sections σ_e implies equality of the total cross sections. Thus for the purposes of this paper it is sufficient that at low energies σ_e in H_2 be very nearly independent of the molecular rotational state. That this is indeed the case stems from the fact that, apart from the small contribution of long-range forces [comparable to the inelastic cross sections in Eq. (20) of I], the zero energy elastic cross section involves only incoming and outgoing s-waves, coupling through the spherical part of the interaction. At zero energy, consequently, the interaction, Eq. (12) of I, can be replaced by its spherical part, which part is readily seen to yield a scattering amplitude independent of J. A more detailed analysis[13] shows that σ_e remains very nearly independent of rotational state at energies less than a few tenths of an electron volt.

In conclusion, we stress: (1) We have not found any experimental comparisons of the total cross sections, and therefore are forced to rely on a theoretical argument. (2) Since our argument is wholly theoretical, and in many respects qualitative, we feel it would be worth while to measure these total cross sections using some suitable and identical procedure, e.g., Varnerin's.[5] (3) The swarm mobility and diffusion measurements[2] which yield λ' also yield a measurement of σ_t, so that the proposed swarm experiments can check, albeit somewhat equivocally[14] our assumption about σ_t.

[13] S. Stein, thesis, University of Pittsburgh, 1955 (unpublished).

[14] Since the swarm experiments determine an effective σ_t from complicated averages over the electron distribution, and since the electron energy distribution is affected by inelastic losses, it is possible for the swarm experiments to indicate unequal σ_t even though the total cross sections in these gases actually are the same. Thus, for definitive experiments detailed knowledge of the distribution functions is required, but we may expect that there should be at least qualitative significance to comparing the magnitudes of $\sigma_{t\,\mathrm{eff}}$ in different gases.

We are happy to thank Dr. Arthur V. Phelps for his very helpful comments.

APPENDIX

The examination of the validity of the theory in H_2 parallels I, Sec. IV. Because both Q and the molecular radius r_0 are smaller in H_2 than in N_2, and because we are here interested in even slower electrons than in the previous paper, the only question requiring detailed examination is the ratio A_1/A_2 of "near-" and "far-" field amplitudes. Using, much as in I, the Wang potential[15] with the nuclei at their equilibrium separation to compute for H_2 the quantity $f_a(\theta)$, we obtain for small k_a

$$A_1 = 1.8(k_a a_0)^2 A_2. \tag{1}$$

Estimating $f_a(\theta)$ from the measured elastic cross section gives nearly the same result, the factor being 2.3 instead of 1.8. At the vibrational threshold, Eq. (1) makes $A_1/A_2 = 0.07$; at 0.075 ev, $A_1/A_2 = 0.01$. These numbers indicate that the cross sections of I are valid for H_2 at electron energies below the vibrational threshold, and are surely valid at the very low energies of interest in the proposed low-temperature experiments.

[15] S. C. Wang, Phys. Rev. **31**, 579 (1928).

PHYSICAL REVIEW VOLUME 98, NUMBER 6 JUNE 15, 1955

Generation of Coulomb Wave Functions by Means of Recurrence Relations

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(Received March 9, 1955)

A discussion of the computation of Coulomb wave functions from their recurrence relations is given. Specifically, it is demonstrated that the regular solution F_L and the irregular solution G_L may be obtained recursively based on the knowledge of the functions for $L=0$.

THE present paper is concerned with the computation of the regular and irregular Coulomb wave functions F_L and G_L, for L a positive integer, with the aid of the recurrence relations satisfied by these functions. Thus if y_L stands for either $F_L(\eta,\rho)$ or $G_L(\eta,\rho)$ we have[1]

$$L\frac{dy_L}{d\rho} = (L^2+\eta^2)^{\frac{1}{2}} y_{L-1} - \left(\frac{L^2}{\rho}+\eta\right) y_L, \tag{1}$$

$$(L+1)\frac{dy_L}{d\rho} = \left[\frac{(L+1)^2}{\rho}+\eta\right] y_L - [(L+1)^2+\eta^2]^{\frac{1}{2}} y_{L+1}, \tag{2}$$

$$L[(L+1)^2+\eta^2]^{\frac{1}{2}} y_{L+1} = (2L+1)\left[\eta+\frac{L(L+1)}{\rho}\right] y_L - (L+1)[L^2+\eta^2]^{\frac{1}{2}} y_{L-1}. \tag{3}$$

[1] J. L. Powell, Phys. Rev. **72**, 626–627 (1947).

In addition, we have the Wronskian relations

$$F_L' G_L - F_L G_L' = 1, \tag{4}$$

$$F_{L-1} G_L - F_L G_{L-1} = L(L^2+\eta^2)^{-\frac{1}{2}}. \tag{5}$$

The method is entirely similar to that employed for the generation of Bessel functions of integral order.[2] This is to be expected in view of the fact that the functions F_L and G_L bear the same relation to each other as the Bessel functions J_n and Y_n or the modified Bessel functions I_n and K_n; namely, for $L\to\infty$, that F_L is a decreasing function of L while G_L is an increasing function of L. The recurrence relation (3) will be stable when applied in decreasing order to F_L and in increasing

[2] Bessel Functions, Part II (British Association for the Advancement of Science, Cambridge, 1952). The method is credited to J. C. P. Miller.

order to G_L. By this we mean no error will be propagated if we generate $F_L(G_L)$ in decreasing (increasing) order. The technique described may be applied for other sets of functions such as the spherical Bessel functions.

To generate the functions G_L we need only be able to calculate G_0 and G_0'. The application of (2) will produce G_1 and then (3) may be used to obtain as many values of G_L as may be desired.

To determine the values of F_L recursively for given η and ρ it would appear that F_L and F_L' (say) would be required as starting values for some $L>0$. Actually, this is not the case as we shall show. We can start with arbitrary values $\bar{F}_L$ and $\bar{F}_{L+1}$ which we consider as the values of the solution of the respective differential equations at the given η and ρ. Then, given $\bar{F}_{L+1}$, there is a one-parameter family of α and β such that

$$\alpha F_{L+1}+\beta G_{L+1}=\bar{F}_{L+1}. \tag{6}$$

In order to fix α and β uniquely we require that

$$\alpha F_L+\beta G_L=\bar{F}_L. \tag{7}$$

Then, by virtue of (5) we can solve this system for α and β. For example, with $\bar{F}_L=1$, $\bar{F}_{L+1}=0$ we get

$$\alpha=G_{L+1}\frac{\{(L+1)^2+\eta^2\}^{\frac{1}{2}}}{(L+1)},\quad \beta=-\alpha\frac{F_{L+1}}{G_{L+1}}. \tag{8}$$

If we now generate a sequence $\bar{F}_m\equiv\alpha F_m+\beta G_m$ for $m=L-1, L-2, \cdots 1, 0$ by the use of (3) we have

$$\bar{F}_m=\alpha\left\{F_m-\frac{F_{L+1}}{G_{L+1}}G_m\right\}. \tag{9}$$

Since $F_L\rightarrow 0$ and $G_L\rightarrow\infty$ as $L\rightarrow\infty$ we may choose L so large that the second factor in the brackets can be made as small as we please and thus

$$\bar{F}_m\sim\alpha F_m \tag{10}$$

where $\alpha=\alpha(L,\eta,\rho)$ and α is defined in (8).

It is now clear that if α is known we can determine F_m from $\bar{F}_m$. However, the knowledge of α implies the knowledge of G_{L+1}. On the other hand, if for some integer m where $0\leq m\leq L'\ll L$ we know F_m, then by virtue of (10) we can determine $\alpha\sim\bar{F}_m/F_m$ and use this value of α to calculate F_k for $0\leq k\leq L'\ll L$ from the corresponding $\bar{F}_k$. In particular we can therefore take $\alpha=\bar{F}_0/F_0$ provided $F_0\neq 0$. In practice, this means that if η and ρ are near a zero of F_0 we should choose some other value for m, say $m=1$.

In those situations where the G_L are also desired the procedure may be modified in the following manner. From G_0 and G_0' we generate the sequence G_m in increasing order, the sequence $\bar{F}_m$ being generated from $\bar{F}_L$, $\bar{F}_{L+1}$ in decreasing order as above. However, instead of computing α from $\bar{F}_0\sim\alpha F_0$ we can now use

$$F_0G_1-F_1G_0=\alpha^{-1}(\bar{F}_0G_1-\bar{F}_1G_0)=(1+\eta^2)^{-\frac{1}{2}},$$

which follows from (5) for $L=1$.

As an illustration of the method let us take $\eta=5$, $\rho=5$ starting with $\bar{F}_{31}=0$, $\bar{F}_{30}=0.1$. Generating the values of $\bar{F}_m$ carrying ten figures, we get

$$\begin{array}{ll}
\bar{F}_{20}=13854\ 08764, & F_{20}=0.0^{13}1883\ 426_8,\\
\bar{F}_{11}=35942\ 29977.\times 10^7, & F_{11}=0.0^{6}4886\ 261_1,\\
\bar{F}_{10}=17217\ 50614.\times 10^8, & F_{10}=0.0^{5}23406\ 746_8,\\
\cdots & \cdots\\
\bar{F}_{1}=16379\ 10103.\times 10^{12}, & F_{1}=0.02226\ 695_7,\\
\bar{F}_{0}=20355\ 68006.\times 10^{12}, & F_{0}=0.02767\ 301_2.
\end{array}$$

To eight significant figures the value of $F_0=0.02767\ 3012$, from which we find $\alpha^{-1}=1.3594737\times 10^{-23}$. An independent calculation shows that the values of F_L indicated above are correct to approximately eight significant figures. Starting now with $G_0=18.1933$ and $G_1=21.7261$ and generating the successive values of G_L, we get ultimately $G_{19}=79310\ 46945\times 10^2$, $G_{20}=62908\ 14544\times 10^3$. Checking the results with the Wronskian relation (5) shows the values of G_{19} and G_{20} to be correct to approximately six significant figures. Similar calculations were made for $\eta=10$, $\rho=1$ and $\eta=1$, $\rho=10$ with comparable success.

A suggested procedure for carrying out the aforementioned technique is as follows. Suppose the values of F_m and G_m are desired for $0\leq m\leq L'$. Then choose two values of L (say, L and $L+5$) with $L\gg\rho$ and $L\gg L'$. Generate the sequences F and G for both values of L and compare. If the results do not agree for $0\leq m\leq L'$ then start with $L+10$ and compare the results with those for $L+5$.

The derivatives F_L' and G_L' can be generated from (1) and (2) to provide a check once the desired F_L and G_L have been obtained. An additional check is obtained from (4).

The values of F_m may also be derived in the following manner. Once the values of $\bar{F}_m$ have been generated, it remains only to determine the normalizing factor α. This may be done with the aid of the relations[3]:

$$\rho\ \cos\rho=\left(\frac{e^{2\pi\eta}-1}{2\pi\eta}\right)^{\frac{1}{2}}\sum_{L=0}^{\infty}(2L+1)\ \cos\delta_L(\eta)\cdot F_L(n,\rho), \tag{11}$$

$$\rho\ \sin\rho=\left(\frac{e^{2\pi\eta}-1}{2\pi\eta}\right)^{\frac{1}{2}}\sum_{L=0}^{\infty}(2L+1)\ \sin\delta_L(\eta)\cdot F_L(\eta,\rho), \tag{12}$$

where

$$\delta_0=0,$$

$$\delta_L(\eta)=\frac{L\pi}{2}+\sum_{k=1}^{L}\arctan(\eta/L).$$

Since this procedure will yield the values of F_0 and F_1, only G_0 need be computed independently. G_1 will be found from (5) with $L=1$, and (3) may then be employed to generate as many additional values of G_L as may be desired.

[3] The authors are indebted to their colleague, Dr. P. Henrici, for having provided them with these relations.