# Isotopic Spin Selection Rule for Electric Dipole Transitions

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The validity of the isotopic spin quantum number in nuclei provides a selection rule on electric dipole transitions. The extent to which this selection rule is violated provides a means of experimentally determining the isotopic spin impurity of nuclear states providing that the radiation widths for uninhibited $E1$ radiation can be predicted. The sources of possible variations of $E1$ matrix elements are discussed with reference to the reliability of predicted $E1$ radiation widths in nuclei for $A \leq 20$ and excitation energy $\sim 15$ Mev. Higher order corrections to the $E1$ selection rules are determined and found to be negligible compared to the effects of isotopic spin impurity. It is concluded that the isotopic spin selection rule on $E1$ transitions provides a sensitive test of charge independence. Isotopic spin impurity determined in this way can be attributed solely to the Coulomb potential.

## INTRODUCTION

THE most striking evidence for the validity of isotopic spin in light nuclei is the existence of the selection rules on electric dipole transitions. These selection rules were first derived on the basis of supermultiplet theory by Trainor,[1] but it was pointed out by Radicati[2] and Christy[3] that these restrictions on electric dipole radiation actually follow more generally from properties of the interaction of the electromagnetic field with a system of nucleons and from the hypothesis of charge-independent nuclear forces. A more complete statement of the isotopic spin selection rules was then given by Gell-Mann and Telegdi[4] who also discussed the effect of these selection rules on the absorption cross section for $\gamma$ rays.

The experimental investigation of the validity of the isotopic spin quantum number was undertaken by Wilkinson[5-7] with various collaborators. Although the operation of the selection rule was verified in several cases by the complete absence of $E1$ transitions which should have been detectable, in a number of cases the forbidden $E1$ transitions were actually found to have small but measurable radiative widths. Knowledge of the radiative width to be expected for $E1$ transitions if uninhibited by any isotopic spin selection rule should then enable the experimentalist to estimate the admixture of other isotopic spin eigenstates to the nuclear states involved in the transition.

In order to determine the extent to which the isotopic spin selection rules are operating, therefore, we must know whether there are likely to be any other causes for variation in a particular $E1$ transition. Actually, as Kinsey and Bartholomew[8] have pointed out, very little has been known about $E1$ transitions until quite recently. For this reason we shall discuss in some detail the magnitude of $E1$ transitions and the several inhibiting factors such as the isotopic spin selection rules, correlation of neutron and proton distributions, and complex coupling of single particle states.

The results of Wilkinson on isotopic spin impurity in light nuclei have been thrown into doubt, however, by the observation of Gell-Mann and Telegdi that higher order terms in the $E1$ matrix elements are perhaps sufficient to promote "forbidden" $E1$ transitions to an even greater extent than the isotopic spin impurity would provide. To investigate this question we shall derive the exact electromagnetic multipole moments for nonrelativistic quantum mechanics and with these proceed to estimate the higher order corrections to the isotopic spin selection rules on $E1$ radiation. We shall find that these higher order corrections are $\sim 1000$ times less effective than isotopic spin impurity in producing violations of the isotopic spin selection rules. The estimates of isotopic spin impurity by Wilkinson still stand, subject only to the uncertainty in $E1$ radiative widths. The investigation of $E1$ radiation provides in fact a sensitive test of the hypothesis of charge independence of nuclear forces.

## INHIBITION OF $E1$ TRANSITIONS

The nonrelativistic Schrödinger Hamiltonian for the interaction of a system of nucleons with an electromagnetic field can be written in the isotopic spin formalism correct to first order in the vector potential $\mathbf{A}$

$$H_I = -\sum_{i=1}^{A}\left\{\frac{e}{Mc}\mathbf{p}_i\cdot\mathbf{A}(\mathbf{r}_i)(\tfrac{1}{2}-t_{\zeta i}) + [\mu_P(\tfrac{1}{2}-t_{\zeta i})+\mu_N(\tfrac{1}{2}+t_{\zeta i})]\boldsymbol{\sigma}_i\cdot[\nabla\times\mathbf{A}(\mathbf{r}_i)]\right\}, \quad (1)$$

where the isotopic spin operator $t_\zeta$ has the eigenvalue

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[1] L. E. H. Trainor, Phys. Rev. **85**, 962 (1952).
[2] L. A. Radicati, Phys. Rev. **87**, 521(L) (1952).
[3] R. F. Christy, Pittsburgh Conference on Medium Energy Nuclear Physics, 1952 (unpublished).
[4] M. Gell-Mann and V. L. Telegdi, Phys. Rev. **91**, 169 (1952).
[5] D. H. Wilkinson, Phil. Mag. **44**, 547, 1019 (1953).
[6] D. H. Wilkinson and A. B. Clegg, Phil. Mag. **44**, 1269, 1322 (1953).
[7] G. A. Jones and D. H. Wilkinson, Phys. Rev. **90**, 722 (1953).
[8] G. A. Bartholomew and B. B. Kinsey, Phys. Rev. **93**, 1260 (1954).

$+\frac{1}{2}$ for a neutron and $-\frac{1}{2}$ for a proton. The neutron-proton mass difference has been neglected as is certainly justified in first approximation. This equation differs from that of Radicati[2] and Gell-Mann and Telegdi[4] by a sign because our $e$ is positive for protons and negative for electrons. The $H_I$ in (1) is the sum of a part $H_0$ which is a scalar in isotopic spin space and a part $H_1$ which is $\zeta$ component of an isotopic spin vector.

$$H_0=-\sum_{i=1}^{A}\left\{\frac{e}{2Mc}\mathbf{p}_i\cdot\mathbf{A}(\mathbf{r}_i)+\tfrac{1}{2}(\mu_P+\mu_N)\boldsymbol{\sigma}_i\cdot[\nabla\times\mathbf{A}(\mathbf{r}_i)]\right\}, \quad (2)$$

$$H_1=\sum_{i=1}^{A}\left\{\frac{e}{Mc}\mathbf{p}_i\cdot\mathbf{A}(\mathbf{r}_i)+(\mu_P-\mu_N)\boldsymbol{\sigma}_i\cdot[\nabla\times\mathbf{A}(\mathbf{r}_i)]\right\}t_{\zeta i}. \quad (3)$$

If one makes the assumption $(kR)\ll 1$, one can demonstrate that the electric dipole transitions induced by $H_0$ are proportional to the square of the electric dipole moment $Q_{1m}$ between the initial and final nuclear states, $\varphi_i(1,\cdots,A)$ and $\varphi_f(1,\cdots,A)$, which depend on the relative coordinates of $A$ nucleons.

$$Q_{1m}^{(0)}=\tfrac{1}{2}\sum_{k=1}^{A}e\int r_kY_{1m}^*(\vartheta_k,\varphi_k)\times\varphi_f^*(1,\cdots,A)\varphi_i(1,\cdots,A). \quad (4)$$

Correspondingly, $E1$ transitions induced by $H_1$ have transition probabilities proportional to the square of

$$Q_{1m}^{(1)}=\sum_{k=1}^{A}et_{\zeta k}\int r_kY_{1m}^*(\vartheta_k,\varphi_k)\times\varphi_f^*(1,\cdots,A)\varphi_i(1,\cdots,A). \quad (5)$$

If we now neglect nuclear recoil on emission of a $\gamma$ ray, the center of mass of the nucleus can be taken as the origin and $\sum_{k=1}^{A}\mathbf{r}_k=0$. Since (4) vanishes only $H_1$ can produce $E1$ transitions.

From the fact that $H_1$ is the $\zeta$ component of a vector in isotopic spin space, the following selection rules on $E1$ radiation follow immediately[9]:

$$\begin{aligned}&\Delta T=0,\pm1,\quad T_\zeta\neq 0;\\ &\Delta T=\pm1,\quad\ \ T_\zeta=0.\end{aligned} \quad (6)$$

The selection rule for $T_\zeta=0$ $(N=Z)$ nuclei is a generalization of Radicati's requirement of no $T=0\rightarrow T=0$ transitions. The absolute validity of (6) for $T_\zeta=0$ nuclei can be impaired either by the impurity of isotopic spin states or by certain higher order terms in the multipole matrix elements.

In addition to the inhibition of $E1$ transitions by the isotopic spin selection rules, there exists the possibility of inhibition of $E1$ matrix elements by the correlation of the neutrons and protons in the nucleus. This effect upon the matrix elements of $H_1$ follows immediately from (5) where we note that the terms of $\sum_{k=1}^{A}t_{\zeta k}\mathbf{r}_k$ referring to neutrons will have opposite signs from those referring to protons. This diminution of $E1$ matrix elements was pointed out by Delbruck and Gamow,[10] and was later derived by Bethe[11] in a more general form to explain the anomalously small $E1$ transition probabilities thought to occur in heavy nuclei. The evidence for small $E1$ transition probabilities at that time was of several kinds: (1) anomalously long lifetimes for low-lying states in heavy nuclei ($10^{-12}$ sec *versus* theoretical estimated $10^{-16}$ for $E1$), (2) radiative widths for $(n,\gamma)$ reactions, (3) $Li^7(p,\gamma)Be^8$, 17.63 Mev $\gamma$ ray, and (4) radiative widths observed in $(p,\gamma)$ capture in $F^{19}$.

Since considerably more experimental information has recently been obtained on isomeric transitions, we shall discuss this evidence for a neutron-proton correlation effect on $E1$ matrix elements. We shall be interested particularly in the implications for inhibition of $E1$ transitions in light nuclei $(A\leq 20)$ for moderate excitation energies $(E\sim 15$ Mev).

As remarked by Bartholomew and Kinsey,[8] the evidence in (1) has lost most of its significance since more accurate tables of internal conversion coefficients have shown that these $E1$ transitions in heavy nuclei are really $E2$ transitions. Although several true $E1$ transitions in heavy nuclei have been detected recently[12,13] and these have low-transition probabilities, it has been suggested that the emitting state is formed by a complex coupling of several nucleons with a consequent reduction of the extreme single-particle matrix element. That complex coupling can reduce transition probabilities by factors of $\sim$100 has been demonstrated by Lane and Radicati.[14] In particular their results show that a variation by a factor of 50 depending on the coupling is possible for the $E1$ matrix element from the 7.48-Mev state of $B^{10}$ to the ground state. The reason for the small number of $E1$ transitions in heavy nuclei and consequent long lifetimes for the low-lying states may be simply the absence of low-lying states of the proper spin and parity.

As stated in (2) we find that the radiative widths derived from $(n,\gamma)$ reactions in heavy nuclei are rather narrower than one should expect on the basis of simple theory. Yet from the existence of a sum rule for $E1$ transitions it is clear that the widths of $E1$ resonances for high-energy $\gamma$ rays cannot be given by the simple theory.[15] Blatt and Weisskopf[16] have suggested in fact that the radiation width for a single level should be inversely proportional to the level density at the posi-

[9] E. P. Wigner, *Gruppentheorie* (Edwards Brothers, Inc., Ann Arbor, Michigan, 1944).

[10] M. Delbruck and G. Gamow, Z. Physik **72**, 492 (1931).
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[15] V. Weisskopf, Phys. Rev. **83**, 1073 (1951).
[16] J. M. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).

tion of the emitting state. Such a dependence of level widths on level spacing does produce a reduction of $E1$ matrix elements which is in agreement with experiment.[8] This reduction of level width cannot be interpreted as a neutron-proton correlation effect, however, since the ratio both of $E2$ to $E1$ level widths and of $M1$ to $E1$ level widths seem to be correctly predicted by the Weisskopf formulas.[8]

In any case, for light nuclei and excitation energies less than ~15 Mev, the dependence of matrix elements on level density should cause no appreciable reduction of widths of $E1$ emission.[6]

The evidence in (3) is that the 17.63-Mev state in $Be^8$ displays a width for radiative decay which is several hundred times smaller than one should expect for an $E1$ transition. More recent experimental results suggest, however, that the 17.63-Mev state is (1+) in spin and parity. Since the ground state of such an even-even nucleus is expected to be (0+), the radiative decay of the 17.63-Mev state to the ground state must be magnetic dipole or electric quadrupole. This is in agreement with experiment.[17]

The experimental evidence mentioned in (4) has been clarified quite recently with the separation of the 6-Mev background from $F^{19}(p,\gamma)O^{16*}$ and identification of the spins and parities of the states in $Ne^{20}$ lying just above the threshold for proton capture.[18] The only capture radiation which has been observed in $F^{19}(p,\gamma)Ne^{20}$ up to 1 Mev above the proton capture threshold is from the 13.51-Mev state, which emits a $\gamma$ ray of ~12 Mev. Since the 1.63-Mev level is now known to be (2+), $E1$ emission by the (2−) levels at 13.70 and 13.44 Mev to this level should be observed also. The reduced widths for alpha-particle emission of both these levels, however, strongly imply that these are $T=0$ states. The isotopic spin selection rule should operate therefore to inhibit $E1$ transitions to any of the low-lying $T=0$ states. This explanation of the $E1$ inhibition has been suggested by Wilkinson[6] and used by him as the basis of a determination for the upper limit of 3 percent intensity of isotopic spin impurity in the 13.70-Mev state. The absence of $E1$ radiation by any of the other states lying just above the proton capture threshold in $Ne^{20}$ is explained by the absence of lower levels in $Ne^{20}$ with the proper spin and parity.

Some evidence for the existence of an effect on $E1$ transitions arising from the collective motion of neutrons and protons can be drawn from the existence of the giant resonances in photoemission cross sections[19–25] which occur at 15–25 Mev excitation of the product nucleus. These resonances have been interpreted by Goldhaber and Teller[26] as due to the relative vibration of two interpenetrating spheres containing separately the neutrons and protons. Levinger and Bethe[27] have shown, however, that many of the results of Goldhaber and Teller follow directly from the sum rules. The principal success of Goldhaber and Teller's model is that of predicting nearly correctly the position in energy of the maximum in the photoemission cross sections as a function of atomic number. Whether a model not embodying the concept of collective motion of the neutrons and protons can also predict this dependence remains to be seen.

The conclusion from this discussion of the $E1$ transitions is that at moderate energies (~15 Mev) there is slight evidence for some inhibition of $E1$ transitions in light nuclei,[5] but that this inhibition is not nearly so pronounced as one thought previously. It is not even certain that this remaining apparent diminution of $E1$ matrix elements is not due to overestimates inherent in the approximate nature of the Weisskopf[15] formulas.

## EXPERIMENTAL RESULTS ON ISOTOPIC SPIN SELECTION RULES

The experimental investigation of the selection rules undertaken by Wilkinson[5–7] began with the determination of a semi-empirical method for predicting uninhibited (by isotopic spin selection rules) $E1$ radiation widths. This formula was then used to check for the allowed $\gamma$ rays expected in nuclei belonging to isobaric triads in which the selection rule should operate. Finding these allowed $\gamma$ rays Wilkinson and collaborators then proceeded to examine the $E1$ transitions in $T_\zeta=0$ nuclei between states of the same isotopic spin $T$. Transitions which were inhibited so strongly that they could not be observed provided upper limits on isotopic spin impurity. Transitions which were observed with greatly reduced radiative widths provided lower limits on isotopic spin impurity. These limits are given in Table I.

As is evident from the necessarily tentative analysis of the experiments, considerable doubt should be attached to the exact values of the impurity and, in some cases, to the order of magnitude. Perhaps the most reliable value is that for $N^{14}$ where the forbidden

TABLE I. Experimental estimates of isotopic spin impurity.

| Nucleus | Level | Energy (Mev) | Limits on isotopic spin impurity |
|---|---|---|---|
| $B^{10}$ | (2−) | 5.11 | $<3\times10^{-3}$ |
| $N^{14}$ | $1^-\rightarrow0^+$ | $8.06 \rightarrow 2.31$ | $<2\times10^{-2}$ |
| $O^{16}$ | $1^-\rightarrow0^+$ | $7.116 \rightarrow$ ground | $>4\times10^{-6}$ |
| | $2^+\rightarrow3^-$ | $6.913 \rightarrow 6.137$ | $<10^{-3}$ |
| | $1^-\rightarrow0^+$ | $12.09 \rightarrow$ ground | $>3\times10^{-2}$ |
| $Ne^{20}$ | $2^-\rightarrow2^+$ | $13.70 \rightarrow 1.63$ | $<3\times10^{-2}$ |

[17] F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **24**, 321 (1954).
[18] The second article of reference 6 contains a brief discussion of the experimental data on the relevant levels of $Ne^{20}$.
[19] W. Bothe and W. Gentner, Z. Physik **112**, 45 (1939).
[20] G. C. Baldwin and G. S. Klaiber, Phys. Rev. **73**, 1156 (1948).
[21] J. L. Lawson and M. L. Perlman, Phys. Rev. **74**, 1190 (1948).
[22] M. L. Perlman and G. Friedlander, Phys. Rev. **74**, 442 (1948).
[23] J. McElhinney *et al.*, Phys. Rev. **75**, 542 (1948).
[24] K. Strauch, Phys. Rev. **81**, 973 (1951).
[25] Montalbetti, Katz, and Goldemberg, Phys. Rev. **91**, 659 (1953).
[26] M. Goldhaber and E. Teller, Phys. Rev. **74**, 1046 (1948).
[27] J. S. Levinger and H. Bethe, Phys. Rev. **78**, 115 (1950).

$8.06 \to 2.31$-Mev $E1$ transition is compared with the allowed $E1$ transition to the ground state. The impurities obtained for the levels of $O^{16}$ are obtained from the $E1$ and $E2$ branching ratio from a level. Among isomeric transitions, however, the $E2$ have been known for some time to be unusual because of their large matrix elements which equal or exceed the Weisskopf estimates. The $E1$ transitions, on the other hand, were found by Wilkinson to have transition probabilities only $\frac{1}{5}$ of those predicted by the Weisskopf formulas.[5] It is therefore quite possible that the upper limit of $10^{-3}$ quoted for isotopic spin impurity for the low-lying levels ($\sim$7 Mev) in $O^{16}$ is too small.

All these experimental results are in agreement with the theoretical estimates of isotopic spin impurity introduced by Coulomb forces.[28,29] Indeed we shall show that the higher order corrections to the electric dipole selection rule are negligible compared to the effect of isotopic spin impurity. The experimental values given by Wilkinson must be interpreted therefore as the isotopic spin impurity.

## HIGHER ORDER CORRECTIONS TO THE ISOTOPIC SPIN SELECTION RULES

The selection rules on isotopic spin changes for $E1$ transitions are based on the vanishing of the $H_0$ matrix element [Eqs. (2), (4)] in the lowest order approximation. There are three higher order effects to be considered: (1) neutron-proton mass difference, (2) lowest order electric dipole matrix element of the spin dependent part of $H_0$, and (3) higher order terms in the expressions for the electric dipole moment.

The first of these effects is easily dismissed by noting that the dipole moment of the nucleus involves $\sum_{k=1}^{A}\mathbf{r}_k$ Taking the center of mass as the origin,

$$m_p\sum_{k=1}^{z}\mathbf{r}_k+m_n\sum_{k=z+1}^{A}\mathbf{r}_k =(m_p-m_n)\sum_{k=1}^{z}\mathbf{r}_k+m_n\sum_{k=1}^{A}\mathbf{r}_k=0, \quad (7)$$

where $m_p$ is the proton mass, $m_n$ the neutron mass. Therefore,

$$\sum_{k=1}^{A}\mathbf{r}_k=\left(\frac{m_p-m_n}{m_n}\right)\sum_{k=1}^{z}\mathbf{r}_k, \quad (8)$$

so that the effect of the center of mass and the center of charge nearly coinciding is to reduce the single-particle transition probability by

$$\left(\frac{m_p-m_n}{m_n}\right)^2\approx 2\times 10^{-6}.$$

Since the isotopic spin impurity produces $E1$ widths which are at least $10^{-3}$ times the single-particle widths, the neutron-proton mass difference can be neglected in considering the violation of the isotopic spin selection rules.

In order to treat the corrections (2) and (3), we shall need expressions for the electromagnetic transition probabilities in terms of the exact multipole moments. The usual expressions for the transition probabilities are derived in the long-wavelength approximation $(kR)\ll 1$, where $k$ is the propagation vector for the emitted photon and $R$ is the nuclear radius. Higher order terms in $(kR)$ will therefore enter the exact electric dipole matrix elements of $H_0$ and will produce transitions which violate the selection rules.

## EXACT NONRELATIVISTIC MULTIPOLE TRANSITION PROBABILITIES

We shall derive the quantum mechanical transition probabilities for emission of electric and magnetic multipole radiation of all orders correct to all orders of $(kR)$. We begin directly with the Schrödinger equation for a charged particle in an electromagnetic field so that our results can be compared directly with the usual formulas. The transition probabilities for the interaction of nucleons with an electromagnetic field as given by Eq. (1) then can be written down immediately if desired. To first order in the vector potential $\mathbf{A}$, the interaction Hamiltonian for a charged particle in a radiation field is

$$H_I=-(e/mc)(\mathbf{A}\cdot\mathbf{p})-\mu\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\times\mathbf{A}), \quad (9)$$

where $\mathbf{A}$ satisfies the gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A}=0$. The transition probability per unit time for emission of radiation is given by

$$T_{a\to b}=(2\pi/\hbar)|H_{ab}|^2\rho(E), \quad (10)$$

where $a$ and $b$ are indices denoting the initial and final states, $\varphi_a$ and $\varphi_b$, respectively of the nucleus and $\rho(E)$ is the density of final states per unit energy. $H_{ab}$ is not the matrix element of $H_I$ but of the part of $H_I$ arising from the positive frequency component of $\mathbf{A}$, i.e., from $\mathbf{A}(\mathbf{r})$ defined in

$$\mathbf{A}=\mathbf{A}(\mathbf{r})e^{-i\omega t}+\mathbf{A}^*(\mathbf{r})e^{i\omega t}. \quad (11)$$

The vector potential is now expanded into a series by use of the vector spherical harmonics $Y_{JL1}{}^M(\vartheta,\varphi)$ which are eigenfunctions of the angular momentum operator $L^2$, $J^2$, and $J_Z$, where

$$\mathbf{L}=(\hbar/i)\mathbf{r}\times\boldsymbol{\nabla}\quad \mathbf{J}=\mathbf{L}+\mathbf{S}\quad \mathbf{S}=i\hbar\mathbf{e}_k\times, \quad (12)$$

and $\mathbf{e}_k$ is a unit vector along the $x_k$ axis. For representing the Maxwell fields it is well known that one can use $X_{JM}(\vartheta,\varphi)=Y_{JJ1}{}^M$ and $\boldsymbol{\nabla}\times\mathbf{X}_{JM}$ to expand $\mathbf{A}(\mathbf{r},t)$. The part of $A$ represented by terms in $X_{JM}$ provides a magnetic field of parity $(-1)^l$ and is called the electric multipole field $\mathbf{A}_E(\mathbf{r},t)$. The sum of terms in $\mathbf{A}(\mathbf{r},t)$ which contain $\boldsymbol{\nabla}\times\mathbf{X}_{JM}$ is called the magnetic

[28] L. A. Radicati, Proc. Phys. Soc. (London) **A66**, 139 (1953).
[29] W. M. MacDonald, Princeton thesis, 1954 (unpublished).

multipole field $\mathbf{A}_M(\mathbf{r},t)$ and has an associated magnetic field of parity $(-1)^{l+1}$.

We now normalize $\mathbf{A}_M(\mathbf{r},t)$ and $\mathbf{A}_E(\mathbf{r},t)$ to energy $(\hbar ck)$ inside a sphere of radius $a$ and volume $V$ on the boundary of which $A$ vanishes. The resulting expressions for the vector potentials for the electric and magnetic multipole fields are

$$\mathbf{A}_E(l,m)=i\left(\frac{\hbar ck}{a}\right)^{\frac{1}{2}}\frac{\mathbf{L}}{[l(l+1)\hbar^2]^{\frac{1}{2}}}j_l(kr)Y_{lm}(\vartheta,\varphi), \quad (13)$$

$$\mathbf{A}_M(l,m)=\frac{1}{k}\left(\frac{\hbar ck}{a}\right)^{\frac{1}{2}}\frac{\nabla\times\mathbf{L}}{[l(l+1)\hbar^2]^{\frac{1}{2}}}j_l(kr)Y_{lm}(\vartheta,\varphi), \quad (14)$$

where

$$\begin{aligned}L^2\mathbf{A}_{E,M}(l,m)&=l(l+1)\hbar^2\mathbf{A}_{E,M}(l,m),\\ J^2\mathbf{A}_{E,M}(l,m)&=l(l+1)\hbar^2\mathbf{A}_{E,M}(l,m),\\ J_Z\mathbf{A}_{E,M}(l,m)&=m\hbar\mathbf{A}_{E,M}(l,m).\end{aligned} \quad (15)$$

From the boundary condition that $\mathbf{A}(a)=0$ the density of final states per unit energy $\rho(E)$ follows immediately as

$$\rho(E)=a/(\hbar c\pi). \quad (16)$$

The matrix element $H_{ab}$ which appears in (10) can be written as

$$H_{ab}=-\frac{1}{c}\int\mathbf{J}_{ab}\cdot\mathbf{A}-\mu\int\varphi_a{}^*\boldsymbol{\sigma}\cdot(\nabla\times\mathbf{A})\varphi_b, \quad (17)$$

where

$$\mathbf{J}_{ab}=(e/2mc)[\varphi_a{}^*(\mathbf{p}\varphi_b)+(\mathbf{p}\varphi_a)^*\varphi_b]$$

is the quantum-mechanical transition current.

By combining Eqs. (14), (16), and (17), we find the magnetic multipole transition probability per unit time and per unit solid angle for the direction of the emitted photon to be

$$T_{ab}{}^M=\frac{2}{\hbar}\frac{k}{l(l+1)\hbar^2}\left|-\frac{i}{c}\int\mathbf{J}_{ab}\cdot\mathbf{L}j_lY_{lm}-i\mu\int\varphi_a{}^*\boldsymbol{\sigma}\cdot(\nabla\times\mathbf{L})j_lY_{lm}\varphi_b\right|^2. \quad (18)$$

Similarly, one finds for the electric transition probability per unit time and per unit solid angle:

$$T_{ab}{}^E=\frac{2}{\hbar}\frac{k}{l(l+1)\hbar^2}\left|-\frac{1}{kc}\int\mathbf{J}_{ab}\cdot(\nabla\times\mathbf{L})j_lY_{lm}-\mu k\int\varphi_a{}^*\boldsymbol{\sigma}\cdot\mathbf{L}j_lY_{lm}\varphi_b\right|^2. \quad (19)$$

We have transformed the second term of $T_{ab}{}^E$ by means of the relation $\nabla\times\mathbf{A}_E(l,m)=-ik\mathbf{A}_M(l,m)$. We now transform the integrals appearing in $T_{ab}{}^M$ and $T_{ab}{}^E$ by integration by parts and the operator identity

$$\nabla\times\mathbf{L}=i\left[\nabla\left(\frac{\partial}{\partial r}r\right)-\mathbf{r}\nabla^2\right].$$

Define the generalized multipole moments:

*Magnetic.—*

$$\mathfrak{M}_{lm}=-\frac{(2l+1)!!}{(l+1)k^l}2\mu\int j_l(kr)Y_{lm}(\vartheta,\varphi)\,\mathrm{div}(\varphi_a\mathbf{L}\varphi_b),$$

$$\mathfrak{M}_{lm}'=-\frac{(2l+1)!!}{(l+1)k^l}\mu\left\{\int\frac{\partial}{\partial r}rj_l(kr)Y_{lm}(\vartheta,\varphi)\,\mathrm{div}(\varphi_a{}^*\boldsymbol{\sigma}\varphi_b)-k^2\int j_l(kr)Y_{lm}(\vartheta,\varphi)(\varphi_a{}^*\boldsymbol{\sigma}\cdot\mathbf{r}\varphi_b)\right\}. \quad (20)$$

*Electric.—*

$$\mathfrak{Q}_{lm}=\frac{(2l+1)!!}{(l+1)k^l}\left\{e\int\left[\frac{\partial}{\partial r}rj_l(kr)Y_{lm}(\vartheta,\varphi)\right]\varphi_a{}^*\varphi_b+\frac{ik}{c}\int j_l(kr)Y_{lm}(\vartheta,\varphi)\mathbf{r}\cdot\mathbf{J}_{ab}\right\},$$

$$\mathfrak{Q}_{lm}'=\frac{(2l+1)!!}{(l+1)k^l}\left(-i\mu k\int j_l(kr)\times Y_{lm}(\vartheta,\varphi)\,\mathrm{div}(\varphi_a{}^*\mathbf{r}\times\boldsymbol{\sigma}\varphi_b)\right), \quad (21)$$

where $(2l+1)!!=(2l+1)(2l-1)\cdots2$, or 1. The transition probabilities per unit time (integrated over solid angle) are

$$T_{ab}{}^M=\frac{8\pi(l+1)}{l[(2l+1)!!]^2}\frac{k^{2l+1}}{\hbar}|\mathfrak{M}_{lm}+\mathfrak{M}_{lm}'|^2, \quad (22)$$

$$T_{ab}{}^E=\frac{8\pi(l+1)}{l[(2l+1)!!]^2}\frac{k^{2l+1}}{\hbar}|\mathfrak{Q}_{lm}+\mathfrak{Q}_{lm}'|^2. \quad (23)$$

The formulas are exact so far in that no assumption about the magnitude of $(kR)$ has been made. If we make the "long-wavelength approximation" $kR\ll1$, we put $j_l(kr)\sim(kr)^l/(2l+1)!!$. Since $\nabla^2(r^lY_{lm})=0$, the multipole moments defined in (20) and (21) become

$$M_{lm}{}^{(0)}=-\frac{1}{l+1}2\mu\int r^lY_{lm}\,\mathrm{div}(\varphi_a{}^*\mathbf{L}\varphi_b),$$

$$M_{lm}{}^{(0)\prime}=-\mu\int r^lY_{lm}\,\mathrm{div}(\varphi_a{}^*\boldsymbol{\sigma}\varphi_b), \quad (24)$$

$$Q_{lm}{}^{(0)}=e\int r^lY_{lm}\varphi_a{}^*\varphi_b,$$

$$Q_{lm}{}^{(0)\prime}=\frac{-ik}{l+1}\mu\int r^lY_{lm}\,\mathrm{div}(\varphi_a{}^*\mathbf{r}\times\boldsymbol{\sigma}\varphi_b). \quad (25)$$

These are precisely the single-particle moments of Blatt and Weisskopf.

Although no derivation of the exact multipole moments for nonrelativistic quantum mechanics has been given, the exact multipole moments for the classical Maxwell field have been given by Wallace[30] and by French and Shimamoto.[31] Quite recently a derivation of the exact multipole moments for the Dirac equation has been given by Stech[32] using a treatment similar to the preceding development. Brennan and Sachs[33] also have discussed higher order corrections to the $E1$ matrix element from a different point of view, and have formally defined the exact multipole moments.

### EFFECT OF ELECTRIC DIPOLE CORRECTIONS

Two types of higher order terms in $H_0$ exist to produce contributions to $E1$ radiation which therefore violate the selection rules in $T_\zeta=0$ nuclei. We estimate first the contribution of the spin-dependent part of $H_0$ to the $E1$ radiation. This contribution appears as $Q_{1m}^{(0)\prime}$ in (25) and is of course neglected in the Weisskopf formulas. We calculate $Q_{1m}^{(0)\prime}$ for $H_0$, noticing first by comparing (2) and (9) that we should use $\mu=\frac{1}{2}(\mu_p+\mu_n)$, $e/mc=e/2Mc$. In using $Q_{1m}^{(0)}$ and $Q_{1m}^{(0)\prime}$, we are of course using the extreme single-particle picture, but this approximation gives the proper orders of magnitude. To evaluate $Q_{1m}^{(0)}$ and $Q_{1m}^{(0)\prime}$, assume that $\phi_a$ and $\phi_b$ are constant over a sphere of radius $R$, the radius of the nucleus, and that $Y_{lm}$ is of order unity. In $Q_{lm}^{(0)\prime}$ we also take $\boldsymbol{\sigma}$ of order unity and $\frac{1}{2}(\mu_p+\mu_n)\sim 0.5e\hbar/2Mc$. The divergence in $Q_m^{(0)\prime}$ is approximated by $(1/R)$. We then obtain

$$|Q_{1m}^{(0)\prime}/Q_{1m}^{(0)}|^2\sim(1/25)(\hbar\omega/Mc^2)^2.$$

For gamma rays of $\sim$10 Mev, $|Q_{1m}^{(0)\prime}/Q_{1m}^{(0)}|^2\sim 4\times10^{-6}$. The contribution of the magnetic moment in $H_0$ to $E1$ transitions therefore provides a rate $4\times10^{-6}$ times an uninhibited $E1$ matrix element. This is negligible compared to the effect of isotopic spin impurities.

The higher order term in $\mathcal{Q}_{1m}'$ arising from the second term of $j_l(kr)$ can be found by using

$$j_l(kr)=\frac{(kr)^l}{(2l+1)!!}-\frac{(kr)^{l+2}}{(2l+3)!!2}.$$

The second term in an expansion of $\mathcal{Q}_{lm}$,

$$\mathcal{Q}_{lm}=Q_{lm}^{(0)}+Q_{lm}^{(1)}+\cdots,$$

found from successive terms in $j_l(kr)$, is

$$Q_{lm}^{(1)}=-\frac{(2l+1)!!}{(2l+3)!!}\frac{k^{l+2}}{2k^l}\left\{\int(l+3)r^{l+2}Y_{lm}\varphi_a{}^*\varphi_b+\frac{ik}{c}\int r^{l+2}Y_{lm}\mathbf{r}\cdot\mathbf{J}_{ab}\right\},$$

or

$$Q_{1m}^{(1)}=-\frac{k^2}{5}\left\{4\int r^3Y_{1m}\varphi_a{}^*\varphi_b+\frac{ik}{c}\int r^3Y_{1m}\mathbf{r}\cdot\mathbf{J}_{ab}\right\}.$$

By the same approximations as before, we obtain the result:

$$|Q_{1m}^{(1)}/Q_{1m}^{(0)}|^2\sim(2/15)^2(kR)^4.$$

From $(kR)\approx\hbar\omega/137$, we find that for 10-Mev photons,

$$|Q_{1m}^{(1)}/Q_{1m}^{(0)}|^2\sim10^{-6}.$$

We can conclude that all the higher order terms in $H_0$ give rise to $\Gamma_\lambda$ widths for $E1$ emission which are of order $10^{-6}$ times the uninhibited widths. The $\Gamma_\lambda$ widths produced by isotopic spin impurities are proportional to the impurity $p$ and are at least $10^{-3}$ times a "normal" $E1$ width.[28,29]

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