward scattering amplitude as an integral over the imaginary part similar to Eq. (2). For this to be as useful as the analogous Kramers-Kronig relations for optical systems it is necessary to know from experiment the imaginary part of the scattering amplitude for all energies, real and virtual. It will be shown that this information is directly available from experiments and that it leads to a relation between the forward elastic cross section and the total cross section for real processes.

THEORY

The elastic forward scattering amplitude $f(E)$, considered as a function of the total energy E of the incident particle with mass m can only be measured for E real and greater than the rest energy m ($\hbar=c=1$). For complex values of E, $f(E)$ is defined by analytic continuation. This function of a complex variable $f(E)$ is now taken to be analytic in the entire upper half plane, $\mathrm{Im}\,E>0$, with the exception of the point at infinity where $f(E)$ may have a pole.

If $g(z)$ is a function analytic in the upper half plane,

$$g(z)=\frac{1}{2\pi i}\oint\frac{g(t)}{t-z}dt \quad (3)$$

for any contour around the point z in the region of analyticity. If, in addition, $g(z)$ is at most of order $1/z$ for large z then

$$g(x)=\lim_{(\epsilon>\delta)\to 0}\frac{1}{2\pi i}\int_{-\infty+i\delta}^{\infty+i\delta}\frac{g(z)}{z-x-i\epsilon}dz, \quad x\text{ real.} \quad (4)$$

To carry out the limiting process, we must have some information about $g(z)$ on the real axis. It is sufficient to treat a $g(z)$ that has simple poles at the points x_i ($i=1, 2, \cdots$) and branch points. If the latter are connected by cuts in the lower half-plane, then they make no direct contribution to the integral when it is taken along the real axis; they only determine the relative phase of the function on the two sides. The poles, however, do contribute as follows:

$$\tfrac{1}{2}\,\mathrm{Re}\,g(x)=\frac{1}{2\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,g(x')}{x'-x}dx' -\frac{1}{2}\sum_i\frac{1}{x_i-x}\,\mathrm{Re}\,[\mathrm{Res}\,g(x')\,|_{x'=x_i}]. \quad (5)$$

To obtain useful consequences from this equation, we must be able to apply it to the forward scattering amplitude and we must have enough information about this function to be able to carry out the indicated mathematical operation.

We shall assume in the following that the total cross section and therefore, by Eq. (1), the imaginary part of $f(E)$, are known from experiment in the range $m<E<\infty$. Since $f(E)$ may not vanish sufficiently rapidly at infinity to satisfy the requirements on g, we introduce a convergence factor and consider

$$g(E)=f(E)/(E^2-E_0^2), \quad (E_0\text{ real}). \quad (6)$$

One can now see from Eq. (5) that $\mathrm{Im}\,f(E)$ in the range $-\infty<E<m$, $f(E_0)$, $f(-E_0)$ and the residues of $f(E)$ at its poles E_i on the real axis must be known.

First, we observe that if $f(E)$ is calculated as the matrix element for forward elastic scattering according to a relativistic field theory, then the replacement of E by $-E$ gives the scattering amplitude for the inverse process.* We therefore take

$$f(-E)=f^*(E) \quad (7)$$

on the real axis. We next remark that that

$$\mathrm{Im}\,f(E)=0, \quad 0<E<m \quad (8)$$

if the scatterer is in the ground state. This requirement is satisfied for scattering by elementary particles. The identity is easily verified by inspection of the partial

[6] That the S-matrix for each angular-momentum states separately has this property follows from the completeness of the asymptotic wave functions as discussed without reference to causality for interactions that vanish outside some finite radius. See W. Heisenberg, Z. Naturforsch. 1, 608 (1946) and N. Hu, Phys. Rev. 74, 131 (1948).

* *Note added in proof.*—We are indebted to Professor Marvin Goldberger for pointing out to us that this assumption applies only to the scattering of neutral particles described by real fields. Therefore, the subsequent equations can be used only in this case. The application of the dispersion relations to the scattering of charged mesons has been described to us by Professor Goldberger in a private communication.

wave expansion of the forward scattering amplitude,

$$f((k^2+m^2)^{\frac{1}{2}})=\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l(k)}-1)/2ik, \tag{9}$$

for the scattering of a particle by a potential. Since $\delta_l(k)$ is an odd function of k, it is imaginary when k is imaginary or $E<m$ and $f(E<m)$ is real. A more general argument is based on the form of the matrix element for forward scattering due to a Hermitian interaction H. It is the sum of terms each of which is of the form

$$\lim_{\delta\to 0^+}\Big(\underbrace{\int\cdots\int di\cdots dl}_{\text{continuum states}}+\underbrace{\sum_i\cdots\sum_l}_{\text{discrete states}}\Big)\times\frac{\langle 0|H|i\rangle\langle i|H|\cdots|H|l\rangle\langle l|H|0\rangle}{(W_i-W_0-i\delta)\cdots(W_l-W_0-i\delta)}, \tag{10}$$

where W_0 is the initial total energy of the system and the W_i, $\cdots$ are energies of intermediate states. This expression contains an imaginary part contributed by energy conserving intermediate states in the continuum. Under our supposition of incident particles with $E<m$ the only energy conserving states could be discrete states in which the incident particle is bound to the scatterer. The imaginary part therefore vanishes. The discrete states only give rise to the poles at E_i. The imaginary part does not vanish when the scatterer is initially sufficiently excited to supply the energy deficiency of the incident particle. Then Eq. (8) must be revised to read

$$\operatorname{Im} f(E)=0, \quad 0<E<m-\Delta W, \tag{8'}$$

where ΔW is the initial excitation of the scattering center.

When Eqs. (6), (7), and (8) are used in conjunction with Eqs. (5) and (1) one may change from the energy variable to the momentum variable $k=(E^2-m^2)^{\frac{1}{2}}$ with the result

$$\begin{aligned}\operatorname{Re}[f(k)-f(k_0)]&=\frac{1}{2\pi^2}P\int_0^\infty dk'\sigma(k')\left[\frac{k^2}{k'^2-k^2}-\frac{k_0^2}{k'^2-k_0^2}\right]\\&\quad+\textstyle\sum_i 2E_i\operatorname{Re}[\operatorname{Res} f(E)|_{E=E_i}]\\&\qquad\times\left[\frac{1}{E_i^2-k^2-m^2}-\frac{1}{E_i^2-k_0^2-m^2}\right].\end{aligned} \tag{11}$$

The evaluation of $\operatorname{Re} f(k)$ therefore depends on the knowledge of this function at some momentum k_0 as well as on the asymptotic behavior of the bound states. The latter information can be replaced by knowledge of $\operatorname{Re} f(k)$ at as many points as there are bound states. For complicated systems such as enter nucleon-nucleus scattering this procedure becomes cumbersome.

Equation (11) is most useful when there are no bound states. With the choice $k_0=0$, the equation becomes

$$\operatorname{Re} f(k)=\operatorname{Re} f(0)+\frac{k^2}{2\pi^2}P\int_0^\infty dk'\frac{\sigma(k')}{k'^2-k^2}. \tag{12}$$

Since a constant cross section contributes zero to the integral, one can increase its convergence by subtracting the limiting value for high energy, $\sigma(\infty)$,

$$\operatorname{Re} f(k)=\operatorname{Re} f(0)+\frac{k^2}{2\pi^2}P\int_0^\infty dk'\frac{\sigma(k')-\sigma(\infty)}{k'^2-k^2}. \tag{13}$$

For comparison with experiment one may evaluate the forward scattering cross section

$$\frac{d\sigma(0^\circ)}{d\Omega}=|f(k)|^2=[\operatorname{Re} f(k)]^2+\frac{k^2}{16\pi^2}[\sigma(k)]^2. \tag{14}$$

APPLICATION

Equation (13) can also be useful in another way. Since the experiment measures only the absolute magnitude of the scattering amplitude, there are always two solutions, say $f(k)$ and $-f^*(k)$, which have opposite signs for the real part. These correspond to the two possible choices for the signs of all the phase shifts. Both must have a positive imaginary part in accordance with the conservation law Eq. (1). The remaining ambiguity may be resolved by a study of interference effects with the scattering due to a known interaction such as Coulomb scattering.

The causality condition, however, must exclude one of the two alternatives, because only one is analytic in the upper half-plane; the other is analytic in the lower half-plane. Equation (13) gives that real part which is consistent with causality requirements. Evaluation of it will lead to the correct forward scattering amplitude so that Eq. (13) will distinguish between two possible sets of phase shifts which differ only in sign. This differentiation is particularly simple in the neighborhood of a sharp maximum in the cross section. If σ changes rapidly with k, the integral in Eq. (13) is large and positive before the maximum and large and negative after it. For sufficiently small $f(0)$ or a sufficiently narrow peak, this is also the behavior of $\operatorname{Re} f(k)$. This property of the dispersion relation enables one to show that the phase shift of the strongly interacting isotopic spin 3/2 state of the pion-nucleon system is an increasing function of the energy. Therefore, it must be positive and does not have a maximum or cusp near the "resonance."

Since the π meson is spinless the amplitude for meson-nucleon elastic scattering is of the form $A+B\boldsymbol{\sigma}\cdot(\mathbf{k}\times\Delta\mathbf{k})$. When $\Delta\mathbf{k}=0$, there is no spin dependence so forward scattering is all elastic. The measured forward dif-

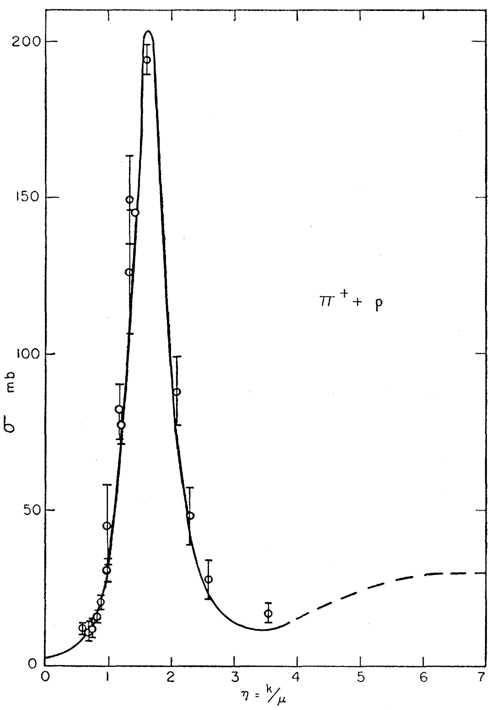

FIG. 1. Total cross section, $\pi^+ + p$, as a function of momentum in the center-of-mass frame.

ferential cross-section should be $|f|^2$ where the real and imaginary parts of f are to be calculated from Eqs. (13) and (1).

The observed[7] dependence of σ on the momentum in the center of mass frame has been approximated in Figs. 1 and 2 by analytic functions for which the required integrals could be calculated simply. The extrapolation of the cross section to zero pion momentum was performed with S-wave phase shifts determined by Orear[8]: $\alpha_3 = -0.11\eta$ and $\alpha_1 = 0.16\eta$, $(\eta = k/\mu c)$.

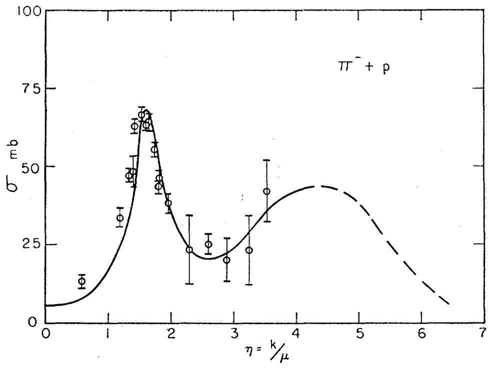

FIG. 2. Total cross section, $\pi^- + p$, as a function of momentum in the center-of-mass frame.

The proton is a state of the $\pi^0 + p$ system with energy less than the sum of the rest masses. Therefore a contribution to f is expected in Eq. (11) from a residue at $E_i \cong \mu^2/2M$ with momentum dependence $k^2/(k^2+\mu^2)$. If conservation of isotopic spin is assumed then the proton can only affect the phase shift for a state of even parity and $J = I = \frac{1}{2}$, i.e., α_{11}. However phase-shift analyses of observed angular distributions indicate that α_{11} is quite small at least up to 100 Mev. Therefore any contribution to α_{11} from the proton state is probably negligible and we have omitted it.

The results of carrying out the integrations are listed in Table I. As was pointed out above, the change of sign

TABLE I. Re $f(k)$ for elastic scattering, $\pi^0 + p$.

$\eta = k/\mu$	0	0.25	0.5	0.75	1.0	1.5	1.75	2.0	3.0	4.0
μ Re $f(k)$	−0.02	−0.017	0.02	0.083	0.19	0.23	−0.36	−0.61	−0.34	−0.13

between $=1.5$ and 1.75 is characteristic of the behavior of Re $f(k)$ near a sharp maximum in the total cross section. Hence the expression for $f(k)$ in terms of the phase shifts,

$$\mathrm{Re}\, f(k) = (1/6k)[\sin 2\alpha_1 + 2\sin 2\alpha_{13} + \sin 2\alpha_{11} + 2\sin 2\alpha_3 + 4\sin 2\alpha_{33} + 2\sin 2\alpha_{31}], \quad (15)$$

must have the same property. This restriction rules out those solutions for the pion-nucleon phase shifts which have a cusp-like energy dependence near the cross section maximum. Near resonance, where the isotopic triplet phase shifts are dominant, a comparison of Eq. (15) and Table I implies the same sign choice for $\sin\alpha$ (positive) as has been inferred from the interference of the nuclear and Coulomb scattering.

We have profited from conversations with Professor M. Gell-Mann and M. Goldberger to whom we are also indebted for prepublication copies of their manuscripts.

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