fraction of normal component including both roton and phonon contributions. But the phonon contribution is

$$x_{ph}=TS_{ph}/c^2, \tag{10}$$

where c is the velocity of first sound. Hence

$$\frac{\partial x_{ph}}{\partial p}=-\frac{T}{c^2}\left(V\alpha_{ph}+2S_{ph}\frac{1}{c}\frac{\partial c}{\partial p}\right), \tag{11}$$

and as S_{ph} and $(1/c)\partial c/\partial p$ are known,[7,8] $\partial x_{ph}/\partial p$ can be obtained and subtracted from $\partial x/\partial p$ to give $\partial x_r/\partial p$.

Differentiating (2) and (3) with respect to ρ, $(\rho/p_0)\times\partial p_0/\partial\rho$ may finally be determined from

$$2\frac{\rho}{p_0}\frac{\partial p_0}{\partial\rho}=\frac{1}{x_rK_T}\frac{\partial x_r}{\partial p}+\frac{\rho}{\Delta}\frac{\partial\Delta}{\partial\rho}\frac{1}{(1+3kT/2\Delta)}+\frac{V\alpha_r}{S_rK_T}, \tag{12}$$

where K_T, the isothermal compressibility, and α_r, the contribution of the rotons to the coefficient of expansion, are known.[3] Taking $(1/u_2)\partial u_2/\partial p$ from the measurements of Peshkov and Zinoveva,[5] $(\rho/p_0)\partial p_0/\partial\rho$ is found to be $+0.26$. However, this result is very sensitive to $(1/u_2)\partial u_2/\partial p$ and, if the measurements of Maurer and Herlin[6] are used, $(\rho/p_0)\partial p_0/\partial\rho$ varies from -1.12 at 1.0°K to -0.23 at 1.6°K. As it should, in principle, be independent of temperature, we have preferred to use the Russian measurements, but it would be desirable to make a further investigation of the variation of u_2 with pressure, placing particular emphasis on the lower pressures to obtain an accurate value for the initial slope. It is also important to note that all the parameters discussed here are sensitive to the values adopted for S and C. We have used the recent values of Hercus and Wilks[4] which are about 10 percent higher than previous values.

If we accept $(\rho/p_0)\partial p_0/\partial\rho=+0.26$, Eqs. (6) and (7) lead to $(\rho/\mu)\partial\mu/\partial\rho=-1.8$. It appears that the effective mass μ varies much more rapidly with ρ than either Δ or p_0.

Equation (5) may now be evaluated to yield $(\rho/T_\lambda)\times\partial T_\lambda/\partial\rho=-0.42$. The experimental value is -0.37. In view of the approximation involved in neglecting roton interactions, the agreement is very satisfactory. It seems that both the negative coefficient of expansion and the negative slope of the λ curve are caused primarily by the negative value of $(\rho/\Delta)\partial\Delta/\partial\rho$.

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Phenomenological Theory of Townsend Breakdown in Dielectrics

P. L. Auer
General Electric Research Laboratory, Schenectady, New York
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The Townsend model of dielectric breakdown is discussed. The transient equations describing the behavior of the electron and positive ion current densities are solved rigorously with boundary conditions appropriate to a simplified Townsend model employing photoelectron production at the cathode as the only secondary mechanism. The correct boundary condition is found to lead to a set of integro-difference equations. Solution of the integro-difference equations allows rigorous specification of the criterion for breakdown. This criterion is found to become equivalent to the familiar Townsend criterion in the asymptotic limit of large time intervals. The results of our development may be applied to the calculation of formative time lags and to the estimation of space-charge distortion effects.

1. INTRODUCTION

When a dielectric material is subjected to an electrical stress of sufficient magnitude it is converted into a conductor. Breakdown occurs when the conductivity of the material becomes sufficient to maintain a current limited only by the external circuit. In his pioneering work on electrical discharges Townsend found that in most instances the conversion of an insulator to conductor is caused by the ionization of the original dielectric material.

The ionization of the dielectric can be initiated by any of several well-known external means, or by the presence of stray electrons in the system. The electrons created by the initial ionization burst become accelerated in the applied electric field and in turn produce additional ionization. Such an avalanche of ionization can, however, produce no breakdown as long as the field remains static and essentially homogeneous since the electrons created in the ionization avalanche are removed from the system by the applied field. In order to achieve a self-sustaining discharge, it becomes necessary to introduce a secondary, regenerative source of electron production; although it is conceivable that in some rare instances breakdown can be achieved purely by some highly effective field distortion mechanism instead of a secondary electron production process.

Various schemes have been proposed to explain the secondary electron production process active in the breakdown of dielectrics by static fields. We shall confine our attentions to what we shall term Townsend breakdown, and we define this phenomenon as the breakdown of dielectric material in the presence of a *homogeneous static* electric field where the only secondary electron producing process active takes place at the cathode surface. In excluding from our analysis the possibility of ionization by positive ions or photons in the bulk of the dielectric, we do not necessarily imply that such processes may not be effective in secondary electron production. However, we are of the opinion that the mechanism we term Townsend breakdown plays an important role in many breakdown studies and a complete analysis of this model may prove to be valuable aid in elucidating the actual mechanisms governing breakdown.

In what follows, we shall present a detailed analysis of the transient growth of the ionization current in a Townsend breakdown and subsequently derive the criterion for breakdown. For the sake of convenience in mathematical operation an especially simplified model has been chosen for the initial analysis. We shall assume that the only active secondary electron production process present is due to the action of photons at the cathode surface and that these photons, created in the primary ionization avalanche, travel through the dielectric material without absorption and with infinite speed. In later sections of this presentation, we shall discuss modifications necessary to include the action of positive ions at the cathode surface and the consequences of photon absorption and finite speeds of travel.

2. DIFFERENTIAL EQUATION OF BREAKDOWN

a. Equation of Continuity

We consider a quantity of dielectric placed between two infinite plane parallel electrodes at δ cm apart. At a given initial time, a voltage of constant magnitude is impressed across the electrodes producing a homogeneous static field in the space occupied by the dielectric. We imagine that at the instant the electric stress is applied to the dielectric there arises an initial current due to the action of the field on either stray electrons present in the system or electrons introduced from an external source. For the sake of convenience in mathematical operation, we restrict the initial current to the origin of our coordinate system, which is conveniently placed at the cathode. As we shall show later, no intrinsic difficulty is met with in considering the initial current to have some arbitrary distribution in space. The number of tedious mathematical operations is greatly increased, however, if we depart from the simple localized picture of the initial current which we shall presently develop in some detail.

We shall find it possible to obtain a complete description of the electron current in terms of the following quantities: α, defined as the relative increase in current density per cm in the direction of the field, β, defined here as the secondary ionization coefficient and used to denote the efficiency of photoelectron production at the cathode; and v, which is the electron drift velocity and will be treated as a vector quantity. These three quantities are functions of the nature of the particular dielectric under consideration. In addition, α is a sensitive function of the electric field strength while β may be influenced by the nature of the cathode material. For the system under discussion, the drift velocity is assumed to be constant in magnitude and direction for any given field strength. We choose a Cartesian frame with the x axis placed in the direction of the field. The cathode is placed at the origin and the anode is placed at $x=\delta$ cm. Denoting the time variable by t, the initial time is chosen such that $t=0$ when the field is first applied and the initial current makes its appearance at the cathode.

We let $i=i(x,t)$ represent the electron current density. Then for the system under discussion the equation of continuity takes the particularly simple form

$$\frac{1}{v}\frac{\partial i}{\partial t}+\frac{\partial i}{\partial x}=\alpha i, \tag{2.1}$$

where v is the drift velocity of the electron and α is the primary ionization coefficient. The solution of the above differential equation must satisfy a boundary condition incorporating the secondary electron generating process. The boundary condition for the system under discussion is given by the relation

$$i(0,t)=i(0,0)+\beta\int_0^\delta i(x,t)dx, \tag{2.2}$$

where β is the secondary ionization coefficient for photoelectron production at the cathode.

The system of equations presented by (2.1) and (2.2) is the one appropriate to discussing Townsend type breakdown in dielectrics. It is essentially the same system that was treated by previous investigators in the field.[1–4] The differences between our model and those employed by the authors cited above will be elaborated on later. For the most part the remainder of this paper will be concerned with presenting a detailed solution of the above system of equations.

b. Steady-State Solution

Before proceeding to the main portion of our investigation, it will be instructive to consider solutions to the system of Eqs. (2.1) and (2.2) at steady state. These solutions are well known and described in most texts and articles on dielectric breakdown. The results for steady state will be made use of in subsequent discussion.

At steady state the time variation disappears and

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[2] H. L. von Gugelburg, Helv. Phys. Acta **20**, 250, 307 (1947).
[3] P. M. Davidson, Brit. J. Appl. Phys. **4**, 173 (1953).
[4] H. W. Bandel, Phys. Rev. **95**, 1117 (1954).

the equation of continuity for the electron current density reduces to

$$di/dx = \alpha i, \quad i = i(x); \tag{2.3}$$

along with the boundary condition

$$i(0) = i_0 + \beta \int_0^\delta i(x)dx. \tag{2.4}$$

In Eq. (2.4) the quantity i_0 must be interpreted as the hypothetical steady-state value of the cathode electron current in the absence of a secondary mechanism.

The solution to the system of Eqs. (2.3) and (2.4) is given by

$$i(x) = \frac{i_0 e^{\alpha x}}{1-(\beta/\alpha)[e^{\alpha\delta}-1]}, \tag{2.5}$$

as may be verified by direct substitution. Townsend was the first to derive the expression given in Eq. (2.5) on the basis of physical arguments. Recognizing that physically this expression predicts a catastrophe whenever the denominator becomes zero, Townsend interpreted the onset of the catastrophe as the criterion for dielectric breakdown. Consequently, it has become customary practice to define breakdown in terms of the relation

$$(\beta/\alpha)[e^{\alpha\delta}-1] = 1. \tag{2.6}$$

It must be recognized that there is no reason to believe steady-state conditions can ever prevail at the point of breakdown and the process of analysis leading to Eq. (2.6) as the criterion for breakdown must be deemed unsatisfactory. The proper criterion for breakdown should be obtained from the solution of Eq. (2.1) instead of (2.3). We shall now proceed to this task.

c. Transient System

The solution of Eq. (2.1) can be written down readily in the form

$$i(x,t) = f(t-x/v)e^{\alpha x}, \tag{2.7}$$

where $f(z)$ is an arbitrary function of its argument in so far as the differential equation is concerned but is required to satisfy Eq. (2.2) in order to represent a solution to our problem. On setting x equal to zero in (2.7) it is seen that $f(t)$ is simply the cathode electron current density at time t.

Substituting the expression (2.7) into Eq. (2.2) we obtain

$$f(t) = f(0) + \beta \int_0^\delta f\left(t - \frac{x}{v}\right) e^{\alpha x} dx; \tag{2.8}$$

and on rearranging terms under the integral sign we obtain the integral equation

$$f(t) = f(0) + \beta v \int_{t-\delta/v}^{t} e^{\alpha v(t-s)} f(s) ds. \tag{2.9}$$

The solution of the above equation represents the central problem in characterizing Townsend breakdown and is the major topic of our investigation.

3. INTEGRAL EQUATION OF TOWNSEND BREAKDOWN

a. System of Integral Equations

The form of Eq. (2.9) suggests that it represents a set of difference equations. This may be demonstrated in the following manner. The transit time of an electron, t_e, from one electrode to the other is simply $\delta/|v|$. The time axis may then be divided in units of electron transit times. We define a set of functions according to the relation

$$\begin{gathered} f(t) \equiv \phi_n(y), \quad n = 1, 2, \cdots, \\ (n-1)t_e \leqslant t \leqslant nt_e, \\ 0 \leqslant y \leqslant 1, \\ t_e = \delta/|v|. \end{gathered} \tag{3.1}$$

Setting the initial time equal to zero, Eq. (3.1) yields the matching conditions:

$$\begin{aligned} &\phi_0(y) = 0, && n = 0 \\ &\phi_n(0) = \phi_{n-1}(1), && n > 0. \end{aligned} \tag{3.2}$$

With the aid of the above definitions, Eq. (2.9) may be written in the form

$$\begin{gathered} \phi_n(y) = f(0) + p\int_y^1 e^{\sigma(y-z)}\phi_{n-1}(z)dz + \lambda \int_0^y e^{\sigma(y-z)}\phi_n(z)dz, \\ p \equiv \lambda e^\sigma, \quad \lambda \equiv \beta\delta, \quad \sigma \equiv \alpha\delta. \end{gathered} \tag{3.3}$$

The quantities, p, λ, σ have been introduced for the sake of convenience in notation.

From Eq. (3.3), we obtain the particular relations:

$$\begin{gathered} \phi_n(1) = f(0) + p\int_0^1 e^{-\sigma z}\phi_n(z)dz, \\ \phi_n(0) = f(0) + p\int_0^1 e^{-\sigma z}\phi_{n-1}(z)dz, \end{gathered} \tag{3.4}$$

and consequently we check that the integral equation (3.3) satisfies the matching conditions of (3.2). On differentiating Eq. (3.3) with respect to y we obtain

$$\phi_n'(y) = q\phi_n(y) - p\phi_{n-1}(y) - \sigma f(0), \quad q \equiv \lambda + \sigma, \tag{3.5}$$

where the quantity q has been introduced for the sake of convenience in notation and the prime denotes differentiation with respect to the argument. An additional matching condition may be obtained by combining (3.2) with (3.5) to yield

$$\phi_n'(0) = \phi_{n-1}'(1), \quad n > 2. \tag{3.6}$$

Thus, the $\phi_n(y)$ represent a function of time which after the second transit time interval is not only continuous but also smooth.

It is interesting to note that the system of equations described by (3.3) bears some resemblance to systems encountered in criticality calculations of nuclear reactors. The resemblance is not too surprising if we recognize that the Townsend model under consideration is a sort of chain reaction itself.

b. Solution for the Initial Time Span

Equation (3.3) takes a particularly simple form for the initial electron transit-time span. Since $\phi_0(y)$ doesn't exist,

$$\phi_1(y)=f(0)+\lambda\int_0^y e^{\sigma(y-z)}\phi_1(z)dz. \quad (3.7)$$

This is an elementary inhomogeneous Volterra integral equation of the second kind. It is instructive to note that with the exception of pathological cases the homogeneous Volterra equation possesses no nonzero solutions. The above is not an exception to this rule and consequently Eq. (3.7) can have no solution other than zero unless $f(0)$, which is the cathode electron current density at zero time, has a value different from zero. We stress this point merely to refute the contention of previous authors that the Townsend model for breakdown can be described by a spontaneous creation equation where the electron current is finite for times greater than the initial time but is zero initially.

Equation (3.7) is readily solved by any of several familiar methods to yield

$$\phi_1(y)=f_0\{1+(\lambda/q)[e^{qy}-1]\}.^5 \quad (3.8)$$

c. Solution for General Time Spans

The solution of the set of equations given in (3.3) may be accomplished by several methods. In Appendix 1 of this presentation we describe a direct solution of the set of integro-difference equations by means of a simple operational technique. Immediately below we shall sketch briefly a solution of the analogous differential-difference equations given by (3.5).

We define two functions in the following manner:

$$\zeta_n(y)=\phi_n(y)-\left(\frac{\sigma}{q-p}\right)f_0, \quad q\neq p,$$
$$G(s,y)=\sum_{j=1}\zeta_j(y)s^{j-1}, \quad (3.9)$$

[5] Reference 4 quotes the results of some calculations made by W. Kunkel. For the model under discussion here, that is, where the only secondary mechanism active is photoelectron production at the cathode, Kunkel's results would be in agreement with our Eq. (3.8) providing his role of positive ion and photon mechanisms were reversed. The time range of validity quoted in reference 4 would then be incorrect, however. Actually, Eq. (10) of reference 4 appears to predict that for our model, steady-state conditions are obtained after a single electron transit time has occurred.

where it is seen that $G(s,y)$ is a generating function and the power series is assumed to converge for small enough values of s. On substituting $\zeta_n(y)$ for $\phi_n(y)$ into Eqs. (3.5) and (3.2), multiplying by appropriate powers of s, and adding, we obtain

$$\frac{\partial G(s,y)}{\partial y}=(q-ps)G(s,y)+\frac{p\sigma}{q-p}f_0,$$
$$G(s,0)=sG(s,1)+\left(1-\frac{\sigma}{q-p}\right)f_0, \quad q\neq p. \quad (3.10)$$

The solution of the above equation is given by

$$G(s,y)=-\left(\frac{p\sigma}{q-p}\right)\frac{f_0}{q-ps}+(1-se^{q-ps})^{-1}\left(1-\frac{\sigma}{q-ps}\right)f_0e^{(q-ps)y}. \quad (3.11)$$

Expansion of the above expression for $G(s,y)$ as a power series in s and association of the appropriate coefficients in the expansion provides the required solution for the $\phi_n(y)$ of Eq. (3.5) with the restriction $q\neq p$.

If we set y equal to zero in Eq. (3.11), the following expression is obtained:

$$\frac{\phi_n(0)}{\phi_1(0)}=\frac{\sigma}{q}\left[\frac{1-(p/q)^n}{1-(p/q)}\right]+\frac{\lambda}{q}A_{n-1}-\frac{\sigma}{q}\sum_{j=0}^{n-2}A_j(p/q)^{n-j-1}, \quad (3.12)$$

where the A_j are the coefficients of s^j in the expansion of the generating function $[1-se^{(q-ps)}]^{-1}$ and are given by the expressions

$$A_0=1, \quad A_n=\sum_{j=0}^{n-1}(-1)^j\frac{(n-j)^j}{j!}p^je^{(n-j)q}. \quad (3.13)$$

The expression for $\phi_n(y)$ may be written in the form

$$\frac{\phi_n(y)}{\phi_1(0)}=\frac{\sigma}{q}\left[\frac{1-(p/q)^n}{1-(p/q)}\right]+e^{qy}\sum_{j=0}^{n-1}(-1)^j\frac{(py)^j}{j!}\epsilon_{n-j},$$
$$\epsilon_n=(\lambda/q)A_{n-1}-(\sigma/q)\sum_{j=0}^{n-1}A_j(p/q)^{n-j-1}. \quad (3.14)$$

The results of Eqs. (3.12) and (3.14) may be applied to the case $p=q$ if we make use of the relation

$$\lim_{x\to1}\frac{1-x^n}{1-x}=n. \quad (3.15)$$

Equations (3.12) to (3.14) represent the required solution to the original integral equation of (2.9). Unfortunately the sums involving the coefficients A_j

have no convenient representation in closed form and the exact calculation of the $\phi_n(y)$ becomes a tedious tabulation process. We may, however, make use of the following properties obtained in usual practice:

$$pe^{-q} \equiv \lambda e^{-\lambda}, \quad \lambda \ll 1. \tag{3.16}$$

After some calculation we obtain from (3.12) and (3.13)

$$\frac{\phi_n(0)}{\phi_1(0)} = \frac{\lambda}{q}\left(\frac{p}{q}\right)^n + \frac{\sigma}{q}\left[\frac{1-(p/q)^n}{1-(p/q)}\right] + p^n O(\lambda^n). \tag{3.17}$$

For any practical purpose the last term on the right of Eq. (3.17) may be neglected with negligible loss in accuracy. Figure 1 presents the behavior of the cathode electron current density with time according to the relation given by (3.17). The curves were obtained by smoothly joining together points obtained from (3.17).

4. CRITERION FOR BREAKDOWN AND THE APPROACH TO STEADY STATE

a. Physical Criterion

It can be shown easily on the basis of Eq. (3.3) that for real positive values of the coefficients α and β the cathode electron current, $f(t)$, must be a monotonic increasing function of time. In analogy with Townsend it may be asked whether there are values which may be assigned to α and β which will provide for arbitrarily large currents in the course of time regardless of the initial current value. In order to ascertain whether such values exist for α and β it is necessary to examine the behavior of Eq. (3.14) in the asymptotic limit of large "n."

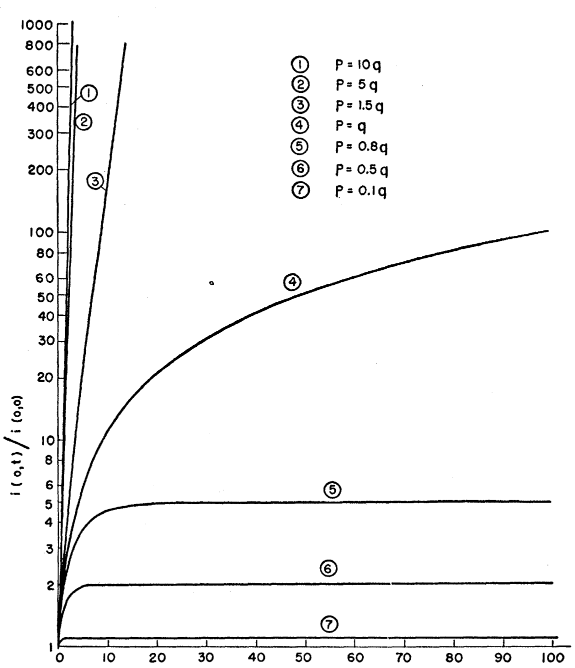

FIG. 1. Cathode electron current density as a function of time, according to the relation given by Eq. (3.17).

There is an alternative and simpler method available for deriving the breakdown criterion. We state that a system is capable of undergoing breakdown only if the current is unable to approach a finite asymptotic value, which may be associated with its steady-state value. This requires that the first time derivative of the current can never become a decreasing function of time. However, it is sufficient to require that the derivative be a constant, and we shall use this as the concept for breakdown criterion.

In particular, the constancy of the first time derivative permits us to require

$$\phi_{n+1}'(0) = \phi_n'(0) \tag{4.1}$$

for all values of "n." Equation (4.1) will now be developed to provide the criterion for breakdown.

b. Mathematical Criterion for Breakdown

From Eq. (3.12) we obtain the relation

$$\phi_{n-1}(0) = \left(\frac{p}{q}\right)\phi_n(0) + \phi_1(0)\left\{\frac{\lambda}{q}A_n - \left(\frac{p}{q}\right)A_{n-1} + \frac{\sigma}{q}\right\}. \tag{4.2}$$

Inserting (4.2) into (3.5) provides

$$\phi_{n+1}'(0) - \phi_n'(0) = \lambda\phi_1(0)\{A_n - (1+e^\sigma)A_{n-1} + e^\sigma A_{n-2}\} = 0. \tag{4.3}$$

The above equation constitutes a set of finite difference equations for the A_n; the necessary boundary conditions may be obtained from (3.13):

$$A_0 = 1; \quad A_1 = e^q. \tag{4.4}$$

The solution to Eq. (4.3) subject to (4.4) is readily obtained by the usual methods to yield

$$A_n = 1 + (e^q - 1)\left[\frac{1-e^{n\sigma}}{1-e^\sigma}\right]. \tag{4.5}$$

It can be readily shown that the coefficients A_n must satisfy the relation

$$\sum_{j=0}^{n}(-1)^j a_j A_{n-j} = 0, \qquad a_0 = 1, \quad a_n = e^q p^{(n-1)}/(n-1)!, \tag{4.6}$$

since it will be recognized that the a_j are simply the expansion coefficients of $(-s)^j$ in the generating function $[1 - se^{(q-ps)}]$. Substitution of (4.5) into (4.6) yields the criterion for breakdown in the form

$$\left(\frac{e^{q}-e^{\sigma}}{1-e^{\sigma}}\right)\left[1-e^{q}\sum_{k=0}^{n}(-1)^{k}\frac{p^{k}}{k!}\right]+e^{n\sigma}\left(\frac{e^{q}-1}{e^{\sigma}-1}\right)\left[1-e^{\lambda}\sum_{k=0}^{n}(-1)^{k}\frac{\lambda^{k}}{k!}\right]=0, \quad (4.7)$$

$$q=\lambda+\sigma, \quad p=\lambda e^{\sigma}.$$

In Appendix B we show that the second term of Eq. (4.7) can be made to approach zero as n approaches infinity. Consequently, the criterion for breakdown becomes asymptotically

$$1-e^{(q-p)}=0, \quad (4.8)$$

which requires that

$$p=q, \quad \beta e^{\alpha\delta}=\alpha+\beta. \quad (4.9)$$

Equation (4.9) is the familiar Townsend criterion cited previously in (2.6). It would have been possible to obtain this criterion by inspection of Eq. (3.17) with the sacrifice of some rigor.

c. Approach to Steady State

It has been established in the previous section that for values of (p/q) less than unity the system can approach a limiting steady-state condition. Since

$$\lim_{n\to\infty}\left[\frac{1-x^{n}}{1-x}\right]=\frac{1}{1-x}, \quad x<1, \quad (4.10)$$

Eq. (3.14) provides

$$\lim_{n\to\infty}\phi_{n}(y)=\left(\frac{\sigma}{q}\right)\frac{\phi_{n}(0)}{1-p/q}=\frac{f_{0}}{1-(\beta/\alpha)(e^{\alpha\delta}-1)}, \quad (4.11)$$

$$p/q<1.$$

Comparison of the above result with Eq. (2.5) verifies that (4.11) predicts the correct asymptotic value for the cathode electron current density in the event the system is below the breakdown threshold.

The value of the cathode electron current for the case $p=q$ can be evaluated readily using Eqs. (3.17) and (3.15). We obtain

$$\phi_{n}(0)=\left(\frac{n\sigma}{q}+\frac{\lambda}{q}\right)\phi_{1}(0)+O(n\lambda); \quad p=q. \quad (4.12)$$

5. ELECTRON AND POSITIVE ION CURRENT DISTRIBUTION IN SPACE

a. Electron Current

In Sec. 3. a. we have shown that the electron current density at the cathode is described for given times by the $\phi_n(y)$ according to the relation of (3.1). Making use of Eq. (2.7) we find that the spatial distribution of the electron current density at a given time may be written as

$$i(x,t)/i(0,t)=e^{\alpha x}\phi_{n}(y-x/\delta)/\phi_{n}(y), \quad (5.1)$$

$$\phi_{n}(-z)=\phi_{n-1}(1-z), \quad 0<z<1.$$

In particular the electron current density at the anode becomes

$$i(\delta,t)=\phi_{n-1}(y)e^{\sigma}, \quad (n-1)t_{e}\leqslant t\leqslant nt_{e}. \quad (5.2)$$

The above relation confirms the fact that no electron current can develop at the anode until a single electron transit time has elapsed.

b. Positive Ion Current

The positive ion current density is denoted $j(x,t)$ and is found to satisfy a relation analogous to (2.1) in the form

$$\frac{1}{w}\frac{\partial j}{\partial t}+\frac{\partial j}{\partial x}=-\alpha i, \quad (5.3)$$

where w is the drift velocity of the positive ion and all other symbols have been defined previously. The solution to the above equation has to satisfy the boundary condition

$$j(\delta,t)=0. \quad (5.4)$$

Solution to Eqs. (5.3) and (5.4) may be constructed in the form

$$i(x,t)=0, \quad (t-|x/v|)<0,$$

$$j(x,t)=\alpha\int_{0}^{\delta}i(s;t-x/w+s/w)ds, \quad (t-|x/v|)\geqslant 0, \quad (5.5)$$

$$1/u=1/v-1/w,$$

where $i(x,t)$ is given by Eq. (2.7).

In particular the positive ion current density at the cathode is given according to Eqs. (5.5) and (2.7) by the relation

$$j(0,t)=\alpha u\int_{t-\delta/u}^{t}f(s)e^{\alpha u(t-s)}ds, \quad (5.6)$$

where $f(t)$ is given by Eq. (2.9).

We shall have no occasion in this presentation to calculate specifically the positive ion current density. However, we wish to point out that the results developed in previous sections permits us to calculate both the electron and positive ion current density distribution in space for arbitrary times. Having made these calculations we can then proceed to find the resulting electric field, $E(x,t)$, from the relation

$$\partial E/\partial t=j(x,t)+i(x,t)+C(t), \quad (5.7)$$

and $C(t)$ may be obtained from the requirement

$$V_0=-\int_0^\delta E(x,t)dx, \tag{5.8}$$

where V_0 is the value of the externally impressed voltage, and we have used mks units in the above equations. Equation (5.7) provides a description of the field under the assumption that field distortions due to space charge are negligible. The consistency of this argument may be tested by solving for the field according to (5.7).

6. DISCUSSION

a. Generalized Townsend Model

In the previous sections we have developed in detail the description of the electron and positive ion current in a Townsend discharge system where the only active secondary mechanism present was the photoelectron production at the cathode. Several other forms of secondary mechanisms have been considered instrumental in Townsend discharge. For present purposes we need only consider secondary mechanisms occurring at the cathode. There are then two important alternatives to consider, photoelectron production and electron production due to positive ion impact at the cathode.

We may generalize and consider that both photons and positive ions are effective in producing electrons at the cathode. The efficiency of electron production due to photons will be denoted by β as before, while the efficiency due to the positive ions will be denoted by the quantity ω/α. We are still required to solve Eq. (2.1) but the boundary condition of (2.2) is now replaced by

$$i(0,t)=i(0,0)+\frac{\omega}{\alpha}j(0,t)+\beta\int_0^\delta i(x,t)dx. \tag{6.1}$$

Making use of Eqs. (2.7) and (5.6), the above may be written

$$f(t)=f(0)+\omega u\int_{t-\delta/u}^{t} e^{\alpha u(t-s)}f(s)ds +\beta v\int_{t-\delta/v}^{t} e^{\alpha v(t-s)}f(s)ds. \tag{6.2}$$

With $\omega=0$, Eq. (6.2) becomes equal to (2.9). With $\beta=0$, it becomes analogous to (2.9) and the solution proceeds in the same manner, the role of v being replaced with u. When the quantities ωu and βv have comparable values, Eq. (6.2) must be solved as stands. The equation may be converted into an integro-difference equation analogous to (3.3) providing the electron transit time $t_e(t_e\equiv\delta/|v|)$ is an integral multiple of the positive ion transit time $t_p(t_p\equiv\delta/|\omega|)$. The solution of (6.2) is then carried out in the same manner as the development presented in Sec. 3.

Additional difficulties are encountered if the simplified model chosen here for discussion is made more realistic. For example, we have assumed the initial electron current is restricted to the cathode. If at the initial time there is a finite distribution of electrons in space, we are required to construct relations analogous to Eq. (2.2) for each localized source of initial current and obtain the net current by superposition. In the event the dielectric material under consideration has an appreciable photon absorption coefficient, Eq. (2.2) must be amended accordingly. The effect of absorption causes no undue difficulty in treatment only as long as the coefficient is a well-defined function of space and time. The fact that photons travel with finite speeds instead of infinite, as we have tacitly assumed, may also be accounted for by amending Eq. (2.2); this correction, however, is negligible in most cases of physical interest.

b. Formative Time Lag

The application of the results given in Sec. 3 to the calculation of formative time lags is obvious. It is only necessary to find the time intervals required by the relation (3.17) to reach some arbitrary value adjudged equal to the value of the electron current density at the cathode during breakdown. In order to compare our results with experimental data it would be highly desirable to estimate accurately the magnitude of the breakdown current at either the cathode or anode.

APPENDIX A.

Consider the integral operator

$$\mathbf{I}=\int_0^y ds e^{\sigma(y-s)}. \tag{A-1}$$

Equation (3.7) of the text may be written in terms of $\mathbf{I}$ as

$$(1-\lambda\mathbf{I})\phi_1(y)=f_0; \tag{A-2}$$

and the solution of this equation provides the relation

$$(1-\lambda\mathbf{I})^{-1}=(1+\lambda\mathbf{J}),\quad \mathbf{J}=\int_0^y ds e^{q(y-s)}. \tag{A-3}$$

After adding and subtracting the term $p\mathbf{I}\phi_{n-1}(y)$ on the right of (3.3) we obtain

$$(1-\lambda\mathbf{I})\phi_n(y)=f_0-p\mathbf{I}\phi_{n-1}(y) +e^{\sigma y}[\phi_n(0)-\phi_1(0)], \tag{A-4}$$

where use has been made of (3.4). Application of the operational algebra of (A-3) to (A-4) yields the difference equation

$$\phi_n(y)-\phi_n(0)e^{qy}=\phi_1(y)-\phi_1(0)e^{qy}-p\mathbf{J}\phi_{n-1}(y); \tag{A-5}$$

and the solution of (A-5) is given by

$$\phi_n(y)=\sum_{k=0}^{n-1}(-1)^k p^k \mathbf{J}^k\left\{\phi_1(0)\frac{\sigma}{q}[1-e^{qy}]+\phi_{n-k}(0)e^{qy}\right\}, \quad \text{(A-6)}$$

where use has been made of Eq. (3.8).

The following relations involving the operater **J** are now required:

$$\mathbf{J}^m(ae^{qy})=\frac{ay^m}{m!}e^{qy},$$
$$\mathbf{J}^m(a)=(-1)^{m-1}aq^{-m}e^{qy}\sum_0^{m-1}(-1)^h\frac{(qy)^h}{h!}+(-1)^m aq^{-m}, \quad \text{(A-7)}$$
$$a=\text{constant}.$$

Substituting the above results into (A-6) and combining terms yields

$$\phi_n(y)=\frac{\sigma}{q}\phi_1(0)\sum_{k=0}^{n-1}(p/q)^k+\sum_{k=0}^{n-1}(-1)^k\frac{(py)^k}{k!}\phi_{n-k}(0)e^{qy}-\frac{\sigma}{q}\phi_1(0)\sum_{k=0}^{n-1}\sum_{h=0}^{k}(-1)^h(p/q)^k\frac{(qy)^h}{h!}e^{qy}. \quad \text{(A-8)}$$

The first sum or the right side of the above equation is a geometric series. We introduce the following notation:

$$R_n(\lambda,\sigma)\equiv\frac{\sigma}{q}\sum_{k=0}^{n-1}(p/q)^k=\frac{\sigma}{q}\left[\frac{1-(p/q)^n}{1-p/q}\right], \quad p\neq q, \quad \text{(A-9)}$$

and note the following useful property of the polynominal $R_n(\lambda,\sigma)$:

$$R_{n-m}=(p/q)^{-m}[R_n-R_m]. \quad \text{(A-10)}$$

The order of summation in the last term of (A-8) is now inverted and the results of (A-9, 10) are applied to obtain the final form

$$\phi_n(y)-\phi_1(0)R_n(\lambda,\sigma)=e^{qy}\sum_{k=0}^{n-1}(-1)^k\frac{(py)^k}{k!}\times[\phi_{n-k}(0)-\phi_1(0)R_{n-k}(\lambda,\sigma)]. \quad \text{(A-11)}$$

A set of relations may be obtained for the $\phi_j(0)$ on setting $y=1$ in expression (A-11). Using the notation

$$\phi_1(0)\Psi_n=\phi_n(0)-\phi_1(0)R_n(\lambda,\sigma), \quad \text{(A-12)}$$

we obtain the set of equations

$$\sum_{k=0}^{n-1}(-1)^k a_k\Psi_{n-k}=\theta_n,$$
$$\begin{cases} a_0=1, & a_k=e^q p^{(k-1)}/(k-1)!, \\ \theta_1=\lambda/q, & \theta_j=-(\sigma/q)(p/q)^{j-1}. \end{cases} \quad \text{(A-13)}$$

The set of linear equations given in (A-13) may be inverted by standard matrix methods. We note, however, that the coefficients a_k are the coefficients of $(-1)^k s^k$ in the expansion of the generating function $F(s)=1-se^{(q-ps)}$. Consequently, we may obtain the elements of the triangular matrix inverse to the matrix of a_k by collecting the coefficients of s^k in the expansion of $[1/F(s)]$. Proceeding in this manner, we obtain

$$\Psi_n=\sum_{j=0}^{n-1}A_j\theta_{n-j},$$
$$A_0=1, \quad \text{(A-14)}$$
$$A_n=e^{nq}\sum_{j=0}^{n-1}(-1)^j\frac{(n-j)^j}{j!}(pe^{-q})^j.$$

Equations (A-11) and (A-14) correspond to Eqs. (3.14) and (3.12) and (3.13) of the text, respectively.

APPENDIX B.

We are indebted to Dr. H. Poritsky for suggesting the method of proof described below. It is required to show that

$$\lim_{n\to\infty}e^{nu}\left\{1-e^s\sum_0^n(-1)^j s^j/j!\right\}=0, \quad u>s. \quad \text{(B-1)}$$

We write, for complex s and t,

$$\sum_0^n(-1)^j s^j/j!=\frac{1}{2\pi i}\oint dte^{-st}\sum_1^{n+1}t^{-j}=\frac{1}{2\pi i}\oint dte^{-st}\left[\frac{1-t^{-(n+1)}}{t-1}\right], \quad \text{(B-2)}$$

where the contour is taken to include the origin and unity on the positive real axis. According to the theory of residues,

$$e^{-s}=\frac{1}{2\pi i}\oint\frac{e^{-st}}{t-1}dt, \quad \text{(B-3)}$$

the contour being extended to infinity. Combining results, we find

$$\lim_{n\to\infty}e^{nu}\left\{1-e^s\sum_0^n(-1)^j s^j/j!\right\}=\lim_{n\to\infty}\frac{e^{-u}}{2\pi i}\oint\left(\frac{e^u}{t}\right)^{n+1}\frac{e^{-s(t-1)}}{(t-1)}dt. \quad \text{(B-4)}$$

Since u is finite and e^{-st} is analytic for positive t at least up to infinity, the quantity on the right of (B-4) may be made to approach zero by choosing t large enough. This completes the required proof.