Nuclear Dispersion Contribution to High-Energy Electron Scattering*

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An expression for the contribution of virtual intermediate states of the nucleus (nuclear dispersion) to the elastic and inelastic scattering of high-energy electrons is obtained. The closure relation for the nuclear states is used to put this in a form which depends only on the properties of the initial and final states. An estimate of this contribution in comparison with the first-order scattering shows that it is expected to be small but not negligible for the light elements and somewhat larger for the heavy elements, when the electrons have several hundred Mev energy. The estimate is only valid for large scattering angles. More detailed knowledge of the one- and two-proton transition charge densities would be required in order to extend the range of validity. It seems likely that the bulk of the dispersion correction would be taken into account if the correct electron wave functions in the static Coulomb field were to be used in place of plane waves.

I. INTRODUCTION

CALCULATIONS of the elastic scattering of high-energy electrons from nuclei have for the most part regarded the nucleus as a rigid structure with no internal degrees of freedom. Such calculations have been done exactly for spherically symmetric charge distributions,[1] and approximately for spheroidal charge distributions that can be set into rotation.[2,3] The internal degrees of freedom of the nucleus, which manifest themselves in the existence of excited nuclear states, give rise to inelastic scattering and to dispersive contributions to both elastic and inelastic scattering. Inelastic scattering has been considered in a number of papers.[4] Dispersion effects, caused by transitions to and from virtual intermediate states of the nucleus-electron system, have recently been discussed in a general way by Lewis.[5]

The object of the present paper is to reduce the rather complicated formula for the dispersion contribution to a form in which an estimate of its order of magnitude can be made in some situations. In doing this, some of Lewis' results are rederived. The following approximating assumptions are made. (1) The electron interacts only with the charges of the nuclear protons. (2) The matrix element is calculated only to second order in this interaction. (3) The sum over intermediate nuclear states is estimated with the help of the closure relation; this means that a suitably averaged virtual excitation energy of the nucleus must be assumed. (4) The reduced wavelength of the electron is taken to to be small in comparison with the nuclear radius.

The resulting expression involves integrals over one- and two-proton transition charge densities between the initial and final states of the nucleus. The order of magnitude of this dispersion contribution in comparison with the first-order scattering can then be found from simple assumptions concerning these densities.

II. EXPRESSION FOR THE CROSS SECTION

The first-order matrix element for the transition from an initial state ψ_0 of the nucleus and electron energy and momentum E_0, $\hbar\mathbf{k}_0$, to a final state ψ_f of the nucleus and electron energy and momentum E_f, $\hbar\mathbf{k}_f$, is

$$V_{f0}^{(1)}=\sum_{i=1}^{Z}\int\int\bar{\psi}_f\exp(-i\mathbf{k}_f\cdot\mathbf{r})a_{f0}e^2|\mathbf{r}-\mathbf{R}_i|^{-1}\psi_0 \times\exp(i\mathbf{k}_0\cdot\mathbf{r})d\tau_N d\tau, \quad (1)$$

where a_{f0} is the matrix element of the Dirac unit operator between initial and final electron spin states, and $\int d\tau_N$ denotes integration over the coordinates of the Z protons and $A-Z$ neutrons. We add and subtract $i(\mathbf{k}_0-\mathbf{k}_f)\cdot\mathbf{R}_i$ in the exponent, and integrate over the new variable $\mathbf{r}-\mathbf{R}_i$ in place of $\mathbf{r}$, to obtain

$$V_{f0}^{(1)}=(4\pi e^2 a_{f0}/q^2)\sum_{i=1}^{Z}\int\bar{\psi}_f\exp(i\mathbf{q}\cdot\mathbf{R}_i)\psi_0 d\tau_N, \quad (2)$$

where $\hbar\mathbf{q}\equiv\hbar(\mathbf{k}_0-\mathbf{k}_f)$ is the momentum transfer from the electron to the nucleus. The differential scattering cross section is

$$\sigma^{(1)}=(k_f/2\pi\hbar c)^2\tfrac{1}{2}S_f S_0|V_{f0}^{(1)}|^2, \quad (3)$$

where S denotes a sum over the two positive energy spin states. In the extreme relativistic region, $\frac{1}{2}S_f S_0|a_{f0}|^2=\cos^2(\frac{1}{2}\theta)$, where θ is the angle between $\mathbf{k}_0$ and $\mathbf{k}_f$, so that the first-order cross section becomes

$$\sigma^{(1)}=(4e^4E_f^2\cos^2(\tfrac{1}{2}\theta)/\hbar^4c^4q^4) \times\left|\sum_{i=1}^{Z}\int\bar{\psi}_f\exp(i\mathbf{q}\cdot\mathbf{R}_i)\psi_0 d\tau_N\right|^2. \quad (4)$$

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[1] See Yennie, Ravenhall, and Wilson, Phys. Rev. **95**, 500 (1954), and earlier papers cited there.

[2] L. I. Schiff, Phys. Rev. **96**, 765 (1954).

[3] Downs, Ravenhall, and Yennie, Bull. Am. Phys. Soc., **29**, No. 8, 29 (1954).

[4] See reference 2 and earlier papers cited there.

[5] Robert R. Lewis, Jr., thesis, University of Michigan, 1954 (unpublished); this contains extensive references to earlier work.

It is convenient to write Eq. (4) in terms of the one-proton transition charge density

$$\rho_{f0}^{(1)}(\mathbf{R}_a)\equiv\int\cdots\int\bar{\psi}_f(\mathbf{R}_1\cdots\mathbf{R}_A)\sum_{i=1}^{Z}\delta(\mathbf{R}_a-\mathbf{R}_i)\times\psi_0(\mathbf{R}_1\cdots\mathbf{R}_A)d\tau_1\cdots d\tau_A, \quad (5)$$

which has the property

$$\int\rho_{f0}^{(1)}(\mathbf{R}_a)d\tau_a=Z\delta_{f0}. \quad (6)$$

In terms of the form factor

$$F(\mathbf{q})\equiv\int\rho_{f0}^{(1)}(\mathbf{R}_a)\exp(i\mathbf{q}\cdot\mathbf{R}_a)d\tau_a, \quad (7)$$

Eq. (4) becomes

$$\sigma^{(1)}=(4e^4E_f^2\cos^2(\tfrac{1}{2}\theta)/\hbar^4c^4q^4)\,|F(\mathbf{q})|^2. \quad (8)$$

The second-order (dispersive) scattering is obtained along with the first-order scattering if $V_{f0}^{(1)}$ in Eq. (3) is replaced by $V_{f0}^{(1)}+V_{f0}^{(2)}$, where

$$V_{f0}^{(2)}=-\sum_n\sum_\nu(\epsilon_n-\epsilon_0+E_\nu-E_0)^{-1}\times\sum_{i=1}^{Z}\int\int\bar{\psi}_f\exp(-i\mathbf{k}_f\cdot\mathbf{r})a_{f\nu}e^2|\mathbf{r}-\mathbf{R}_i|^{-1}\psi_n\times\exp(i\mathbf{k}_\nu\cdot\mathbf{r})d\tau_Nd\tau\times\sum_{j=1}^{Z}\int\int\bar{\psi}_n'\exp(-i\mathbf{k}_\nu\cdot\mathbf{r}')a_{\nu0}e^2\times|\mathbf{r}'-\mathbf{R}_j'|^{-1}\psi_0'\exp(i\mathbf{k}_0\cdot\mathbf{r}')d\tau_N'd\tau'. \quad (9)$$

Here, the subscript ν refers to the intermediate electron state, and n to the intermediate nuclear state of energy ϵ_n. $\sum_\nu$ consists of a summation S' over all four (positive and negative energy) spin states, and an integration $\int_C d\mathbf{k}_\nu/8\pi^3$, where the contour C passes under the pole on the positive k_ν-axis.

The differential cross section through second order is $\sigma^{(1)}+\sigma^{(2)}$, where $\sigma^{(1)}$ is given by Eq. (8) and $\sigma^{(2)}$ arises from the cross terms between $V_{f0}^{(1)}$ and $V_{f0}^{(2)}$:

$$\sigma^{(2)}=(k_f/2\pi\hbar c)^2\tfrac{1}{2}S_fS_0\bar{V}_{f0}^{(1)}V_{f0}^{(2)}+\text{c.c.}; \quad (10)$$

here, "c.c." denotes complex conjugate. In the extreme relativistic region:

$$\begin{aligned}&\tfrac{1}{2}S_fS_0S_\nu'\bar{a}_{f0}(\epsilon_n-\epsilon_0+E_\nu-E_0)^{-1}a_{f\nu}a_{\nu0}\\&=\tfrac{1}{2}(E_\nu^2-E_n^2)^{-1}S_fS_0S_\nu'a_{0f}a_{f\nu}(E_\nu+E_n)a_{\nu0}\\&=\tfrac{1}{2}(E_\nu^2-E_n^2)^{-1}[E_n(1+\cos\theta)+|E_\nu|(\cos\theta_{0\nu}+\cos\theta_{f\nu})],\end{aligned} \quad (11)$$

where $E_n\equiv E_0+\epsilon_0-\epsilon_n$, $\theta_{0\nu}$ is the angle between $\mathbf{k}_0$ and $\mathbf{k}_\nu$, and $\theta_{f\nu}$ is the angle between $\mathbf{k}_f$ and $\mathbf{k}_\nu$. We therefore obtain from substitution of Eqs. (9) and (11) into (10), and Eqs. (5) and (7) into (2):

$$\sigma^{(2)}=-(k_f/2\pi\hbar c)^2(4\pi e^2/q^2)\bar{F}(\mathbf{q})\times\int_C(d\mathbf{k}_\nu/8\pi^3)\cdot\tfrac{1}{2}\sum_n(E_\nu^2-E_n^2)^{-1}\times[E_n(1+\cos\theta)+|E_\nu|(\cos\theta_{0\nu}+\cos\theta_{f\nu})]\times\sum_{i=1}^{Z}\int\int\bar{\psi}_fe^2|\mathbf{r}-\mathbf{R}_i|^{-1}\psi_n\times\exp[i(\mathbf{k}_\nu-\mathbf{k}_f)\cdot\mathbf{r}]d\tau_Nd\tau\times\sum_{i=1}^{Z}\int\int\bar{\psi}_n'e^2|\mathbf{r}'-\mathbf{R}_j'|^{-1}\psi_0'\times\exp[i(\mathbf{k}_0-\mathbf{k}_\nu)\cdot\mathbf{r}']d\tau_N'd\tau'+\text{c.c.} \quad (12)$$

The two most promising methods for dealing with Eq. (12) are (1) to first perform the $\mathbf{r},\mathbf{r}'$ integrations, and (2) to first perform the $\mathbf{k}_\nu$ integration. The first procedure can be carried through by adding and subtracting $i(\mathbf{k}_\nu-\mathbf{k}_f)\cdot\mathbf{R}_i$ in the first exponent, and integrating over a new variable $\mathbf{r}-\mathbf{R}_i$ in place of $\mathbf{r}$. The integration over the first set of nuclear coordinates then gives a form factor with the argument $\mathbf{k}_\nu-\mathbf{k}_f$. The second (primed) set of coordinates is treated in the same way. The difficulty here is that the resulting integration over $\mathbf{k}_\nu$ cannot be performed analytically even for extremely simple assumptions about the form factor,[6] and more important, we were unable to find approximations that would enable us to estimate orders of magnitude.

We have therefore investigated the second procedure. The $\mathbf{k}_\nu$ integration is easily carried through with the help of a convergence factor, and the remaining integrations over space coordinates can then be estimated by making use of the closure relation for the nuclear states and the assumed disparity in magnitude between $1/k_0$ and the nuclear radius. The $\mathbf{k}_\nu$ integration leads to

$$\sigma^{(2)}=-(e^6k_f^2/8\pi^2\hbar^3c^3q^2)\bar{F}(\mathbf{q})\sum_n\sum_{i=1}^{Z}\sum_{j=1}^{Z}\int\int\int\int\times[K_n(1+\cos\theta+\cos\theta_{0\rho}+\cos\theta_{f\rho})+(i/\rho)(\cos\theta_{0\rho}+\cos\theta_{f\rho})]\times\exp i(K_n\rho+\mathbf{k}_0\cdot\mathbf{r}'-\mathbf{k}_f\cdot\mathbf{r})(\rho|\mathbf{r}-\mathbf{R}_i||\mathbf{r}'-\mathbf{R}_j'|)^{-1}\times\bar{\psi}_f\psi_n\bar{\psi}_n'\psi_0'd\tau_Nd\tau_N'd\tau d\tau'+\text{c.c.} \quad (13)$$

Here, $K_n\equiv E_n/\hbar c$, $\boldsymbol{\varrho}\equiv\mathbf{r}-\mathbf{r}'$, $\theta_{0\rho}$ is the angle between $\mathbf{k}_0$ and $\boldsymbol{\varrho}$, and $\theta_{f\rho}$ is the angle between $\mathbf{k}_f$ and $\boldsymbol{\varrho}$.

[6] The corresponding integrals with unit form factor (point charge) are evaluated by R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951), who also cites earlier papers; this and other cases are discussed in reference 5.

III. ESTIMATION PROCEDURE

The closure relation for the nuclear wave functions states that

$$\sum_n \int \psi_n \bar{\psi}_n' g(\mathbf{R}_j') d\tau_N' = g(\mathbf{R}_j),$$

for any function g that is antisymmetric in the coordinates of identical nucleons. This cannot be applied directly to Eq. (13) since the summand there also depends on n through the quantity K_n. However, we assume that the main terms in the summand correspond to moderate excitation of the nucleus, for which K_n is not greatly different from k_0. Then K_n and $\exp(iK\rho)$ are slowly varying functions of n, and we assume that we can write to good approximation:

$$\sum_n \int f(K_n) \psi_n \bar{\psi}_n' g(\mathbf{R}_j') d\tau_N' = f(K) g(\mathbf{R}_j), \quad (14)$$

where we expect K to be of the order of and probably somewhat less than k_0.[7] With this assumption, Eq. (13) becomes

$$\sigma^{(2)} = -(e^6 k_f^2/8\pi^2\hbar^3 c^3 q^2) \bar{F}(h) \sum_{i=1}^{Z} \sum_{j=1}^{Z} \int\int\int [K] \times \exp\{i(K\rho + \mathbf{k}_0 \cdot \mathbf{r}' - \mathbf{k}_f \cdot \mathbf{r})\} \times (\rho |\mathbf{r} - \mathbf{R}_i| |\mathbf{r}' - \mathbf{R}_j'|)^{-1} \bar{\psi}_f \psi_0 d\tau_N d\tau d\tau' + \text{c.c.}, \quad (15)$$

where $[K]$ denote the square bracket of Eq. (13) with K_n replaced by K.

We now break up Eq. (15) into two parts, according as $i=j$ or $i \neq j$. The first (one-proton) part can be rewritten with the help of Eq. (5) as

$$-(e^6 k_f^2/8\pi^2\hbar^3 c^3 q^2) \bar{F}(\mathbf{q}) \int\int\int [K] \times \exp\{i(K\rho + \mathbf{k}_0 \cdot \mathbf{r}' - \mathbf{k}_f \cdot \mathbf{r})\} (\rho |\mathbf{r} - \mathbf{R}_a| |\mathbf{r}' - \mathbf{R}_a|)^{-1} \times \rho_{f0}^{(1)}(\mathbf{R}_a) d\tau_a d\tau d\tau' + \text{c.c.} \quad (16)$$

The second (two-proton) part can be expressed in terms of the two-proton transition charge density

$$\rho_{f0}^{(2)}(\mathbf{R}_a, \mathbf{R}_b) \equiv \int \cdots \int \bar{\psi}_f(\mathbf{R}_1 \cdots \mathbf{R}_A) \sum_{i=1}^{Z} \sum_{j \neq i}^{Z} \delta(\mathbf{R}_a - \mathbf{R}_i) \times \delta(\mathbf{R}_b - \mathbf{R}_j) \psi_0(\mathbf{R}_1 \cdots \mathbf{R}_A) d\tau_1 \cdots d\tau_A, \quad (17)$$

which has the properties

$$\rho_{f0}^{(2)}(\mathbf{R}_a, \mathbf{R}_b) = \rho_{f0}^{(2)}(\mathbf{R}_b, \mathbf{R}_a), \qquad \int \rho_{f0}^{(2)}(\mathbf{R}_a, \mathbf{R}_b) d\tau_b = (Z-1) \rho_{f0}^{(1)}(\mathbf{R}_a). \quad (18)$$

[7] Note that even when it is assumed that the right side of Eq. (14) is separable, as written above, K need not lie within the range of variation of K_n, since $\psi_n \bar{\psi}_n'$ is not always of one sign as n changes for fixed $\mathbf{R}_i$, $\mathbf{R}_j'$.

The result is

$$-(e^6 k_f^2/8\pi^2\hbar^3 c^3 q^2) \bar{F}(\mathbf{q}) \int\int\int\int [K] \times \exp\{i(K\rho + \mathbf{k}_0 \cdot \mathbf{r}' - \mathbf{k}_f \cdot \mathbf{r})\} \times (\rho |\mathbf{r} - \mathbf{R}_a| |\mathbf{r}' - \mathbf{R}_b|)^{-1} \times \rho_{f0}^{(2)}(\mathbf{R}_a, \mathbf{R}_b) d\tau_a d\tau_b d\tau d\tau' + \text{c.c.} \quad (19)$$

Equation (16) can be simplified by adding and subtracting $i(\mathbf{k}_0 - \mathbf{k}_f) \cdot \mathbf{R}_a$ in the exponent, and replacing the integrations over $\mathbf{r}$ and $\mathbf{r}'$ by integrations over new variables $\mathbf{s} \equiv \mathbf{r} - \mathbf{R}_a$ and $\mathbf{s}' \equiv \mathbf{r}' - \mathbf{R}_a$. Then with the help of (7), Eq. (16) becomes

$$-(e^6 k_f^2/8\pi^2\hbar^3 c^3 q^2) |F(\mathbf{q})|^2 \int\int [K] \times \exp\{i(K\rho + \mathbf{k}_0 \cdot \mathbf{s}' - \mathbf{k}_f \cdot \mathbf{s})\} (\rho s s')^{-1} d\tau d\tau' + \text{c.c.}, \quad (20)$$

where $\boldsymbol{\rho} = \mathbf{s} - \mathbf{s}'$. Equation (20), omitting the factor $|F(\mathbf{q})|^2$, can now be recognized as being close to a special case of Eq. (13). In the latter we assume for the moment that only the $n=0$ term contributes to the sum, that the initial and final states are the same, and that ψ_0 represents a single point charge. This corresponds to elastic scattering from an infinitely massive point proton, in which case $K_n = k_0$ and $\bar{F}(\mathbf{q}) = 1$. Then Eq. (13) is known from previous work[6] to be equal to

$$\sigma^{(2)}(\text{point}) = +(\pi e^6/\hbar^3 c^3 q^2)(\operatorname{cosec}^2 \tfrac{1}{2}\theta - 1). \quad (21)$$

If now we assume that K in Eq. (20) is nearly equal to k_0, Eq. (16) or (20) is approximately equal to $|F(\mathbf{q})|^2$ multiplied by Eq. (21).[8] Combining this with Eq. (8), we find that the sum of the first-order scattering and the one-proton part of the second-order scattering is

$$(4e^4 E_f^2/\hbar^4 c^4 q^4) |F(\mathbf{q})|^2 \times [\cos^2 \tfrac{1}{2}\theta + (\pi e^2/\hbar c)(\sin^2 \tfrac{1}{2}\theta - \sin^2 \tfrac{1}{2}\theta)]. \quad (22)$$

To this must be added the two-proton part of the second-order scattering, which we now proceed to estimate from Eq. (19).

We start by changing the four variables of integration from $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{R}_a$, $\mathbf{R}_b$ to

$$\mathbf{s} \equiv \mathbf{r} - \mathbf{R}_a, \quad \mathbf{s}' \equiv \mathbf{r}' - \mathbf{R}_b, \quad \mathbf{S} \equiv \tfrac{1}{2}(\mathbf{R}_a + \mathbf{R}_b), \quad \boldsymbol{\rho} \equiv \mathbf{r} - \mathbf{r}';$$

the Jacobian of this transformation is equal to unity. The factor $\mathbf{k}_0 \cdot \mathbf{r}' - \mathbf{k}_f \cdot \mathbf{r}$ in the exponential then becomes

$$\mathbf{q} \cdot \mathbf{S} - \mathbf{K}_0 \cdot \boldsymbol{\rho} + \tfrac{1}{2}\mathbf{q} \cdot \mathbf{s} + \tfrac{1}{2}\mathbf{q} \cdot \mathbf{s}', \quad \text{where} \quad \mathbf{K}_0 \equiv \tfrac{1}{2}(\mathbf{k}_0 + \mathbf{k}_f).$$

Equation (19) is now equal to

$$-(e^6 k_f^2/8\pi^2\hbar^3 c^3 q^2) \bar{F}(\mathbf{q}) \int\int\int\int [K] \times \exp\{i(\mathbf{q} \cdot \mathbf{s} + K\rho - \mathbf{K}_0 \cdot \boldsymbol{\rho} + \tfrac{1}{2}\mathbf{q} \cdot \mathbf{s} + \tfrac{1}{2}\mathbf{q} \cdot \mathbf{s}')\} \times (\rho s s')^{-1} \rho_{f0}^{(2)}(\mathbf{R}_a, \mathbf{R}_b) d\tau_S d\tau_\rho d\tau d\tau' + \text{c.c.} \quad (23)$$

[8] As pointed out by Lewis (reference 5, pp. 69–70), this approximate equality becomes exact in his static limit.

In order to estimate Eq. (23), we assume that $\rho_{f0}^{(2)}$ is a slowly varying function of its arguments, and consider first the dependence of the integrand on $\mathbf{s}$. If $\frac{1}{2}q$ is large in comparison with $1/R$, where R is the nuclear radius (more precisely, R is the distance over which $\rho_{f0}^{(2)}$ varies appreciably), the rapid oscillations of $\exp(\frac{1}{2}i\mathbf{q}\cdot\mathbf{s})$ prevent contributions to the integral from points much more distant from each other than $2/q$ about the point at which the coefficient of $\exp(\frac{1}{2}i\mathbf{q}\cdot\mathbf{s})$ is a maximum, namely $\mathbf{s}=0$. This conjecture is readily verified by examining the integral

$$\int s^{-1}\exp(\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{s}-\alpha s)d\tau=4\pi/(\tfrac{1}{4}q^2+\alpha^2).$$

As long as $\alpha\ll\frac{1}{2}q$, this is independent of α; in other words, as long as $\exp(-\alpha s)$ is not appreciably different from unity within a distance of order $2/q$ of the origin, it makes little difference how it behaves. The same type of behavior can be demonstrated for other smoothly varying coefficients, such as $\exp(-\beta s^2)$ in place of $\exp(-\alpha s)$.

We thus conclude that the bulk of the $\mathbf{s}$ integration comes from a region of approximate linear dimensions $2/q$ about the origin. The same is true of the $\mathbf{s}'$ integration. In the case of the $\boldsymbol{\rho}$ integration, the length that corresponds to $2/q$ above is $1/(K-K_0)$, since $\exp i(K\rho-\mathbf{K}_0\cdot\boldsymbol{\rho})$ varies most slowly when $\boldsymbol{\rho}$ is parallel to $\mathbf{K}_0$. If we assume that k_f and K are roughly equal to k_0, then $q=2k_0\sin\frac{1}{2}\theta$ and $K-K_0=k_0(1-\cos\frac{1}{2}\theta)$. Thus the present approximation requires that both $k_0R\sin\frac{1}{2}\theta$ and $k_0R(1-\cos\frac{1}{2}\theta)$ be large in comparison with unity. Since the first of these quantities is never smaller than the second, we require that

$$k_0R(1-\cos\tfrac{1}{2}\theta)\gg1. \tag{24}$$

The $\mathbf{s}$, $\mathbf{s}'$ and $\boldsymbol{\rho}$ integrations in (23) may now be carried out separately with the help of convergence factors, and $\mathbf{R}_a$ and $\mathbf{R}_b$ in $\rho_{f0}^{(2)}$ replaced by their values when $\mathbf{s}=\mathbf{s}'=\boldsymbol{\rho}=0$, namely $\mathbf{R}_a=\mathbf{R}_b=\mathbf{S}$. The $\mathbf{s}$ and $\mathbf{s}'$ integrals are each equal to $16\pi/q^2$, and the $\boldsymbol{\rho}$ integral is equal to $(8\pi/k_0)\{1-[K^2+Kk_0\cos^2(\frac{1}{2}\theta)]/(K^2-K_0^2)\}$ if we assume that $k_f\cong k_0$. If we further assume that $K\cong k_0$, this last becomes $-(16\pi/k_0)\cot^2(\frac{1}{2}\theta)$. With these approximations, Eq. (23) becomes

$$+(1024\pi e^6k_f^2/\hbar^3c^3k_0q^6)\cot^2(\tfrac{1}{2}\theta)$$
$$\times\mathrm{Re}\left[\bar{F}(\mathbf{q})\int\rho_{f0}^{(2)}(\mathbf{S},\mathbf{S})\exp(i\mathbf{q}\cdot\mathbf{S})d\tau_S\right], \tag{25}$$

where Re denotes the real part of the following square bracket.

As a further estimate, we assume that the value of $\rho_{f0}^{(2)}(\mathbf{R}_a,\mathbf{R}_b)$ when $\mathbf{R}_b=\mathbf{R}_a$ is approximately equal to its average value as $\mathbf{R}_b$ varies over the nucleus, which is its integral over $\mathbf{R}_b$ divided by the nuclear volume:

$$\rho_{f0}^{(2)}(\mathbf{R}_a,\mathbf{R}_a)\cong(3/4\pi R^3)\int\rho_{f0}^{(2)}(\mathbf{R}_a,\mathbf{R}_b)d\tau_b. \tag{26}$$

Because of the second of Eqs. (18), the right side of (26) is equal to $[3(Z-1)/4\pi R^3]\rho_{f0}^{(1)}(\mathbf{R}_a)$; thus with the help of Eq. (7), we find that

$$\rho_{f0}^{(2)}(\mathbf{S},\mathbf{S})\exp(i\mathbf{q}\cdot\mathbf{S})d\tau_S\cong[3(Z-1)/4\pi R^3]F(\mathbf{q}). \tag{27}$$

Substitution of (27) into (25) gives as our final estimate for the two-proton part of the second-order scattering

$$[768e^6k_f^2(Z-1)/\hbar^3c^3R^3k_0q^6]|F(\mathbf{q})|^2\cot^2(\tfrac{1}{2}\theta); \tag{28}$$

this is to be added to Eq. (22).

IV. DISCUSSION

The importance of the dispersion contribution to the scattering cross section is measured by the ratio of the second to the first bracket term in Eq. (22) for the one-proton part, and by the ratio of (28) to (8) for the two-proton part. The first of these is less than or of order $\pi/137$ for all angles, and hence at most of the order of a few percent. The second ratio is

$$Q\equiv[48(Z-1)e^2/\hbar ck_0^3R^3]\operatorname{cosec}^4(\tfrac{1}{2}\theta). \tag{29}$$

Equation (29) is however only a valid estimate if the inequality (24) is satisfied. If θ is not too large, (24) can be multiplied through by $1+\cos\frac{1}{2}\theta\approx2$ to yield $\operatorname{cosec}^2(\frac{1}{2}\theta)\ll\frac{1}{2}k_0R$. Substitution into (29) then shows that Q must be small in comparison with

$$P\equiv12(Z-1)/137k_0R$$

in order for the estimate to be valid. For 200-Mev electrons, and assuming that $R=1.2A^{\frac{1}{3}}\times10^{-13}$ cm, $P=0.16$ for carbon and $P=0.97$ for gold; for other electron energies, P is inversely proportional to the energy. Thus whenever our estimate (29) is valid, the dispersion correction to the first-order scattering is reasonably small, for electrons in the several hundred Mev region.

The main difficulty with our estimate is its restricted range of validity. If, for example, we interpret (24) as meaning that the left side must be at least equal to 2, then for 200-Mev electrons our estimate is valid for $\theta\geqq148°$ in the case of carbon and $\theta\geqq88°$ in the case of gold, with the limiting angle decreasing as the electron energy increases. For these two limiting angles, the ratio (29) is $Q=0.098$ and $Q=0.35$, respectively. It is apparent from the derivation of (24) that the validity range cannot be increased without more detailed knowledge of the two-proton transition charge density $\rho_{f0}^{(2)}$.

It is worth noting that insofar as our estimation procedure is reliable, much of the dispersion contribution would be included in a first-order calculation that starts from a correct initial wave function. To see this, we write down that part of the first-order perturbed initial state wave function that arises from the nuclear ground state only and from all electron states; in the

notation of Eq. (9), this is

$$\psi_0\Big\{u_0\exp(i\mathbf{k}_0\cdot\mathbf{r})-\sum_\nu(E_\nu-E_0)^{-1}u_\nu\exp(i\mathbf{k}_\nu\cdot\mathbf{r})$$
$$\times\sum_{j=1}^{Z}\int\int\bar{\psi}_0'\exp(-i\mathbf{k}_\nu\cdot\mathbf{r}')a_{\nu 0}e^2|\mathbf{r}'-\mathbf{R}_j'|^{-1}\psi_0'$$
$$\times\exp(i\mathbf{k}_0\cdot\mathbf{r}')d\tau_N'd\tau'\Big\},\quad(30)$$

where u_0 and u_ν are constant spinors. Equation (30) is what would be obtained to first order by regarding the nucleus as a rigid charge distribution, and so is included in the numerically calculated electron wave function.[1] If now Eq. (30) is inserted in the first-order matrix element (1) in place of $\psi_0u_0\exp(i\mathbf{k}\cdot\mathbf{r})$, we obtain Eq. (1) together with that part of Eq. (9) that arises from the $n=0$ term in the summation over intermediate nuclear states n. Thus if we were to make a first-order calculation using the correct initial electron function in place of a plane wave,[9] the correction due to dispersion would not be given by Eq. (13), but by (13) with the term $n=0$ omitted from the summation over n. In such a case we must therefore subtract from (23) the quantity

$$-(e^6k_f^2/8\pi^2\hbar^3c^3q^2)\bar{F}(\mathbf{q})\int\int\int\int[k_0]$$
$$\times\exp\{i(\mathbf{q}\cdot\mathbf{S}+k_0\rho-\mathbf{K}_0\cdot\boldsymbol{\varrho}+\tfrac{1}{2}\mathbf{q}\cdot\mathbf{s}+\tfrac{1}{2}\mathbf{q}\cdot\mathbf{s}')\}$$
$$\times(\rho ss')^{-1}\rho_{f0}^{(1)}(\mathbf{R}_a)\rho_{00}^{(1)}(\mathbf{R}_b)d\tau_S d\tau_\rho d\tau d\tau'+\text{c.c.}\quad(31)$$

in order to obtain the dispersion contribution.

Equation (31) can now be estimated in the same way as Eq. (23), by setting $\mathbf{R}_b=\mathbf{R}_a$, and either approximating $\rho_{f0}^{(1)}(\mathbf{R}_a)$ by $(3/4\pi R^3)\int\rho_{f0}^{(1)}(\mathbf{R}_c)d\tau_c$ or approximating $\rho_{00}^{(1)}(\mathbf{R}_a)$ by $(3/4\pi R^3)\int\rho_{00}^{(1)}(\mathbf{R}_c)d\tau_c$. Because of Eq. (6), the first of these is $3Z\delta_{f0}/4\pi R^3$ and the second is $3Z/4\pi R^3$. If we average these two results, we see that our estimate for Eq. (31) is the same as (28) except that $Z-1$ is replaced by $\frac{1}{2}Z(1+\delta_{f0})$. Thus, very roughly, the use of the correct initial electron wave function is expected to reduce the dispersion correction to inelastic scattering by about a factor of two, and to nearly cancel the dispersion correction to elastic scattering. Another way of arriving at the same conclusion is to assume that in Eq. (13), products of the form $\bar{\psi}_n\psi_m$ are rapidly varying functions of the nuclear coordinates unless $n=m$. Then the principal terms in the sum will arise from $n=0$ for elastic scattering (this is included if the elastic scattering is calculated numerically), or from $n=0$ and $n=f$ for inelastic scattering. In the latter case, $|\psi_f|^2$ should not be greatly different from $|\psi_0|^2$, so that something like half of (13) is obtained by keeping only the term with $n=0$. This suggests that the first-order inelastic scattering calculations could be further improved by using correct electron wave functions for both the initial and final states[9]; the final state function must of course represent asymptotically a plane plus ingoing spherical wave. If this were done, both the $n=0$ and $n=f$ terms in Eq. (13) would be retained.

It should be emphasized that the remarks in the last paragraph are little more than an indication that the situation is improved by using correct electron wave functions instead of plane waves, which is almost self-evident anyhow. Reliable estimates of the degree of improvement cannot be made without more detailed knowledge of the one- and two-proton transition charge densities.

[9] A useful approximation for this purpose has been found by Yennie, Ravenhall, and Downs, Bull. Am. Phys. Soc. No. 8, 29 (1954).

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