Equivalent Two-Body Method for the Triton*

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A method for developing equivalent two-body problems for the binding energy of the triton has been derived from the variational principle. The method has been applied to two central-force problems. In the first, the exponential well of Rarita and Present is discussed and the method is shown to give excellent results. In the second, the central-force term in the Lévy potential including the repulsive core is investigated. It is estimated that the Lévy potential including the tensor term will give the correct order of binding energy to the triton.

INTRODUCTION

THE problem of solving the quantum-mechanical three-body problem has received from the very outset considerable attention from nuclear physicists. This is natural inasmuch as the binding energy of the triton provides a first test of any nuclear potential that has been found to predict the experimentally known deuteron and nucleon-nucleon scattering data. It is also now known to be a sensitive test of the particular shape of the potential that is assumed.

The mathematical difficulty that is immediately encountered is the lack of separability of Schrödinger's equation. Since the classic work of Hylleraas[1] on the ground state of the helium atom, the problem has been almost exclusively attacked by use of the Ritz variational method. Among the large-scale applications of this method to the triton problem, we may mention the work of Rarita and Present,[2] and, more recently, in connection with the use of tensor forces, the work of Schwinger, Feshbach, and co-workers.[3–8] The last-named work[8] represents at this time the culmination of this effort in providing a Yukawa-type tensor interaction that fits all the known low-energy two-body experimental data and also predicts within the error of the variational method the experimental values of the triton- and alpha-particle binding energies. Among other methods of attack on the problem that have been tried, the work of Svartholm[9] is noteworthy: he has made use of the more powerful variation-iteration method. A direct numerical procedure for solving the associated integral equation[10] has also been attempted.

Because of the tremendous amount of labor that is involved in these computations, more approximate methods for obtaining reasonably accurate solutions have always been desired. Furthermore, the large state of flux of the nuclear two-body potential has, unfortunately, rendered obsolete a vast amount of the work that has been accomplished in the past, and the recent introduction of a hard-core meson-theoretic potential[11] threatens to do the same for much of the recent work, insofar as the use of a particular well shape is concerned.

This paper is designed to provide a simple means of providing approximate eigenvalue solutions for any given potential. We shall work with the S state only. The extensions required for the inclusion of tensor forces are not difficult to outline, though the required analysis would be very lengthy. The most notable effort in this direction in the past was Feenberg's "equivalent" two-body method.[12] This method is based on plausibility arguments which at best can be shown to hold for a restricted class of potentials.[13] It is not known whether it gives an upper or lower bound to the energy, although the latter has been surmised from the results of various examples.

In what follows, an "equivalent" two-body method is introduced in which the essential approximation employed is to give the wave function a particular functional form.[14] This leads to a true two-body equation, of which the lowest eigenvalue is an upper bound of the lowest eigenvalue of the original three-body equation. The equation is given once and for all except that the "effective" two-body potential energy that has to be introduced is an integral of the original three-body potential energy.

EFFECTIVE TWO-BODY METHOD

It is assumed that the center-of-mass motion is removed, and for simplicity only central potentials will be considered so that, consequently, the triton in its ground state is in a $^2S_{\frac{1}{2}}$ state. The variational principle

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14 It should be mentioned that such an approach was already suggested by the pioneer work of E. P. Wigner [Phys. Rev. **43**, 252 (1933)] in this field.

for Schrödinger's equation may then be written as

$$0=\delta\int_0^\infty dr_1\int_0^\infty dr_2\int_{|r_1-r_2|}^{r_1+r_2} dr_3 r_1 r_2 r_3$$
$$\times\Bigg\{\left(\frac{\partial\psi}{\partial r_1}\right)^2+\left(\frac{\partial\psi}{\partial r_2}\right)^2+\left(\frac{\partial\psi}{\partial r_3}\right)^2$$
$$+\frac{r_1^2+r_3^2-r_2^2}{2r_1r_3}\frac{\partial\psi}{\partial r_1}\frac{\partial\psi}{\partial r_3}+\frac{r_1^2+r_2^2-r_3^2}{2r_1r_2}\frac{\partial\psi}{\partial r_1}\frac{\partial\psi}{\partial r_2}$$
$$+\frac{r_2^2+r_3^2-r_1^2}{2r_2r_3}\frac{\partial\psi}{\partial r_2}\frac{\partial\psi}{\partial r_3}$$
$$+\frac{m}{\hbar^2}[V_1(r_1)+V_2(r_2)+V_3(r_3)]\psi^2-\frac{m}{\hbar^2}E\psi^2\Bigg\}, \quad (1)$$

where (r_1,r_2,r_3) represent the three nucleon interdistances (see Fig. 1) of the triton and ψ is a function of (r_1,r_2,r_3) only. The potentials $V_i(r_i)$ represent the result of taking the appropriate spin average. Again for simplicity we have taken the spin dependence of the wave function as $(1/\sqrt{2})\alpha(1)[\alpha(2)\beta(3)-\beta(2)\alpha(3)]$. Introducing the other possible spin dependence, $(1/\sqrt{6})\{\alpha(1)[\alpha(2)\beta(3)+\beta(2)\alpha(3)]-2\beta(1)\alpha(2)\alpha(3)\}$, and its associated spatial wave function would lead eventually to a pair of coupled two-body Schrödinger equations. For most potentials considered up to now, this additional complication does not lead to significant effects and for this reason we have not included it. If Eq. (1) is considered as a variational principle for the energy, it is known that the trial function $\psi=\exp[-\frac{1}{2}\lambda(r_1+r_2+r_3)]$ yields excellent results for the binding energy, with λ considered as a variational parameter. Such a trial function bears out the idea that the triton in its ground state is spherical and essentially symmetric in the behavior of its constituent particles. This is generalized by letting

$$\psi=\varphi(\tfrac{1}{2}(r_1+r_2+r_3)), \quad (2)$$

where φ is an unspecified arbitrary function of the perimetric length of the triangle formed by the three particles. It is now desirable to change variables so that this length becomes one of the independent variables.

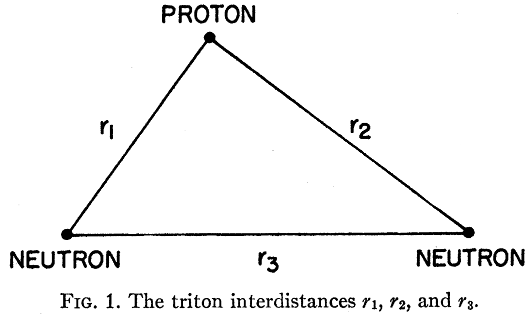

FIG. 1. The triton interdistances r_1, r_2, and r_3.

Thus, let

$$R=\tfrac{1}{2}(r_1+r_2+r_3), \quad R_2=r_2, \quad R_3=r_3. \quad (3)$$

Then, since φ is a function of R only, the integration over the other variables (R_2,R_3) may be performed. The volume element is now

$$\int_0^\infty dr_1\int_0^\infty dr_2\int_{|r_1-r_2|}^{r_1+r_2} dr_3 r_1 r_2 r_3$$
$$=\int_0^\infty dR\int_0^R dR_2\int_{R-R_2}^R dR_3(2R-R_2-R_3)R_2R_3, \quad (4)$$

and Eq. (1) becomes

$$0=\delta\int_0^\infty dR\frac{R^5}{8}\left\{\left(\frac{d\varphi}{dR}\right)^2+\frac{m}{\hbar^2}V_{\text{eff}}\varphi^2-\frac{m}{\hbar^2}E\frac{14}{15}\varphi^2\right\}, \quad (5)$$

where

$$V_{\text{eff}}=\frac{8}{R^5}\int_0^R dR_2\int_{R-R_2}^R dR_3(2R-R_2-R_3)R_2R_3$$
$$\times\{V_1(r_1)+V_2(r_2)+V_3(r_3)\}. \quad (6)$$

If the potentials are now assumed to be all the same shape, so that $V_i(r_i)=V_i v(r_i)$, with V_i constant, Eq. (6) becomes

$$V_{\text{eff}}=(V_1+V_2+V_3)\cdot\frac{8}{R^5}$$
$$\times\int_0^R dR_2(R^2R_2^2-RR_2^3+\tfrac{1}{6}R_2^4)v(R_2). \quad (7)$$

A particular form of $\varphi(R)$ may, of course, still be specified. The choice $\varphi=e^{-\lambda R}$ naturally reduces to the previously mentioned calculation in a very simple way. The best choice of the form of $\varphi(R)$ is the solution of the equation obtained from the variation indicated in Eq. (5):

$$\frac{1}{R^5}\frac{d}{dR}\left(R^5\frac{d\varphi}{dR}\right)-\frac{14}{15}k^2\varphi+U_{\text{eff}}\varphi=0, \quad (8)$$

where $k^2=(m/\hbar^2)|E|$ and $U_{\text{eff}}=-(m/\hbar^2)V_{\text{eff}}$. Letting $\varphi(R)=F(R)/R^{5/2}$, this becomes

$$\frac{d^2F}{dR^2}-\frac{14}{15}k^2F-\frac{15}{4}\frac{1}{R^2}F+U_{\text{eff}}F=0. \quad (9)$$

It may now be noted that this is exactly the form of the Schrödinger equation for the deuteron in which $\psi=[F(R)/R]Y_{lm}$. The reduced mass has the value $(14/15)m$ instead of m, and there is a centrifugal potential energy term $[+(15/4)R^{-2}]$ corresponding to setting the orbital quantum number $l=\frac{3}{2}$. This is a very important term and is contributed by the kinetic energy, independent of the choice of potential energy.

The potential energy that is to be inserted in this equation is not the actual two-body potential, but the "averaged" potential given by Eq. (7). It might perhaps be more apt to call the combination $[-U_{\rm eff}+(15/4)R^{-2}]$ the true effective potential of the problem, as can be readily seen by the form of the equation. This completes the reduction to an "effective" two-body equation. It may perhaps bear repeating that the lowest eigenvalue associated with Eq. (9) remains an upper bound of the true eigenvalue of the original triton problem.

For purposes of illustration, assume Eq. (9) is to be solved for an exponential well nucleon-nucleon interaction. Let $V_i(r_i)=-V_ie^{-r_i/R_0}$, and let R_0 be chosen as the unit of length. For this case $U_{\rm eff}$ becomes, after performing the integration indicated in Eq. (7),

$$U_{\rm eff}=(14/15)(U_1+U_2+U_3)R_0^2u(R),$$

where

$$u(R)=\frac{8}{R^5}\frac{15}{14}\left\{4-6R+2R^2+e^{-R}\times\left[-4+2R+2R^2+\frac{R^3}{3}-\frac{R^4}{6}\right]\right\}. \quad (10)$$

The function $u(R)$ decreases monotonically from the value 1 at the origin to zero value at infinity. The attractive potential $(-U_{\rm eff})$ is plotted in Fig. 2, as well as the combination $[-U_{\rm eff}+(15/4)R^{-2}]$, with $(U_1+U_2+U_3)R_0^2$ set equal to 5.128. This corresponds

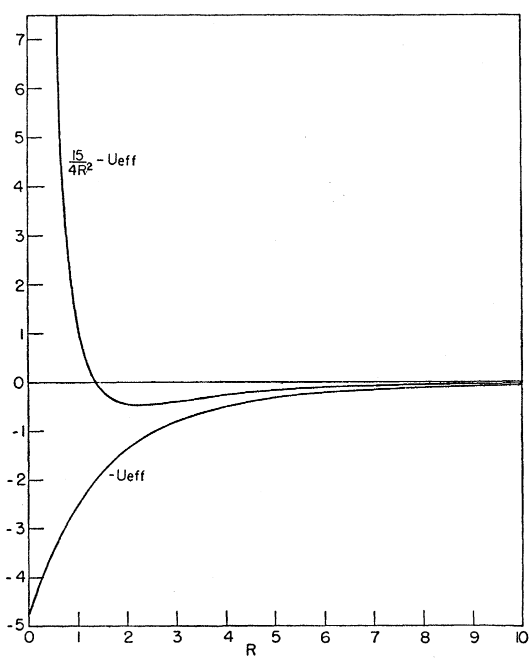

FIG. 2. The effective potential $(-U_{\rm eff})$ and the net effective potential $[(15/4)R^{-2}-U_{\rm eff}]$ with $(U_1+U_2+U_3)R_0^2=5.128$. The unit of length is $R_0=0.678\times10^{-13}$ cm, and unit well depth corresponds to 90.3 Mev.

to a choice of neutron-proton and neutron-neutron singlet and triplet well depths which fit the low-energy scattering data.[15] Numerical integration of the equation for this case yields a binding energy value of 11.02 Mev, compared to the experimental value of 8.50 Mev. This is, surprisingly, only about 1 percent deeper than the value that is obtained with the trial function $e^{-\lambda R}$, which is probably a reflection of the "goodness" of the latter trial function.

As a second illustration, it will easily be shown that a two-body potential of infinitesimally small range will produce infinite binding of the triton.[16] For, suppose the nucleon-nucleon potential is a delta function,

$$V_2(r_2)=-V_2\delta(\mathbf{r}_2),$$

of such a strength that

$$\frac{1}{4\pi}\int d\tau_2V_2(r_2)=-V_2.$$

It follows directly from Eq. (7) for the effective potential that $U_{\rm eff}=(U_1+U_2+U_3)\cdot8/R^3$, and such a singular potential is well known to have an infinite binding energy.

APPLICATION TO HARD-CORE POTENTIALS; THE LÉVY POTENTIAL

Recent calculations based on various approximations in the meson theory of nuclear forces have indicated that as the distance between nucleons decreases, the force between nucleons eventually becomes very strongly repulsive. A similar result has been suggested by the success of a hard-core model proposed on a phenomenological basis by Jastrow.[17] We shall be interested here in formulating an equivalent two-body method for S potentials of this type and shall apply the method to a potential proposed by Lévy[11] which has the merit of giving an adequate description of low-energy two-nucleon phenomena. The potential under consideration here is:

$$V(r)=\begin{cases}\infty & r<r_0\\ V_0v(r) & r>r_0,\end{cases}$$

where r is the distance between any two nucleons. The wave function must satisfy the following boundary conditions:

$$\psi(r_1,r_2,r_3)=0 \quad\text{if}\quad r_i\leq r_0,\quad i=1,2,3.$$

A few qualitative remarks are in order at this point. In the absence of a hard core it is well known that an attractive two-body potential will lead to a triton structure which is considerably more compact than the deuteron, with a corresponding increase in the binding energy per particle. This is illustrated by the calculation

[15] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).
[16] L. H. Thomas, Phys. Rev. **47**, 903 (1935).
[17] R. Jastrow, Phys. Rev. **78**, 135 (1950).

in the preceding section where the two-body exponential potential required to yield the correct deuteron binding energy gave a triton binding energy far in excess of the experimental value. A simple explanation of this effect will now be given so that the modification introduced by hard-core potentials may be understood.

Let $u(r)$ be the solution of the deuteron problem with potential $V(r)$. Insert then the trial function $\psi=u(r_1)u(r_2)$ into the variational principle [Eq. (1)] for the binding energy of the triton, letting $V_i(r_i)=V(r_i)$. We find immediately that the resultant variational energy equals twice the binding energy of the deuteron plus an average over potential $V(r_3)$ which will always increase the calculated binding energy if the usual monotonic potentials are employed for V. This potential energy term is not balanced by a corresponding kinetic energy term, so that for the narrow deep potentials customarily used this term will be large. For hard-core potentials the physical situation is changed drastically. Since the wave function ψ must go to zero whenever $r_i\leq r_0$ and in particular $r_3\leq r_0$, the function $u(r_1)u(r_2)$ is no longer an admissible trial function for the three-body problem. New kinetic energy terms are introduced because the wave function must be so distorted as to be zero not only whenever r_1 and r_2 equal r_0 but also when $r_3=r_0$. It is no longer possible to make even the mild statement that the binding energy of the triton is at least twice the binding energy of the deuteron. As we shall see below this kinetic energy effect is large. Of course similar remarks apply to any potential which is not essentially monotonic, the hard-core potential being an extreme case where this effect is probably a maximum.

Let the potential in variational principle (1) be

$$V_i(r_i)=\begin{cases}\infty & r_i<r_0\\ V_i v(r_i) & r_i>r_0.\end{cases} \tag{11}$$

A trial function which obeys the boundary conditions and which is still simple in form is

$$\psi(r_1,r_2,r_3)=\begin{cases}0 & r_i\leq r_0\\ (r_1-r_0)(r_2-r_0)(r_3-r_0)\varphi(r_1+r_2+r_3) & \text{elsewhere.}\end{cases} \tag{12}$$

As we did in the previous section, we introduce appropriate new coordinates:

$$R=r_1+r_2+r_3-3r_0,\quad R_2=r_2,\quad R_3=r_3, \tag{13}$$

and integrate over (R_2,R_3). A complication arises from the limits of integration, for the volume element is now

$$\int_{r_0}^{\infty}dr_1\int_{r_0}^{\infty}dr_2\int_{[|r_1-r_2|,r_0]}^{r_1+r_2}dr_3,$$

where the brackets signify "the greater of the two quantities." However, the integration over (R_2,R_3) is straightforward and is carried through once and for all for the kinetic energy. It is convenient to introduce the nondimensional variable $\rho=R/r_0$. The equation analogous to (5) is then

$$0=\delta\int_0^{\infty}d\rho\, a_2(\rho)\left\{\left(\frac{d\varphi}{d\rho}\right)^2+\frac{a_1(\rho)}{a_2(\rho)}\varphi\frac{d\varphi}{d\rho}+\frac{a_0(\rho)}{a_2(\rho)}\varphi^2+\frac{k^2r_0^2n(\rho)}{a_2(\rho)}\varphi^2-r_0^2U_{\text{eff}}\varphi^2\right\}, \tag{14}$$

where

$$a_2(\rho)=\rho^8[0.2\rho^3+2.6\rho^2+9\rho+9],\quad 0\leq\rho\leq1,$$
$$=\frac{1}{256}[39.8\rho^{11}+517.4\rho^{10}+2367\rho^9+3639\rho^8-1194\rho^7-3301.2\rho^6+4796.4\rho^5+822\rho^4-3825\rho^3+819\rho^2+1092.6\rho-448.2],\quad \rho>1;$$

$$a_1(\rho)=\rho^7[\rho^3+18\rho^2+69\rho+72],\quad 0\leq\rho\leq1,$$
$$=\frac{1}{256}[253\rho^{10}+3402\rho^9+16\,557\rho^8+28\,824\rho^7-1638\rho^6-25\,956\rho^5+20\,706\rho^4+9432\rho^3-12\,087\rho^2-342\rho+1809],\quad \rho>1;$$

$$a_0(\rho)=\rho^6[\rho^3+45\rho^2+198\rho+210],\quad 0\leq\rho\leq1,$$
$$=\frac{1}{256}[451\rho^9+6723\rho^8+40\,572\rho^7+87\,948\rho^6+21\,546\rho^5-76\,230\rho^4+12\,012\rho^3+32\,796\rho^2-2133\rho-7461],\quad \rho>1;$$

$$n(\rho)=\rho^8\left[\frac{0.6}{11}\rho^3+0.6\rho^2+2\rho+2\right],\quad 0\leq\rho\leq1,$$
$$=\frac{1}{256}\left[\frac{101.4}{11}\rho^{11}+119.4\rho^{10}+542\rho^9+812\rho^8-306\rho^7-747.6\rho^6+1142.4\rho^5+180\rho^4-915\rho^3+201\rho^2+262.8\rho-\frac{1195.2}{11}\right],\quad \rho\geq1; \tag{15}$$

$$U_{\text{eff}}=(U_1+U_2+U_3)\frac{1008}{a_2(\rho)}\int_1^{\rho+1}dx\,x(x-1)^2\times(\rho+1-x)^5\left[\frac{1}{14}(\rho+1-x)^2+\frac{1}{3}(\rho+1-x)+\frac{1}{3}\right]v(x),\quad 0\leq\rho\leq1,$$

$$=(U_1+U_2+U_3)\frac{10\,080}{a_2(\rho)}\int_1^{(\rho+1)/2}dx\cdot x^2(x-1)^2\times\left\{\frac{1}{140}x^6-\left[\frac{1}{10}\left(\frac{\rho+1}{2}\right)+\frac{1}{30}\right]x^5+\left[\frac{3}{5}\left(\frac{\rho+1}{2}\right)^2+\frac{2}{5}\left(\frac{\rho+1}{2}\right)+\frac{1}{30}\right]x^4-\left(\frac{\rho+1}{2}\right)\left[2\left(\frac{\rho+1}{2}\right)^2+2\left(\frac{\rho+1}{2}\right)+\frac{1}{3}\right]x^3+\left(\frac{\rho+1}{2}\right)^2\left[\frac{7}{2}\left(\frac{\rho+1}{2}\right)^2+\frac{14}{3}\left(\frac{\rho+1}{2}\right)+\frac{4}{3}\right]x^2-\left(\frac{\rho+1}{2}\right)^3\left[3\left(\frac{\rho+1}{2}\right)^2+5\left(\frac{\rho+1}{2}\right)+2\right]x+\left(\frac{\rho+1}{2}\right)^4\left(\frac{\rho+1}{2}+1\right)^2\right\}v(x)$$
$$+(U_1+U_2+U_3)\frac{1008}{a_2(\rho)}\times\int_{(\rho+1)/2}^{(\rho+1)/2+1}dx\,x(x-1)^2(\rho+1-x)^5\times\left[\frac{1}{14}(\rho+1-x)^2+\frac{1}{3}(\rho+1-x)+\frac{1}{3}\right]v(x),\quad \rho>1;\qquad(16)$$

with $x=r_2/r_0$. All the numbers given in these equations are exact. It should be noted that the point $\rho=0$ corresponds in the original problem to the point $r_1=r_2=r_3=r_0$, while the point $\rho=1$ corresponds to $R=4r_0$. The associated differential equation is

$$\frac{d}{d\rho}\left(a_2\frac{d\varphi}{d\rho}\right)+\left[\frac{1}{2}\frac{da_1}{d\rho}-a_0\right]\varphi-k^2r_0^2n(\rho)\varphi+U_{\text{eff}}r_0^2a_2\varphi=0.\qquad(17)$$

Or, setting $[a_2(\rho)]^{\frac{1}{2}}\varphi=F(\rho)$, the effective two-body equation becomes

$$\frac{d^2F}{d\rho^2}-k^2r_0^2\frac{n(\rho)}{a_2(\rho)}F-c(\rho)F+U_{\text{eff}}r_0^2F=0,\qquad(18)$$

where $c(\rho)=a_0/a_2-\frac{1}{4}(a_1/a_2)^2$. The ratio $n(\rho)/a_2(\rho)$ is remarkably constant for the entire range of values of ρ and varies monotonically from a minimum value of $2/9=0.2222\cdots$ at $\rho=0$ to a maximum value of $0.2316\cdots$ at $\rho=\infty$. Hence the effective mass is smaller by a factor of about 4.5 than that of a corresponding deuteron equation. The binding energy will therefore be magnified by the factor 4.5 (as contrasted with the factor 15/14 in the previous case). Of greater importance is the term U_{eff} and the centrifugal repulsive term $c(\rho)$. For small values of ρ, $c(\rho)\sim22/3\rho^2$ so that F goes to zero at $\rho=0$ like $\rho^{2.25}$, where the exponent is approximate. Note that the wave function φ and therefore ψ is singular at $\rho=0$, since

$$\varphi=F(\rho)/a_2^{\frac{1}{2}}\underset{\rho\to0}{\longrightarrow}\rho^{-1.75}.$$

Of course ψ is nevertheless quadratically integrable because of the high dimensionality of the configuration space for the three particles.

We have obtained U_{eff} and integrated Eq. (18) numerically for the central-force term in the Lévy potential. Since the Lévy potential contains noncentral tensor terms as well, the calculation gives at best an upper bound to the energy. A lower bound may be obtained by increasing the strength of the central Lévy potential until it, unaided by a tensor term, would bind the deuteron with the experimental binding energy. This may be accomplished by increasing the well depth by 42.5 percent. We expect that such a potential would give a greater binding in the three-body problem than

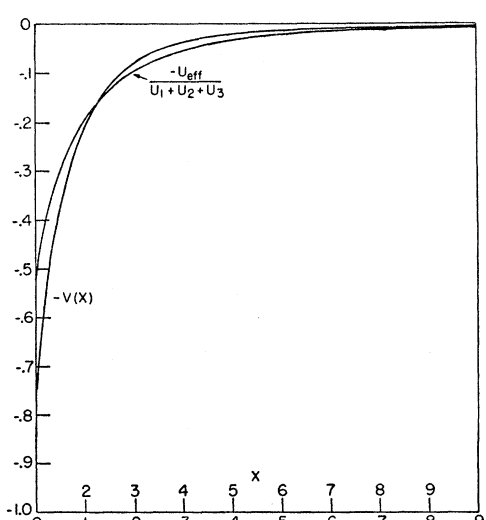

FIG. 3. The Lévy well-shape function $-v(x)$ as compared to the effective two-body well shape $-U_{\text{eff}}(\rho)/(U_1+U_2+U_3)$. The unit of length is the core radius $r_0=0.38(1.40\times10^{-13})$ cm, and unit well depth corresponds to 147 Mev.

the original Lévy potential including the tensor term, since the tensor term is expected to be less effective in binding H^3 than in binding H^2.

To summarize, we have calculated the binding energy of H^3 using two values of $U_i r_0^2$, (1) 10.57 and (2) 15.06, with a $v(x)$ equal to

$$v(x)=\frac{1}{x^2}\left[\frac{2}{\pi}K_1(0.76x)+0.075\left(\frac{2}{\pi}K_1(0.38x)\right)^2 +0.013xe^{-0.38x}\right],\quad x\geq 1, \tag{19}$$

and with $r_0=0.38(1.4\times 10^{-13})$ cm. We expect the binding energy to be between the two results. The potential $v(x)$ is a very deep but narrow potential hole. The well depth at $\rho=1$ gives a value of V_i of 1189 Mev and 1690 Mev for the two cases. In Fig. 3, we have plotted $-v(x)$ as well as the effective potential well shape $-U_{\text{eff}}/(U_1+U_2+U_3)$.

In Fig. 4, we give the total equivalent two-body potential including the centrifugal term for the Lévy central potential (case 1) and also the stronger adjusted Lévy central potential, case (2). We see that the effective potential is almost completely canceled by the kinetic repulsion for the original Lévy central potential. As we expected from our earlier qualitative discussion, the insistence on making the wave function go to zero at $r_i=r_0$ is costly in kinetic energy; the stronger centrifugal term, as compared to the case in the preceding section, arises from the linear terms inserted in the variational form (12) in order to meet the boundary conditions. As we see from Fig. 4, only a very small attractive potential pocket remains. One can immediately infer that there is no binding in case (1). In case (2) the well is considerably deeper and broader and binding is obtained. The binding energy calculated by integrating Eq. (18) numerically is found to be 13.3 Mev.

From this we conclude that the binding energy calculated with the complete Lévy potential including the tensor term would be between zero and 13.3 Mev, in short, of the right order of magnitude. This is of some interest since it indicates that, if the Lévy potential is correct,[18] strong three-body forces are not needed in the understanding of the triton. It indicates that the kinetic effects of the repulsive core are large and may be enough to at least partially insure saturation of nuclear forces in heavier nuclei.

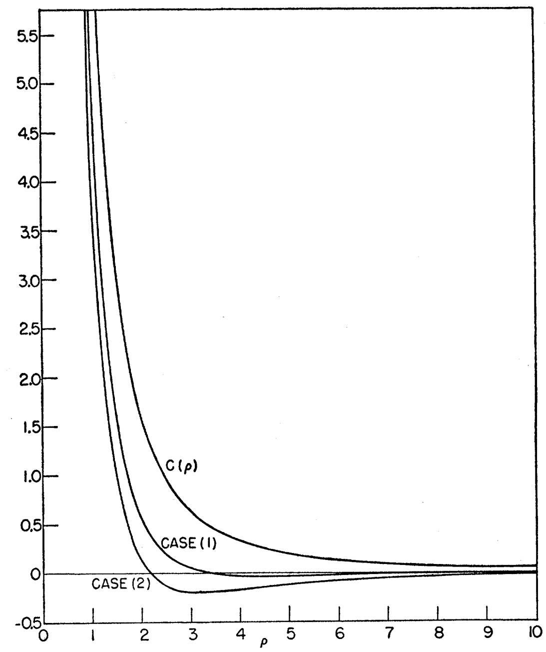

FIG. 4. The centripetal repulsion function $c(\rho)$ and the net effective potentials for (1) the Lévy central potential, and (2) the Lévy central potential with well depth adjusted to give the correct value of the binding energy of the deuteron. The units are the same as for Fig. 3.

The authors wish to take this opportunity to thank Mrs. Barbara Levine of the Joint Computing Group for performing the necessary numerical computations.

[18] The original derivation from meson theory was found to be faulty. See A. Klein, Phys. Rev. 89, 1158 (1953).