An alternative parameterization[1,6] is to consider scattering amplitudes corresponding to particular values of the total angular momentum; these are related to our parameters by

$$\mu_0=C+\tfrac{1}{3}B,$$
$$\mu_2=\tfrac{1}{3}(2/5)^{\frac{1}{2}}B,$$
$$\mu_1=(2/3)^{\frac{1}{2}}iH.$$

Thus our parameter B contributes both to the $J=0$ and $J=2$ scattering. This is because the term is expressed in conventional vector notation instead of in terms of the spherical harmonics $Y_{2m}(\mathbf{k})$. Other parameters defined by Rosenfeld are

$$r_0=|r_0|e^{i\tau_0}=-\mu_0/\sqrt{5}\mu_2=-(3C+B)/\sqrt{2}B,$$
$$r_1=|r_1|e^{i\tau_1}=-i\sqrt{3}\mu_1/\sqrt{5}\mu_2=3H/B,$$
$$r_0+(\tfrac{1}{2})^{\frac{1}{2}}=|r_0+(\tfrac{1}{2})^{\frac{1}{2}}|e^{i\psi}=-3C/\sqrt{2}B.$$

From these equations it may be seen that our definitions [Eqs. (5)] of X and $(\psi-\tau_1)$ are consistent with those of Rosenfeld. The complex ratio of C to B is clearly determined by two quantities: one of these is a ratio of magnitudes for which we have chosen X; the other is a phase factor which may be chosen as ψ, γ, or τ_0. Since τ_0 and τ_1 may be related to p-p scattering results[6] it is of interest to relate ψ, γ, and τ_0

$$\tan\tau_0=\frac{3(X^2+X)^{\frac{1}{2}}\sin\gamma}{1-X+(X^2+X)^{\frac{1}{2}}\cos\gamma},$$
$$\cot\psi=\cot\gamma-\csc\gamma[(X+1)/X]^{-\frac{1}{2}}.$$

Looking at Eq. (10d), we note that if $\langle S_y\rangle$ turns out to be large, that is, $\sin\gamma$ is not far from unity, then the measured value of $\langle S_y\rangle$ would be almost directly proportional to $\tan\tau_0$ independent of the value of X as long as $X\ll 1$.

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Energy Levels of π-Mesonic Atoms*

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The shift of the energy levels of the π-mesonic atom which results from the strong nuclear absorption of the meson is estimated and shown to be comparable with the shift associated with the scattering interaction. It is also shown that the level broadening due to the finite lifetime of the meson in the nucleus is almost entirely produced by the absorption process.

In a recent paper by Deser, Goldberger, Baumann, and Thirring,[1] the nuclear interaction of the π meson was shown to give rise to appreciable shifts and broadening of the energy levels of the π-mesonic atom.[2] The particular interaction which these authors investigated was that associated with the small s-wave meson scattering observed for free nucleons. It is the purpose of this note to point out that another effect exists which not only gives level shifts comparable with those originating in the simple scattering interaction but also is almost entirely responsible for the broadening of the levels.† This effect arises from the strong absorption of the meson from the low S- and P-orbits which leads to nuclear stars[3]; it has been previously described and analyzed in detail by Brueckner, Serber, and Watson.[4] It is quite apparent from the experimental study[3] of the capture of π mesons in light nuclei that this is by far the most important absorption process, the charge exchange scattering being almost completely absent except in hydrogen. We will give an estimate of the level broadening due to this effect and also determine the level shift associated with this interaction.

The mechanism we are particularly interested in is:

$$\pi^-+\text{nucleus}\rightarrow\text{star}. \tag{1}$$

This process, which is responsible for very nearly all of the absorption, can take place only through the cooperative effect of at least two nucleons. As shown by BSW, if the reasonable assumption is made that the process involves only two nucleons, then fairly quantitative conclusions can be drawn about the nuclear ground state structure. The results show that the ground state is rather strongly correlated with appreciable high-momentum components present in the wave function. Similar evidence is also available from other high-energy processes.[5] A summary[6] of the data shows

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[2] Stearns, DeBenedetti, Stearns, and Leipuner, Phys. Rev. **93**, 1123 (1954); Phys. Rev. **96**, 804 (1954); Stearns *et al.* (unpublished).
† This effect was mentioned by Deser *et al.*, but its contribution to the level shift and broadening was not determined.
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[6] Brueckner, Eden, and Francis (to be published).

that the Fourier transform of the dependence of the ground-state wave function on the relative coordinate r_{12} is:

$$\psi(p)=\int\psi(r_{12})e^{i\mathbf{p}\cdot\mathbf{r}_{12}}d\mathbf{r}_{12}$$
$$=N\exp(-p^2/2\alpha^2). \quad (2)$$

where N is a normalization constant and $\alpha^2/2M=14$ Mev. This choice fits well the π capture data and we shall use it in the following.

We introduce an operator R,[5] which, acting on a two nucleon and one meson system, leads to meson absorption and two fast final nucleons. For the capture process, the transition matrix element is

$$H_{\text{cap}}=\int\frac{e^{-i\mathbf{p}_1\cdot\mathbf{r}_1'}}{(2\pi)^3}e^{-i\mathbf{p}_2\cdot\mathbf{r}_2'}(r_1'r_2'|R|r_1r_2z)\times\psi_I(r_1r_2)\varphi(z)d\mathbf{r}_1d\mathbf{r}_2d\mathbf{r}_1'd\mathbf{r}_2'. \quad (3)$$

If the meson has low momentum and long wavelength (for example, in the lowest K-orbit), we can replace the meson wave function $\varphi(z)$ by its value at the origin $\varphi(0)$. In addition, if the final nucleon momenta are large compared with the original momenta, it is reasonable to assume that R depends only on the final momenta and, possibly, the spins and isotopic spins of the nucleons. In this approximation, going over to the center-of-mass system and relative coordinates, we find

$$H_{\text{cap}}\cong\varphi(0)\int e^{-i\mathbf{p}_1\cdot\mathbf{r}}R(p_1)\psi_I(r)d\mathbf{r}$$
$$=\varphi(0)R(p_1)\psi(p_1). \quad (4)$$

The corresponding transition rate is

$$R_{\text{abs}}=2\pi|R(p_1)|^2|\varphi(0)\psi(p_1)|^2\frac{d\mathbf{p}_1}{dE}\frac{1}{(2\pi)^3}, \quad (5)$$

if one assumes an average over spin and isotopic spins.[7]

Next we consider the scattering of a meson associated with the absorption process, which can occur since the operator R acting twice can lead to scattering. For this process, making approximations similar to those made in the absorption process, the matrix element is

$$H_S=|\varphi(0)|^2\int\psi^2(p)|R(p)|^2\frac{1}{E-p^2/M+i\epsilon}\frac{d\mathbf{p}}{(2\pi)^3}$$
$$=\frac{|\varphi(0)|^2}{(2\pi)^3}\left[P\int\psi^2(p)|R(p)|^2\frac{1}{E-p^2/M}d\mathbf{p}-\pi i\psi^2(p_E)\frac{d\mathbf{p}_E}{dE}|R(p_E)|^2\right], \quad (6)$$

[7] Lacking knowledge of the spin dependence, we neglect it here, but may introduce an error in so doing.

where the two terms come from breaking up the δ_+ function into principal value and delta function, i.e.,

$$\frac{1}{E-p^2/M+i\epsilon}=P\frac{1}{E-p^2/M}-\pi i\delta(E-p^2/M). \quad (7)$$

By using Eq. (5) for the absorption rate, this can also be written as

$$H_S=\frac{|\varphi(0)|^2}{(2\pi)^3}P\int\psi^2(p)\frac{|R(p)|^2}{E-p^2/M}d\mathbf{p}-\frac{i}{2}R_{\text{abs}}$$
$$=\Delta E_R+i\Delta E_{Im}. \quad (8)$$

The first term is the energy shift, the second leads to the level broadening.

To proceed, we first separate the energy shift into two parts associated with low and high momenta, i.e., we write

$$\Delta E_R=\frac{|\varphi(0)|^2}{(2\pi)^3}\left[\int\frac{\psi^2(p)|R(0)|^2}{E}d\mathbf{p}+P\int\psi^2(p)\left[\frac{|R(p)|^2}{E-p^2/M}-\frac{|R(0)|^2}{E}\right]d\mathbf{p}\right]$$
$$=\Delta E_R^{(S)}+\Delta E_R^*. \quad (9)$$

The first term is independent of the correlation effects in the wave function; it is very closely associated with the scattering of mesons by single nucleons, in which case the momentum space wave function is simply a delta function at zero momentum. Thus we shall identify the first term with the interaction due to scattering of mesons by individual nucleons and use the analysis of Deser *et al.* to determine its magnitude. For the second term we have, using the result given for the absorption rate [Eq. (5)],

$$\Delta E_R^*/R_{\text{abs}}=P\int\psi^2(p)d\mathbf{p}\left[\frac{|R(p)|^2}{E-p^2/M}-\frac{|R(0)|^2}{E}\right]\times\left[2\pi\psi^2(p_E)\frac{d\mathbf{p}_E}{dE}|R(p_E)|^2\right]^{-1}. \quad (10)$$

This can be further simplified if an explicit form is assumed for $R(p)^2$. A reasonably general form is

$$|R(p)|^2=R_0+R_1p^2, \quad (11)$$

where R_0 and R_1 are constants. In this case the expression for ΔE_R^* simplifies to

$$\Delta E_R^*/R_{\text{abs}}=P\int\frac{\psi^2(p)d\mathbf{p}p^2/EM}{E-p^2/M}\bigg/2\pi\psi^2(p_E)\frac{d\mathbf{p}_E}{dE}, \quad (12)$$

where we have used $p_E^2=ME$. This integral has been evaluated, using the Gaussian wave function of Eq. (2), to give a ratio of (taking $E=\mu$)

$$\Delta E_R^*=1.09R_{\text{abs}}. \tag{13}$$

This result completely specifies the real and imaginary parts of the energy shift associated with the absorption process, if the absorption rate is known. To determine the rate, we use directly the results of BSW or, equivalently, their results for the mean free path for absorption of the mesons in nuclear matter, which is simply related to the absorption rate. The result for the ratio of the energy shift to the K-shell binding energy $E_Z=\frac{1}{2}\mu(\alpha Z)^2$ is

$$\frac{\Delta E^*}{E_Z}=\left(\frac{Z^2}{985}+i\frac{Z^2}{2150}\right). \tag{14}$$

It is interesting to note that the equivalent potential felt by the meson in uniform nuclear matter at normal density is

$$V^*=(3.66+1.68i)\text{ Mev.} \tag{15}$$

To the energy shift associated with absorption is to be added the effect from scattering. The result given by Deser *et al.* for the effect from scattering is:

$$\frac{\Delta E^{\text{sc}}}{E_Z}=-\frac{4Z\alpha\mu}{3}[2Za_1+(3N+Z)a_3], \tag{16}$$

where a_1 and a_3 are the scattering amplitudes for the isotopic spin $\frac{1}{2}$ and $\frac{3}{2}$ states, with values of $0.16/\mu$ and $-0.11/\mu$ respectively. There can be no contribution to the level broadening from this term, contrary to the result of Deser *et al.* since the charge-exchange scattering is energetically forbidden in most light nuclei. Combining these results, we find (for $N=Z$)

$$\frac{\Delta E}{E_Z}=\frac{Z^2}{456}+\frac{Z^2 i}{2150}. \tag{17}$$

This final result for the level shift is about twice as large as the result obtained when absorption is neglected; in addition, as remarked above, almost all of the level width comes from the absorption effect leading to star formation.

The experiments[2] seem to show a somewhat smaller shift than calculated here; the discrepancy may arise from the assignment of the low-energy meson scattering phase shifts which have been used.

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Applications of Causality to Scattering

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The optical dispersion relations are extended to the scattering of massive particles and are applied to the nuclear interaction of the pion-nucleon system. The sign of the forward scattering amplitude is unambiguously inferred from measured total cross sections and found to agree with that determined from Coulomb interference.

INTRODUCTION

THERE is a considerable history of investigations on those relations among cross sections and scattering amplitudes which are independent of the underlying model. For quantum mechanical systems it has proved convenient to remove all reference to the details of the interactions by expressing these relations in terms of Heisenberg's[1] S-matrix. The requisites of special relativity and of conservation of probabilities then are succinctly stated as the Lorentz invariance and unitarity of the S-matrix. Unitarity leads to the familiar relation

$$4\pi\,\text{Im}\,f=k\sigma, \tag{1}$$

where f is the amplitude for elastic scattering in the forward direction, σ is the total cross section, and k the wave number of the incident particle.

In recent years there has been a renewed interest in the consequences for scattering of causality,[2,3] the fact that signals cannot propagate faster than with the speed of light.

Kramers[4] and Kronig[5] showed the consequence that,

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